Reducible but indecomposable

[For Yvan Saint-Aubin, in celebration of his retirement.]

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October 6, 2022

Integrable systems, exactly solvable models and algebras
CRM, Montréal, Canada

Memories

I first met Yvan in 2006–7, when I was Pierre Mathieu's postdoc.

I wanted to break into logCFT, but didn't know where to start.

Yvan gave a beautiful talk about his work on non-local observables in the Ising model, explaining that their critical exponents live in an extended Kac table.

I was hooked — but he wisely cautioned me to start with an easier model: percolation!

Following his advice, I soon had my first logCFT paper...

Hiking near Canberra, Australia, 2012.

Having fun, somewhere along the Molonglo River trail.

I will also be eternally grateful to Yvan for gifting me with a one-year postdoc in 2009-10.

In Montréal, I took a little time off from logCFT and profited (bien!) from investigating the representation theory of the Temperley-Lieb algebras.

Our review of this beautiful subject eventually appeared (arXiv: 2012, ATMP: 2014) and seems to be appreciated in the community (83 GS citations).

It also heavily influenced my subsequent work in logCFT.

I'll talk a bit about that soon. But first, a big Merci! to Yvan: supervisor, mentor, collaborator, colleague and friend.

Bonne retraite mon ami, tu l'as bien mérité.

Yvan's birthday party, 2013. I have it on good authority that he's a huge fan of Lightning McQueen...

Outline

Symmetry and reducibility

Weyl and Wigner introduced representation theory into quantum physics. According to wikipedia:

> *"... the different quantum states of an elementary particle give rise to an irreducible representation of the Poincaré group."*

In other words, a quantum particle is modelled by an irreducible representation of some symmetry group or algebra.

The irreducibility here captures the fundamental indivisibility of an elementary particle.

But what if we want our particle to have some internal structure, *eg.* a model that allows a one-way transition from an unstable to a stable state?

We'd then need a more refined version of irreducibility...

Recall that an irreducible representation is one with exactly two invariant subspaces: the trivial one $\{0\}$ and the representation space itself.

Completely reducible representations then decompose as a direct sum of irreducible ones: $V = \bigoplus_i V_i$.

Often, all (interesting) representations are completely reducible.[1] *eg.*, finite groups, compact Lie groups, semisimple Lie algebras.

But, this is rare. Normally, one also has indecomposable representations. These may have a non-trivial proper invariant subspace $0 \subset W \subset V$ with no complement: $V \neq W \oplus W'$ for any invariant subspace W'.

Such reducible but indecomposable representations are more complicated mathematically (sometimes much more so!), but may be necessary to construct a good model of a physical system.

[1] I'm going to assume throughout that my field is $\mathbb{C}$, because I'm being quantum.

Ex. 1: Lie superalgebras

Unlike simple Lie algebras, the simple Lie superalgebras typically have reducible but indecomposable finite-dimensional representations.

A (non-simple but typical) example is $\mathfrak{gl}(1|1) = \mathrm{span}\{E, N \mid \psi^+, \psi^-\}$:

$$[N, \psi^\pm] = \pm\psi^\pm, \quad \{\psi^+, \psi^-\} = E.$$

Even the adjoint representation is reducible but indecomposable:

$$\begin{tikzcd}
& N \arrow[dl] \arrow[dr] & \\
\psi^- \arrow[dr] & & \psi^+ \arrow[dl] \\
& E &
\end{tikzcd}$$

Lie superalgebras are essential to much of modern mathematical physics, especially when supersymmetry is involved.

Ex. 2: Quantum groups

Drinfeld–Jimbo quantum groups are q-deformations of the universal enveloping algebra of a simple Lie (super)algebra.

A neat example is $\mathsf{U}_q(\mathfrak{sl}_2) = \langle E, F, K \rangle$:

$$KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

Reducible but indecomposable representations arise for q a root of unity:

$$\begin{tikzcd} v \arrow[d, bend left] \\ w \arrow[u, bend left] \end{tikzcd} \otimes \begin{tikzcd} v \arrow[d, bend left] \\ w \arrow[u, bend left] \end{tikzcd} = \begin{tikzcd} & v \otimes w - \mathrm{i} w \otimes v \arrow[dl] \arrow[dr] & \\ v \otimes v \arrow[dr] & & w \otimes w \arrow[dl] \\ & v \otimes w + \mathrm{i} w \otimes v & \end{tikzcd} \qquad (q = \mathrm{i})$$

Quantum groups govern quantum integrable systems. The TT, TQ and QQ relations are reflections of reducible but indecomposable structure.

Ex. 3: Temperley–Lieb algebras

The Temperley–Lieb algebra is a ubiquitous ingredient for constructing 2D integrable lattice models.

It is the unital associative algebra $TL_n(\beta) = \langle u_1, \dots, u_{n-1} \rangle$:

$$u_i^2 = \beta u_i, \qquad u_i u_{i\pm 1} u_i = u_i, \qquad u_i u_j = u_j u_i \quad \text{if } |i-j| \geqslant 2.$$

Reducible but indecomposable representations arise for certain $\beta \in \mathbb{C}$, eg

$$TL_6(\mathfrak{i}) = 5 \begin{pmatrix} \begin{tikzcd} 5 \arrow[dr] & \\ & 4 \arrow[dl] \\ 5 & \end{tikzcd} \end{pmatrix} \oplus 4 \begin{pmatrix} \begin{tikzcd} & 4 \arrow[dl] \arrow[dr] & \\ 5 \arrow[dr] & & 1 \arrow[dl] \\ & 4 & \end{tikzcd} \end{pmatrix} \oplus \begin{pmatrix} \begin{tikzcd} & 1 \arrow[dl] \\ 4 \arrow[dr] & \\ & 1 \end{tikzcd} \end{pmatrix}.$$

These algebras underlie Q-state Potts and RSOS models. In particular, $\beta = 1$ corresponds to percolation and $\beta = \sqrt{2}$ to the Ising model.

Ex. 4: Virasoro algebras

The Virasoro algebra controls 2D conformal field theories.

It is the Lie algebra $\mathfrak{Vir} = \langle L_n, C \mid n \in \mathbb{Z} \rangle$:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C.$$

Here, the eigenvalue of C on a representation is its central charge.

Irreducibles $\mathcal{L}_h$ of $\mathfrak{Vir}$ are labelled by a minimal conformal dimension h (eigenvalue of L_0). Fusion can produce reducible but indecomposable representations from irreducibles, *eg.*

$$\mathcal{L}_{1/3} \times \mathcal{L}_{1/3} \cong \mathcal{L}_{1/3} \oplus \left(\begin{tikzcd} & \mathcal{L}_2 \arrow[dl] \arrow[dr] & \\ \mathcal{L}_0 \arrow[dr] & & \mathcal{L}_5 \arrow[dl] \\ & \mathcal{L}_2 & \end{tikzcd} \right). \qquad (c = 0)$$

This indecomposable arises in the scaling limit of critical percolation.

Diamonds in the rough

The reducible but indecomposable representations that we've exhibited all have a diamond shape (though sometimes truncated).

In each case, the algebra contains an element C that is central and self-adjoint, even though its action is not diagonalisable:

$$\begin{tikzcd}
& \bullet \arrow[dl] \arrow[dr] \arrow[dd, "C"'] & \\
\bullet \arrow[dr] & & \bullet \arrow[dl] \\
& \bullet &
\end{tikzcd}$$

C is the quadratic Casimir for $\mathfrak{gl}(1|1)$ and $\mathsf{U}_q(\mathfrak{sl}_2)$, while it is the "braid transfer matrix" for $TL_n(\beta)$ and $\cos(2\pi L_0)$ for $\mathfrak{Vir}$.

In each case, C has Jordan blocks of rank at most 2.

[This is evidently the simplest non-trivial possibility and therefore the nicest...]

In the examples discussed, these "diamond representations" have further nice mathematical properties. They are:

- Projective — they cannot appear as a quotient of a strictly larger reducible but indecomposable representation.
- Injective — they cannot appear as an invariant subspace of a strictly larger reducible but indecomposable representation.

[This requires a (sometimes not obvious) choice for the physically relevant representation category.]

These diamonds are thus maximal reducible but indecomposable representations.

Moreover, this identification allows one to start applying the machinery of homological algebra to compute many more interesting things about the representations, *eg.* extension groups.

It also generalises the situation for completely reducible representations, for which every irreducible is both projective and injective.

These examples also have the property that the irreducibles are in bijection with the projectives and injective "diamond representations".

Define a matrix C, the Cartan matrix, whose (i,j) entry is the number of times the i-th irreducible appears in the j-th diamond.

Then, C factorises as $DD^{\intercal}$, where D is called the decomposition matrix.

It counts the number of times the i-th irreducible appears in a j-th intermediate, called a standard representation. In our examples:

- For $\mathfrak{gl}(1|1)$, the standard representations are the Kac representations (in this case, the same as the Verma representations) of [Kac].
- For $\mathsf{U}_q(\mathfrak{sl}_2)$, they are the Verma representations.
- For $TL_n(\beta)$, they are the cell representations of [GL].
- For $\mathfrak{Vir}$, they are the Kac representations of [PRZ].

Along with another similar property (that is also satisfied), this makes the "diamond representations" into tilting representations.

The factorisation $C = DD^{\intercal}$ is then referred to as BGG reciprocity (*cf.* irreducibles, Verma and projectives in the category $\mathscr{O}$ of representations of a semisimple Lie algebra [BGG]).

The corresponding representation categories are then highest-weight categories in the sense of Cline–Parshall–Scott and BGG categories in the sense of Irving.

These features are of course extremely rare from a mathematical sense. It is therefore quite interesting to see them arising in physically relevant representation theories.

Outlook

Reducible but indecomposable representations are notoriously difficult to classify (tame vs wild). However, classifying physically relevant indecomposables may be easier.

The "diamond representations" that we've exhibited here are just the tip of the iceberg. More complicated physical models often lead to much more complicated indecomposables.

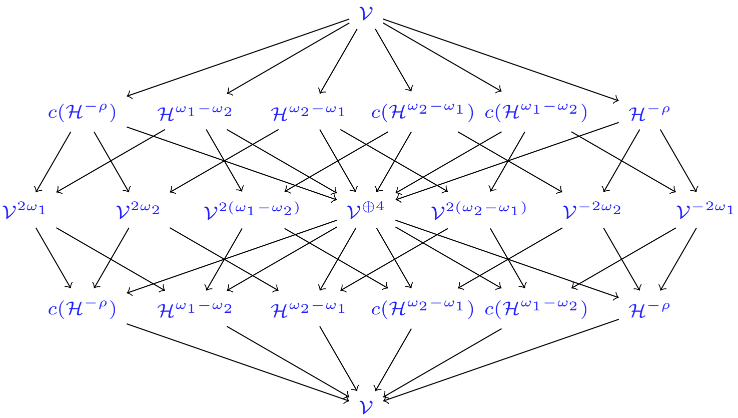
Interest in this has been steadily growing, even though the language needed is not part of the traditional lexicon.

I regard this as 21st century mathematical physics and would like to thank Yvan (among many others) for his work in making this beautiful new world accessible to those who wish to join him in this modern world.

"*Only one who attempts the absurd is capable of achieving the impossible.*"

— Miguel de Unamuno

Finally, as an example of going beyond diamonds, here is a recent conjecture for the structure of the projective cover of the vacuum module in the logCFT corresponding to $\mathfrak{sl}_3$ at level $k = -\frac{3}{2}$.



[arXiv:2112.13167 [math.RT]]