Vertex Operator Algebras and their Representations

David Ridout
University of Melbourne

Emerging Azores Representation Theory

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1. What is a vertex-operator algebra (VOA)?

Short answer: An axiomatisation of the chiral symmetries of a 2D conformal field theory.

Long answer: A $\mathbb{C}$-graded vector space $V = \bigoplus_{\Delta \in \mathbb{C}} V_\Delta$ equipped with a linear map

formal power series

$$Y(-,z) : V \to \mathrm{End}(V)[[z, z^{-1}]],$$

vertex map

$$Y(A,z) \equiv A(z) = \sum_{n \in \mathbb{Z} - \Delta_A} A_n z^{-n-\Delta_A} \qquad (A \in V_{\Delta_A}),$$

field OR vertex operator

satisfying:

truncation

- $\forall A, B \in V$, $\exists N \in \mathbb{R}$ s.t. $A_n B = 0$ $\forall n$ with $\mathrm{Re}(n) > N$.
- $\exists |0\rangle \in V_0$ s.t. $|0\rangle(z) = \mathbb{1}^V$, ie $|0\rangle_n = \delta_{n,0} \mathbb{1}^V$. vacuum

- $\exists L \in V_2$ s.t. conformal vector OR energy-momentum tensor stress-energy tensor
  - $L_0 A = \Delta_A A \quad \forall A \in V_\Delta$, grading
  - $[L_{-1}, A(z)] = \partial A(z) \quad \forall A \in V_\Delta$, translation
  - $[L_m, L_n] = (m-n) L_{m+n} + \frac{m^3 - m}{12} \delta_{m+n,0}\, c\, \mathbb{1}^V \quad \forall m, n \in \mathbb{Z}$, conformal

    for some $c \in \mathbb{C}$. central charge
- $\forall A \in V$, $A(z)|0\rangle\big|_{z=0}$ exists and is $=$ to A. state-field correspondence
- Given $A, B \in V$, $\exists k \in \mathbb{Z}$ s.t. $(z-w)^k [A(z), B(w)] = 0$. locality

This defines an algebra in the sense that there is a multiplication from $V \otimes V$ to $V((z, z^{-1}))$: lower-bounded formal power series

$$A \otimes B \longmapsto A(z)B = \sum_{n \in \mathbb{Z} - \Delta_A} A_n B\, z^{-n-\Delta_A}, \qquad A \in V_\Delta.$$

Affine VOAs

- Pick a (complex) Lie algebra $\mathfrak{g}$ with a non-degenerate symmetric invariant bilinear form $\kappa(-,-)$.

- Form the loop algebra $\mathfrak{g}_{loop} = \mathfrak{g}[t, t^{-1}]$ by setting

$$[At^m, Bt^n] = [A,B]\, t^{m+n}, \qquad A, B \in \mathfrak{g}, \quad m, n \in \mathbb{Z}.$$

- Centrally extend to get the (untwisted) affine Kac-Moody algebra $\hat{\mathfrak{g}}$:

$$\begin{array}{c|c} 0 \to \mathbb{C} \to \hat{\mathfrak{g}} \to \mathfrak{g}_{loop} \to 0 & [A_m, \mathbb{1}] = 0, \\ 1 \mapsto \mathbb{1} & [A_m, B_n] = [A,B]_{m+n} \\ A_m \mapsto At^m & \quad + m\kappa(A,B)\delta_{m+n,0}\mathbb{1}. \end{array}$$

- Note that $\mathfrak{g} \hookrightarrow \hat{\mathfrak{g}}$, as Lie algebras, via $A \mapsto A_0$. horizontal subalgebra

Examples

① $\mathfrak{g} = \mathfrak{gl}_1 = \mathrm{span}\{a\}$, so $[a,a]=0$ and $\kappa(a,a) = k \neq 0$.

$\therefore \hat{\mathfrak{gl}}_1 = \mathrm{span}\{a_n, \mathbb{1} : n \in \mathbb{Z}\}$, where $[a_m, \mathbb{1}] = 0$ and

$$[a_m, a_n] = m k \delta_{m+n,0} \mathbb{1} \quad \forall m,n \in \mathbb{Z}.$$ Heisenberg

The $\hat{\mathfrak{gl}}_1$ with different $k \neq 0$ are all isomorphic.

② $\mathfrak{g} = \mathfrak{sl}_2 = \mathrm{span}\{e,f,h\}$, $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h$$

and $\kappa(A,B) = k\,\mathrm{tr}(AB)$, $k \neq 0$. Level

$\therefore \hat{\mathfrak{sl}}_2 = \mathrm{span}\{e_n, f_n, h_n, \mathbb{1} : n \in \mathbb{Z}\}$, where $\mathbb{1}$ is central and

$[e_m, e_n] = 0$, $\quad [h_m, e_n] = 2e_{m+n}$, $\quad [e_m, f_n] = h_{m+n} + mk\delta_{m+n,0}\mathbb{1}$,

$[f_m, f_n] = 0$, $\quad [h_m, f_n] = -2f_{m+n}$, $\quad [h_m, h_n] = 2k\delta_{m+n,0}\mathbb{1}$.

Continuing:

- Affine Kac-Moody algebras have conformal (parabolic) triangular decompositions: $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_< \oplus \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_>$,

$\mathfrak{g}_{\gtrless} = \mathrm{span}\{A_n : A \in \mathfrak{g}, n \gtrless 0\}$, $\quad \mathfrak{g}_0 = \mathrm{span}\{A_0, \mathbb{1} : A \in \mathfrak{g}\} \simeq \mathfrak{g} \oplus \mathbb{C}\mathbb{1}$.

- Extend the trivial $\mathfrak{g}$-module $\mathbb{C}|0\rangle$ vacuum $(A_0|0\rangle = 0 \ \forall A \in \mathfrak{g})$ to a $\hat{\mathfrak{g}}_{\geqslant}$-module $(\mathbb{1}|0\rangle = |0\rangle$ and $A_n|0\rangle = 0 \ \forall n > 0)$.
- Induce it to a $\hat{\mathfrak{g}}$-module: $V^k(\mathfrak{g}) = \mathrm{Ind}_{\hat{\mathfrak{g}}_{\geqslant}}^{\hat{\mathfrak{g}}} \mathbb{C}|0\rangle$.

Thm: [Witten '84, Frenkel-Zhu '92; Mohammedi '93, Figueroa-O'Farrill–Stanciu '94, Lian '94]

$V^k(\mathfrak{g})$ is a VOA!

(This doesn't mean much yet — let's see how it's a VOA...)

Normal ordering

The naive field product $A(z)B(z)$ is usually not defined ...

Given homogeneous $A, B \in V$, their normally ordered product $:AB: \in V$ is

$$:AB:(z) \equiv \,:A(z)B(z):\, = \sum_{n \in \mathbb{Z} - \Delta_A - \Delta_B}' :AB:_n \, z^{-n-\Delta_A-\Delta_B}, \quad \text{so } \Delta_{:AB:} = \Delta_A + \Delta_B,$$

$$\text{where} \quad :AB:_n = \sum_{r \leq -\Delta_A}' A_r B_{n-r} + \sum_{r > -\Delta_A} B_{n-r} A_r .$$

not commutative

This product arises in the state-field correspondence.

① $V^K(\mathfrak{gl}_1)$, $K(a,a) = k \neq 0$, $[a_m, a_n] = mk\,\delta_{m+n,0}\,\mathbb{1}$.

- Let $a(z) = \sum_{n \in \mathbb{Z}}' a_n z^{-n-1}$ (so $\Delta_a = 1$). Choose vacuum $|0\rangle$. Then,

$$a(z)|0\rangle\Big|_{z=0} = \sum_{n \leq -1}' a_n z^{-n-1} |0\rangle \Big|_{z=0} = a_{-1}|0\rangle, \quad \text{so} \quad a = a_{-1}|0\rangle \ \&\ Y(a,z) = a(z).$$

- $\partial a(z) = -\sum_{n\in\mathbb{Z}} (n+1) a_n z^{-n-2} \quad (\Delta_{\partial a} = 2)$.

$\partial a(z)|0\rangle\big|_{z=0} = -\sum'_{n\leq -2} (n+1) a_n z^{-n-2}|0\rangle\big|_{z=0} = a_{-2}|0\rangle \Rightarrow Y(a_{-2}|0\rangle, z) = \partial a(z)$.

- Easy : $Y(a_{-n-1}|0\rangle, z) = D^n a(z)$, where $D^n = \frac{1}{n!}\partial^n$.

- $:a(z)a(z):|0\rangle\big|_{z=0} = \sum'_{n\in\mathbb{Z}} :aa:_n z^{-n-2}|0\rangle\big|_{z=0}$

$$= \sum_{n\in\mathbb{Z}} \left[\sum_{r\leq -1} a_r a_{n-r} + \sum_{r\geq 0} a_{n-r} \cancel{a_r} \right] z^{-n-2}|0\rangle\big|_{z=0}$$

$$= \sum_{n\leq -2} \sum'_{r\leq -1} a_r a_{n-r}|0\rangle z^{-n-2}\big|_{z=0}$$

$$= \sum_{r\leq -1} a_r a_{-r-2}|0\rangle$$

$$= a_{-1}^2|0\rangle$$

$$\Rightarrow Y(a_{-1}^2|0\rangle, z) = :aa:(z).$$

- Messy: $Y(a_{-n_1-1}\cdots a_{-n_j-1}|0\rangle, z) = :D^{n_1}a\cdots D^{n_j}a:(z)$, for all $j\in\mathbb{N}$ and $n_1,\dots,n_j\in\mathbb{Z}_{\geq 1}$. Here,

$$:ABC: \equiv :A:BC::.$$ not associative

This is the state-field correspondence of $V^k(\mathfrak{gl}_1)$.

- Exercise: Why is the truncation axiom obviously satisfied?

- Set $L = \frac{1}{2k}:aa:$, so $\Delta_L = 2$ and $L = L_{-2}|0\rangle = \frac{1}{2k}a_{-1}^2|0\rangle$. Then,

$$[L_m, a_n] = -na_{m+n} \quad \text{(easy)}$$

conformal with

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\mathbb{1} \quad \text{(hard)}$$

central charge $c=1$

- SF corr. $\Rightarrow L_0|0\rangle = 0$ so

$$L_0(a_{-n_1}\cdots a_{-n_j}|0\rangle) = [L_0, a_{-n_1}\cdots a_{-n_j}]|0\rangle = \underbrace{(n_1+\cdots+n_j)}_{\text{grade of}}\,\underbrace{a_{-n_1}\cdots a_{-n_j}|0\rangle}_{\text{this}}.$$

grading

- SF corr. $\Rightarrow L_{-1}|0\rangle = 0$ so

$$[L_{-1}, a(z)] = \sum_{n\in\mathbb{Z}} [L_{-1}, a_n] z^{-n-1} = -\sum_{n\in\mathbb{Z}} n a_{n-1} z^{-n-1} = -\sum_{n\in\mathbb{Z}} (n+1) a_n z^{-n-2}$$

$$= \partial a(z),$$ translation

(Messy: extend to $Y(a_{-n_1} \cdots a_{-n_j}|0\rangle, z)$.)

- $$[a(z), a(w)] = \sum_{m,n\in\mathbb{Z}} [a_m, a_n] z^{-m-1} w^{-n-1} = \sum_{m,n\in\mathbb{Z}} mk\,\delta_{m+n,0}\,\mathbb{1}\, z^{-m-1} w^{-n-1}$$

$$= k \sum_{m\in\mathbb{Z}} m z^{-m-1} w^{m-1}$$

$$\Rightarrow \quad (z-w)[a(z), a(w)] = k\sum_{m\in\mathbb{Z}} m z^{-m} w^{m-1} - k\sum_{m\in\mathbb{Z}} m z^{-m-1} w^{m}$$

$$= k\sum_{m\in\mathbb{Z}} (m+1) z^{-m-1} w^{m} - k\sum_{m\in\mathbb{Z}} m z^{-m-1} w^{m}$$

$$= k\sum_{m\in\mathbb{Z}} z^{-m-1} w^{m}$$

$$\Rightarrow \quad (z-w)^2 [a(z), a(w)] = k\sum_{m\in\mathbb{Z}} z^{-m} w^{m} - k\sum_{m\in\mathbb{Z}} z^{-m-1} w^{m+1} = 0.$$ locality

- Easy: extend to locality of derivatives.

Hard: extend to locality of normally ordered products. "Dong's lemma" (due to Li)

② Exercise: Check that $V^k(\mathfrak{sl}_2)$ is a VOA with:

- vacuum $|0\rangle$.
- "generators" $e(z) = \sum_{n\in\mathbb{Z}} e_n z^{-n-1}$, $f(z) = \sum_{n\in\mathbb{Z}} f_n z^{-n-1}$, $h(z) = \sum_{n\in\mathbb{Z}} h_n z^{-n-1}$.
- state-field correspondence

$$Y(f_{-l_1-1}\cdots f_{-l_r-1} h_{-m_1-1}\cdots h_{-m_s-1} e_{-n_1-1}\cdots e_{-n_t-1}|0\rangle, z)$$
$$= :D^{l_1}f \cdots D^{l_r}f\, D^{m_1}h \cdots D^{m_s}h\, D^{n_1}e \cdots D^{n_t}e:(z).$$

- conformal vector and central charge

$$L = \frac{1}{2(k+2)}\left(\frac{1}{2}:hh: + :ef: + :fe:\right), \quad c = \frac{3k}{k+2}.$$

Sugawara

Operator product expansions (OPEs)

In practice, VOAs are specified by generators and relations ...

For $V^k(\mathfrak{g})$, the (strong) generators are

$$\{A : A \in \mathfrak{g}\} = V^k(\mathfrak{g})_1, \quad \text{ie} \quad A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1}.$$

Given $A, B \in V^k(\mathfrak{g})$, we compute the product

$$A(z)B(w) = \sum_{m,n\in\mathbb{Z}} A_m B_n z^{-m-1} w^{-n-1}$$

put in normal order

$$= \sum_{n\in\mathbb{Z}} \left(\sum_{m\leq -1} A_m B_n z^{-m-1} + \sum_{m\geq -1} \left(B_n A_m + [A_m, B_n] \right) z^{-m-1} \right) w^{-n-1}$$

$$= \sum_{m,n\in\mathbb{Z}} :AB:_{m+n} z^{-m-1} w^{-n-1} + \sum_{n\in\mathbb{Z}} \sum_{m\geq 0} \left([A,B]_{m+n} + m\kappa(A,B)\,\delta_{m+n,0}\,\mathbb{1} \right) z^{-m-1} w^{-n-1}$$

$$= :A(z)B(w): + \sum_{\ell\in\mathbb{Z}} [A,B]_\ell \, w^{-\ell-1} \sum_{m\geq 0} z^{-m-1} w^m + \kappa(A,B) \sum_{m\geq 1} m z^{-m-1} w^{m-1} \, \mathbb{1}$$

$\ell = m+n$

$$A(z)B(w) = :A(z)B(w): + [A,B](w)\,\frac{1}{z}\,\frac{1}{1-w/z} + \kappa(A,B)\,\partial_w\,\frac{1}{z}\,\frac{1}{1-w/z}\,\mathbb{1}$$

$$= :A(z)B(w): + \frac{[A,B](w)}{z-w} + \frac{\kappa(A,B)\,\mathbb{1}}{(z-w)^2} \qquad (|z|>|w|).$$

Taylor-expanding gives a Laurent series:

$$A(z)B(w) = \frac{\kappa(A,B)\,\mathbb{1}}{(z-w)^2} + \frac{[A,B](w)}{z-w} + :A(w)B(w): + :DA(w)B(w):(z-w)$$
$$+ :D^2A(w)B(w):(z-w)^2 + \cdots.$$

This is an OPE:

$$\mathrm{End}(V)[\![z,z^{-1}]\!] \otimes \mathrm{End}(V)[\![w,w^{-1}]\!] \longrightarrow \mathrm{End}(V)[\![w,w^{-1}]\!]((z-w)).$$

- It is the multiplication of a VOA.
- Because of singularities when $z=w$, $A(z)B(z)$ is usually undefined.
- The regular terms sum to the normally ordered product and so are often omitted, eg. $A(z)B(w) \sim \dfrac{\kappa(A,B)\,\mathbb{1}}{(z-w)^2} + \dfrac{[A,B](w)}{z-w}$.

- As written, OPEs require expansion when $|z|>|w|$. We can also expand when $|z|<|w|$ by swapping the order, eg.

$$B(w)A(z) = \frac{K(B,A)\mathbb{1}}{(w-z)^2} + \frac{[B,A](z)}{w-z} + :B(z)A(z): + \cdots$$

$$= \frac{\overset{\text{symmetry}}{K(A,B)\mathbb{1}}}{(z-w)^2} + \frac{\overset{\text{antisymmetry}}{[A,B](z)}}{z-w} + :BA:(z) + \cdots$$

Taylor expand

$$= \frac{K(A,B)\mathbb{1}}{(z-w)^2} + \frac{[A,B](w)}{z-w} + \underbrace{\partial[A,B](w) + :BA:(w)}_{//} + \cdots$$

$$= \frac{K(A,B)\mathbb{1}}{(z-w)^2} + \frac{[A,B](w)}{z-w} + :AB:(w) + \cdots,$$

since $\partial[A,B](w) + :BA:(w) \xleftrightarrow[\text{corr.}]{SF} [A,B]_{-2}|0\rangle + B_{-1}A_{-1}|0\rangle$

$$= [A,B]_{-2}|0\rangle + A_{-1}B_{-1}|0\rangle + [A,B]_{-2}|0\rangle = A_{-1}B_{-1}|0\rangle \xleftrightarrow[\text{corr.}]{SF} :AB:(w).$$

In fact, $\overset{|z|<|w|}{B(w)A(z)}$ is the analytic continuation of $\overset{|z|>|w|}{A(z)B(w)}$. commutative + assoc.

The threefold way (of VOAs)

$$\text{States} \xleftrightarrow[\text{correspondence}]{\text{state-field}} \text{Fields} \xleftrightarrow[\text{expansion}]{\text{Laurent}} \text{Modes}$$

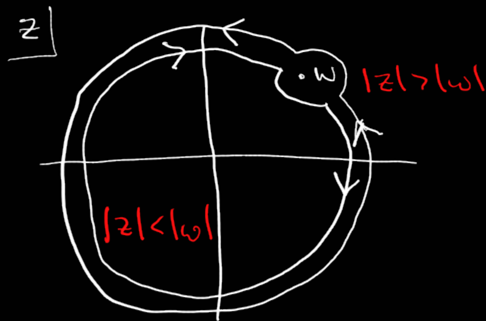
$$\begin{array}{c}\text{mode on state}\\ \text{representation}\end{array} \xleftrightarrow{?} \begin{array}{c}\text{field on field}\\ \text{OPE}\end{array} \xleftrightarrow{?} \begin{array}{c}\text{mode on mode}\\ \text{commutator}\end{array}$$

- OPE $\longrightarrow$ commutator $(A, B \in \mathfrak{g})$.

$$A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1} \implies A_n = \oint_0 A(z)\, z^n \frac{dz}{2\pi i}$$

$$\therefore [A_m, B_n] = \oint_0 \oint_0 \overbrace{A(z)B(w)}^{|z|>|w|} z^m w^n \frac{dz}{2\pi i}\frac{dw}{2\pi i}$$

$$- \oint_0 \oint_0 \underbrace{B(w)A(z)}_{|z|<|w|} z^m w^n \frac{dz}{2\pi i}\frac{dw}{2\pi i}$$



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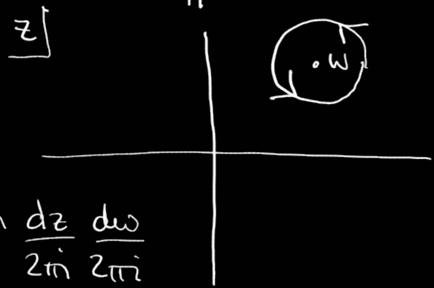


Cauchy

$$\overset{!}{=} \oint_0 \oint_w \left[\frac{K(A,B)\mathbb{1}}{(z-w)^2} + \frac{[A,B](w)}{z-w} + \cancel{\cdots} \right] z^m w^n \frac{dz}{2\pi i}\frac{dw}{2\pi i}$$

$$[A_m, B_n] = \oint_0 \Big[\kappa(A,B)\,\mathbb{1}\, m w^{m+n-1} + [A,B](w)\, w^{m+n} \Big] \frac{dw}{2\pi i}$$

$$= m\,\kappa(A,B)\,\delta_{m+n,0}\,\mathbb{1} + [A,B]_{m+n}. \qquad \checkmark$$

- Commutator (+ representation) $\longrightarrow$ OPE $(A, B \in \mathfrak{g})$.

Ansatz: $A(z)B(w) = \sum_{n\in\mathbb{C}}' C^{(n)}(w)\,(z-w)^{-n-\Delta_A}$ [cf. Huang!]

SF corr. $\Rightarrow$ $A(z)B = A(z)B(w)|0\rangle\big|_{w=0} = \sum_{n\in\mathbb{C}}' C^{(n)}(w)\,(z-w)^{-n-\Delta_A}|0\rangle\big|_{w=0}$

$$\Rightarrow \sum_{n\in\mathbb{Z}} A_n B\, z^{-n-\Delta_A} = \sum_{n\in\mathbb{C}}' C^{(n)}\, z^{-n-\Delta_A}.$$

$\therefore$ $C^{(n)} = A_n B = A_n B_{-1}|0\rangle$ $\forall n \in \mathbb{Z}$ and $C^{(n)} = 0$ otherwise.

- $n > 1$ $\Rightarrow$ $C^{(n)} = [A_n, B_{-1}]|0\rangle = [A,B]_{n-1}|0\rangle = 0$, so $C^{(n)}(w) = 0$.
- $n = 1$ $\Rightarrow$ $C^{(1)} = [A_1, B_{-1}]|0\rangle = \big([A,B]_0 + \kappa(A,B)\mathbb{1}\big)|0\rangle = \kappa(A,B)|0\rangle$,

so $C^{(1)}(w) = \kappa(A,B)\,\mathbb{1}$.

- $n=0 \Rightarrow C^{(0)} = [A_0, B_{-1}]|0\rangle = [A,B]_{-1}|0\rangle$, so $C^{(0)}(w) = [A,B](w)$.
- $n=-1 \Rightarrow C^{(-1)} = A_{-1}B_{-1}|0\rangle$, so $C^{(-1)}(w) = :A(w)B(w):$.
- $n=-2 \Rightarrow C^{(-2)} = A_{-2}B_{-1}|0\rangle$, so $C^{(-2)}(w) = :DA(w)B(w):$.

etc

$$\therefore \quad A(z)B(w) = \frac{K(A,B)\,\mathbb{1}}{(z-w)^2} + \frac{[A,B](w)}{z-w} + :A(w)B(w): + :DA(w)B(w): + \cdots$$

<u>Remark</u>: For $A, B \in V$, we defined normal ordering by

$$:A(w)B(w): = \sum_{n\in\mathbb{Z}-\Delta_A-\Delta_B} :AB:_n\, w^{-n-\Delta_A-\Delta_B}, \qquad :AB:_n = \sum_{r\leq -\Delta_A} A_r B_{n-r} + \sum_{r>-\Delta_A} B_{n-r}A_r .$$

Why? Because it is <u>equivalent</u> to defining it to be the constant term in the OPE:

$$\begin{cases} A(z)B(w) & |z|>|w| \\ B(w)A(z) & |z|<|w| \end{cases}$$

$$:A(w)B(w): = \oint_w \frac{\mathrm{OPE}[A(z),B(w)]}{z-w}\frac{dz}{2\pi i}.$$

2. Representation theory of VOAs

As you'd expect, a module M of a VOA V is a $\mathbb{C}$-graded vector space on which V acts (in a manner consistent with its algebraic structure).

- $\exists\, Y^M(-,z) : V \to \mathrm{End}(M)[[z,z^{-1}]]$, $\quad A \mapsto Y^M(A,z) \equiv A(z) = \sum_{n \in \mathbb{Z}-\Delta_A} A_n^M z^{-n-\Delta_A}$.
- $\forall A \in V,\ v \in M,\ \exists N \in \mathbb{R}$ s.t. $A_n^M v = 0\ \ \forall n$ with $\mathrm{Re}(n) > N$. truncation
- $Y^M(|0\rangle, z) = \mathbb{1}^M$. identity
- $M = \bigoplus_{\Delta \in \mathbb{C}} M_\Delta$, where $M_\Delta = \{ v \in M : (L_0^M - \Delta \mathbb{1}^M)^n v = 0$ for some $n > 0\}$. grading
- $\forall A \in V$, $[L_{-1}^M, Y^M(A,z)] = Y^M(L_{-1}A, z) = \partial Y^M(A,z)$. translation
- All OPEs, with Y replaced by Y^M, are satisfied when acting on $v \in M$.

[We normally drop all the "M" superscripts unless making a point.]

It follows that any relation satisfied by the modes A_n of V must also be satisfied by the A_n^M acting on a V-module M.

Some of these relations are commutation relations.

For $A, B \in V$, the state-field correspondence gives the general OPE

$$A(z)B(w) = \sum_{l \in \mathbb{Z} - \Delta_A} \underbrace{(A_l B)(w)}_{= Y(A_l B, w)} (z-w)^{-l-\Delta_A}.$$

This then gives the general commutation relations

$$[A_m, B_n] = \oint_0 \oint_w \sum_{l \in \mathbb{Z} - \Delta_A} (A_l B)(w)(z-w)^{-l-\Delta_A} z^{m+\Delta_A - 1} w^{n+\Delta_B - 1} \frac{dz}{2\pi i} \frac{dw}{2\pi i}$$

$$= \sum_{l \geq -\Delta_A} \binom{m+\Delta_A - 1}{l+\Delta_A - 1} \oint_0 (A_l B)(w) w^{m+n+\Delta_B - l - 1} \frac{dw}{2\pi i}$$

$$= \sum_{l \geq 0} \binom{m+\Delta_A - 1}{l} (A_{-\Delta_A + l + 1} B)_{m+n}.$$

For $A, B \in V^k(\mathfrak{g})_1$ (so $\Delta_A = 1$),

$$[A_m, B_n] = \sum_{\ell \geq 0} \binom{m}{\ell} (A_\ell B)_{m+n} = (A_0 B)_{m+n} + m (A_1 B)_{m+n} + \cancel{\binom{m}{2}(A_2 B)_{m+n} + \cdots}$$

$A_0 B = A_0 B_{-1}|0\rangle = [A_0, B_{-1}]|0\rangle = [A,B]_{-1}|0\rangle = [A,B]$, $\quad A_1 B = A_1 B_{-1}|0\rangle = [A_1, B_{-1}]|0\rangle$

$$= [A,B]_{m+n} + m k(A,B)\,\delta_{m+n,0}\,\mathbb{1}, \quad = \cancel{[A,B]_0|0\rangle} + k(A,B)|0\rangle$$

as expected. In fact, the mode algebra of $V^k(\mathfrak{g})$ is (a completion of) $U(\hat{\mathfrak{g}})$.

<u>Thm</u>: [Frenkel-Zhu '92] M is a $V^k(\mathfrak{g})$-module <u>iff</u> it is a graded $\hat{\mathfrak{g}}$-module satisfying $\underbrace{\forall A \in \mathfrak{g} \text{ and } v \in M,\ \exists N \text{ s.t. } A_n v = 0\ \forall n > N.}_{\text{smooth}}$

When $\mathfrak{g}$ is simple, $\kappa = \frac{k}{2h^\vee} \cdot \text{Killing}$ for some $\overset{\text{level}}{k} \in \mathbb{C} \setminus \{-h^\vee\}$. Thus, M is a $V^k(\mathfrak{g})$-module <u>iff</u> it is a smooth graded level-k $\hat{\mathfrak{g}}$-module.

Identifying VOA-modules

Model: highest-weight modules.

Irreducibles are classified by their highest weights, which are data associated to their highest-weight vectors.

But, what if a module is not highest weight?

!! Assume from now on that $\Delta_A \in \mathbb{Z}$ $\forall\, A \in V$. !!

- A VOA module is lower bounded if the real parts of its conformal weights (L_0-eigenvalues) are bounded below.
- If M is lower bounded (and the bound is achieved), then its top space M^{top} is the generalised L_0-eigenspace(s) corresponding to the eigenvalue(s) of minimal real part.

- Note that $v \in M^{\mathrm{top}} \Rightarrow \underbrace{A_n v = 0 \quad \forall\, A \in V \text{ and } n > 0}$.
- A weight vector v satisfying this condition is called a *relaxed* highest-weight vector.
- The top space of a lower-bounded weight module is thus spanned by relaxed highest-weight vectors.
- Better yet, the top space of an *irreducible* lower-bounded weight module *is* the span of its relaxed highest-weight vectors.

So top spaces are "good" generalisations of highest-weight vectors, at least when the VOA module is irreducible and lower bounded.

But, how do we identify top spaces?

Observation: $\forall A \in V$ and $v \in M^{top}$, $A_0 v \in M^{top}$.

$\Rightarrow$ Idea: Identify the top space as a module of the algebra of zero modes $\{A_0 : A \in V\}$.

$\Rightarrow$ Define: The <u>Zhu algebra</u> $\mathrm{Zhu}[V]$ of a VOA V is the (unital associative) algebra of zero modes A_0, $A \in V$, acting on an arbitrary top space.

① $V = V^{\kappa}(\mathfrak{gl}_1)$, $\kappa(a,a) = 1$.

State	Field	Zero mode	Action on M^{top}
$\lvert 0\rangle$	$\lvert 0\rangle(z) = \mathbb{1}$	$\mathbb{1}$	id
$a_{-1}\lvert 0\rangle$	$a(z)$	a_0	a_0
$a_{-2}\lvert 0\rangle$	$Da(z)$	$(Da)_0 = -a_0$	$-a_0$
$a_{-1}^2\lvert 0\rangle$	$:a(z)a(z):$	$:aa:_0 = \sum_{r\leq -1} a_r a_{-r} + \sum_{r>-1} a_{-r} a_r$	a_0^2

Continuing, the PBW basis of $V^k(\mathfrak{gl}_1)$ is mapped to polynomials in a_0.

$$\therefore \ \mathrm{Zhu}[V^k(\mathfrak{gl}_1)] = \mathbb{C}[a_0] \quad \text{(as a vector space)}.$$

Exercise: ② $\mathrm{Zhu}[V^k(\mathfrak{sl}_2)] = U(\mathfrak{sl}_2)$ & $\mathrm{Zhu}[V^k(\mathfrak{g})] = U(\mathfrak{g})$ as vector spaces. $\mathrm{Zhu}[L]$ is the quadratic Casimir of $\mathfrak{g}$!

What is the algebra structure on $\mathrm{Zhu}[V]$? Obvious guess:

$$[A_0, B_0] = \sum_{l \geq 0}' \binom{\Delta_A - 1}{l} (A_{-\Delta_A+l+1} B)_0 .$$

But, this would be a Lie algebra, not a unital associative algebra.

Worse: this commutator formula holds for arbitrary zero modes, independent of whether they are acting on a top space or not!

A better product comes from the generalised commutation relation [Zamolodchikov '85, Borcherds '86]

$$\sum_{\ell \geq 0} \left(A_{-\ell} B_{\ell} + B_{-\ell-1} A_{\ell+1} \right) = \sum_{\ell \geq 0} \binom{\Delta_A}{\ell} \left(A_{-\Delta_A+\ell} B \right)_0 .$$

When acting on a top space, the LHS dramatically simplifies:

$$A_0 B_0 = \sum_{\ell \geq 0} \binom{\Delta_A}{\ell} \left(A_{-\Delta_A+\ell} B \right)_0 .$$

This is the associative product on $Zhu[V]$ (Zhu's $*$-product).

The GCR is derived from

$$\oint_0 \oint_w \frac{\mathrm{OPE}\left[A(z), B(w)\right] z^{m+\Delta_A-1} w^{n+\Delta_B-1}}{z-w} \frac{dz}{2\pi i} \frac{dw}{2\pi i} .$$

Exercise: Show that the commutator defined by this associative product is the commutator we derived earlier.

Sanity checks

- $|0\rangle_0 B_0 = \sum_{l\geq 0} \binom{0}{l} (|0\rangle_l B)_0 = (|0\rangle_0 B)_0 = (\mathbb{1} B)_0 = B_0,$

$$A_0 |0\rangle_0 = \sum_{l\geq 0} \binom{\Delta_A}{l} \big(\underbrace{A_{-\Delta_A+l}|0\rangle}_{=0 \ \forall l>0 \ (\text{SF corr.})}\big)_0 = (A_{-\Delta_A}|0\rangle)_0 = A_0. \qquad \text{unital!}$$

- $[L_0, B_0] = \sum_{l\geq 0} \binom{1}{l} (L_{l-1}B)_0 = (L_{-1}B)_0 + (L_0 B)_0 = (\partial B)_0 + \Delta_B B_0$

central $\qquad = -\Delta_B B_0 + \Delta_B B_0 = 0. \qquad \partial B(z) = \sum_{n\in\mathbb{Z}} (-n-\Delta_B) B_n z^{-n-\Delta_B}$

- $V = V^k(\mathfrak{g})$, $\Delta_A = \Delta_B = 1$ gives (in $\mathrm{Zhu}[V^k(\mathfrak{g})]$)

$$A_0 B_0 = \sum_{l\geq 0} \binom{1}{l} (A_{l-1}B)_0 = (A_{-1}B)_0 + (A_0 B)_0 = (A_{-1}B_{-1}|0\rangle)_0 + (A_0 B_{-1}|0\rangle)_0$$

$$= :AB:_0 + ([A,B]_{-1}|0\rangle)_0 = \sum_{r\leq -1} A_r \overset{0\ \forall r}{B_{-r}} + \sum_{r\geq 0} B_{-r} \overset{0\ \forall r>0}{A_r} + [A,B]_0$$

$$= B_0 A_0 + [A,B]_0 .$$

Zhu's theorem [Zhu '90, '96; Li '94; Gepner-Witten '86; Feigin-Nakanishi-Ooguri '92]

- If M is an irreducible lower-bounded V-module, then M^{top} is an irreducible $\mathrm{Zhu}[V]$-module.
- If N is an irreducible $\mathrm{Zhu}[V]$-module, then there exists a unique (up to $\cong$) irreducible lower-bounded V-module M with $M^{top} \cong N$.
- The isomorphism classes of irreducible $\mathrm{Zhu}[V]$-modules and irreducible lower-bounded V-modules are in bijective correspondence.

Proof is very technical:

- A $\mathrm{Zhu}[V]$-module has an action of zero modes.
- Make it a top space by letting positive modes act as 0.
- "Induce" to get an action of the negative modes.
- Take the irreducible quotient. Check that the top space is preserved.

① $V = V^\kappa(\mathfrak{gl}_1)$, $\kappa(a,a)=1$, $\mathrm{Zhu}[V] = \mathbb{C}[a_0]$.

- $\mathrm{Zhu}[V]$ has countable dimension $\Rightarrow$ irreducibles do too.
- $\mathrm{Zhu}[V]$ commutative $\Rightarrow$ irreducibles are 1-dim: [Dixmier-Schur]

$$\mathbb{C}|p\rangle, \quad a_0|p\rangle = p|p\rangle \quad (p \in \mathbb{C}).$$

- Zhu induction is here just Lie algebra induction:
  - $a_n|p\rangle = 0 \ \forall n>0$ hw vector
  - $\mathcal{F}_p = \mathrm{Ind}_{\hat{\mathfrak{gl}}_1^{\geq}}^{\hat{\mathfrak{gl}}_1} \mathbb{C}|p\rangle$ is a smooth Verma $\hat{\mathfrak{gl}}_1$-module, so a $V^\kappa(\mathfrak{gl}_1)$-mod.
  - $\kappa(a,a) \neq 0 \Rightarrow \mathcal{F}_p$ is irreducible with $\mathcal{F}_p^{\mathrm{top}} = \mathbb{C}|p\rangle$.

The irreducible lower-bounded $V^\kappa(\mathfrak{gl}_1)$-modules are thus the Fock spaces $\mathcal{F}_p$, $p \in \mathbb{C}$.

- Since $L_0 = \frac{1}{2}a_0^2$ in $\mathrm{Zhu}[V^\kappa(\mathfrak{gl}_1)]$, $L_0|p\rangle = \frac{1}{2}p^2|p\rangle$. conformal weight
- There are also irreducible $V^\kappa(\mathfrak{gl}_1)$-modules that are not lower bounded.

② $V = L_1(sl_2)$, the simple quotient of $V^\kappa(sl_2)$ with $\kappa(A,B) = tr(AB)$.

Exercise: $e_{-1}^2|0\rangle$ is singular in $V^\kappa(sl_2)$, hence is 0 in $L_1(sl_2)$.

Fact: $e_{-1}^2|0\rangle$ generates the maximal ideal of $V^\kappa(sl_2)$.

Theorem: [Frenkel-Zhu '92] If I is an ideal of a VOA $\bar{V}$, then

$$\mathrm{Zhu}[\bar{V}/I] \cong \frac{\mathrm{Zhu}[\bar{V}]}{\mathrm{Zhu}[I]}.$$

- For $\bar{V} = V^\kappa(sl_2)$ and $I = \langle e_{-1}^2|0\rangle\rangle$, we get

$$\mathrm{Zhu}[L_1(sl_2)] \cong \frac{U(sl_2)}{\mathrm{Zhu}[I]}.$$

- The image of $e_{-1}^2|0\rangle$ in $\mathrm{Zhu}[V^\kappa(sl_2)] \cong U(sl_2)$ is

$$e_{-1}^2|0\rangle \xrightarrow[\text{corr.}]{\text{SF}} :e(z)e(z): \xrightarrow[\text{mode}]{\text{zero}} :ee:_0 = \sum_{r\leq -1} e_r e_{-r} + \sum_{r\geq 0} e_{-r}e_r \xrightarrow[\text{top}]{\text{act on}} e_0^2,$$

so $e_0^2 \in \mathrm{Zhu}[I]$.

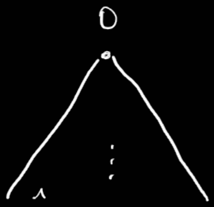
Exercise: $\mathrm{Zhu}[I] = \langle e_0^2 \rangle$. (not usually true that $\mathrm{Zhu}[I]$ is generated by the Zhu-images of the generators of I!)

- A $\mathrm{Zhu}[L_1(sl_2)]$-module is thus an sl_2-module that is annihilated by $e^2 \in U(sl_2)$.

- There are therefore only 2 irreducible $\mathrm{Zhu}[L_1(sl_2)]$-modules:

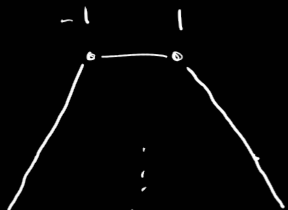
$$\overset{0}{\circ} \qquad \underline{\text{AND}} \qquad \overset{-1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{1}{\circ}\,.$$

- There are therefore only 2 irreducible lower-bounded $L_1(sl_2)$-modules:



$\hat{\mathcal{L}}_0 \simeq L_1(sl_2)$

<u>AND</u>



$\hat{\mathcal{L}}_1$

- $L = \frac{1}{6}\left[\frac{1}{2}:hh: + :ef: + :fe:\right] \Rightarrow L_0 = \frac{1}{6}\left[\frac{1}{2}h_0^2 + f_0e_0 + e_0f_0\right]$ in $\mathrm{Zhu}[L_1(sl_2)]$.
  The conformal weights of $\hat{\mathcal{L}}_0$ and $\hat{\mathcal{L}}_1$ are thus 0 and $\frac{1}{4}$, respectively.

- Fact: There are no irreducible $L_1(sl_2)$-modules that are not lower bounded.

[Dong-Li-Mason ?]

3. Classifications via coherent families

The story is similar for $L_\kappa(\mathfrak{g})$ when $\mathfrak{g}$ is (semi)simple and $\kappa = \frac{k}{2h^\vee} \cdot \text{Killing}$, for $k \in \mathbb{N}$. There are finitely many $L_\kappa(\mathfrak{g})$-modules and they are all highest-weight with finite-dim. top spaces.

The category $L_\kappa(\mathfrak{g})$-mod is very nice: it is a modular fusion category.

But sometimes we need to consider $k \notin \mathbb{N}$. Why?

- Coset constructions of non-unitary W-algebras. [Goddard-Kent-Olive '85]
- Quantum hamiltonian reduction constructions of W-algebras. [Feigin-Frenkel '92, Kac-Roan-Wakimoto '03]
- 4D/2D duality [Beem-Lemos-Liendo-Peelaers-Rastelli-van Rees '13]
- Examples of <u>non-semisimple</u> (non-finite) modular tensor categories.

③ $L_{-4/3}(sl_2)$ is the simple quotient of $V^k(sl_2)$ with $k = -\frac{1}{3}\cdot$ Killing.

The maximal ideal of $V^k(sl_2)$ is generated by

$$|\chi\rangle = (8e_{-3} - 12e_{-2}h_{-1} - 6e_{-1}h_{-2} + 9e_{-1}h_{-1}^2 + 36e_{-1}^2 f_{-1})|0\rangle.$$

Its image in $\mathrm{Zhu}[V^k(sl_2)]$ is then

$$\begin{aligned}\chi_0 &= 8e_0 + 12h_0e_0 + 6h_0e_0 + 9h_0^2e_0 + 36f_0e_0^2 \\ &= (18h_0 + 9h_0^2 + 36f_0e_0 + 8)e_0 = (24L_0 + 8)e_0,\end{aligned}$$

because $L = \frac{3}{4}\left[\frac{1}{2}{:}hh{:} + {:}ef{:} + {:}fe{:}\right]$ has Zhu image

$$L_0 = \frac{3}{4}\left[\frac{1}{2}h_0^2 + f_0e_0 + e_0f_0\right] = \frac{3}{4}\left[\frac{1}{2}h_0^2 + h_0 + 2f_0e_0\right].$$

Again, $\mathrm{Zhu}[\langle\chi\rangle] = \langle\chi_0\rangle$, so

$$\mathrm{Zhu}[L_{-4/3}(sl_2)] \simeq \frac{U(sl_2)}{\langle(24L+8)e\rangle}.$$

An sl_2-module is thus a $\mathrm{Zhu}[L_{-4/3}(sl_2)]$-module when e acts as 0 or L acts as $-\frac{1}{3}$. The irreducibles are thus:

- $e = 0 \Rightarrow$ $\circ \quad \xrightarrow{\text{Zhu-ind.}}$  vacuum module $\hat{\mathcal{L}}_0 \cong L_{-4/3}(sl_2)$.

finite-dim top

- $L = -\frac{1}{3}$:
  - $\circ$ highest-weight 

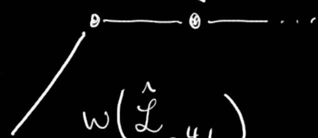
$e_0|\lambda\rangle = 0 \Rightarrow L_0|\lambda\rangle = \frac{3}{4}\left[\frac{1}{2}h_0^2 + h_0\right]|\lambda\rangle$

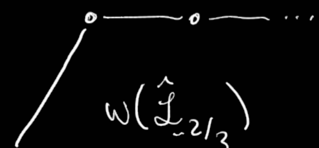
$$\Rightarrow \frac{3}{8}\lambda(\lambda+2) = -\frac{1}{3} \Rightarrow \lambda = -\frac{4}{3}, -\frac{2}{3}.$$

$\xrightarrow{\text{Zhu-ind.}}$ 

∞-dim top!

  - $\circ$ lowest-weight $f_0|\lambda\rangle = 0 \Rightarrow \cdots$ or better apply Weyl reflection.

$\xrightarrow{\text{Zhu-ind.}}$ 



∞-dim top!

Note that the hw and lw $\mathrm{Zhu}[L_{-4/3}(\mathfrak{sl}_2)]$-mods "fit together":

This is <u>not</u> a coincidence!

$\exists$ irreducible weight $\mathfrak{sl}_2$-modules that are neither hw nor lw.

They are said to be dense/torsion-free/cuspidal. [Gabriel '50s, ?]

These modules are <u>classified</u> by their <u>weight support</u> $[\lambda] \in \mathbb{C}/2\mathbb{Z} = \mathfrak{h}^*/Q$ and the eigenvalue $q \in \mathbb{C}$ of the quadratic Casimir.

For each $q \in \mathbb{C}$, they are irreducible for all but either 1 or 2 classes $[\lambda] \in \mathbb{C}/2\mathbb{Z}$.

• dense $\cdots - \overset{\lambda-2}{\circ} - \overset{\lambda}{\circ} - \overset{\lambda+2}{\circ} - \cdots \xrightarrow{\text{Zhu-ind.}} \cdots - \circ - \circ - \circ - \cdots$

$\hat{\mathcal{D}}_{[\lambda];-1/3}$

irreducible unless $[\lambda] = [-\frac{4}{3}], [-\frac{2}{3}]$.

normalise quad. Casimir to be L_0

<u>Thm</u>: Classification of irred. lower-bded. wt. $L_{-4/3}(\mathfrak{sl}_2)$-mods.

- hw.: $\hat{\mathcal{L}}_0$, $\hat{\mathcal{L}}_{-2/3}$, $\hat{\mathcal{L}}_{-4/3}$. [Dong-Li-Mason '95, Adamović-Milas '95]
- conjugates: $w(\hat{\mathcal{L}}_{-2/3})$, $w(\hat{\mathcal{L}}_{-4/3})$. [nb. $w(\hat{\mathcal{L}}_0) \cong \hat{\mathcal{L}}_0$]
- relaxed: $\hat{\mathcal{D}}_{[\lambda];-1/3}$, $[\lambda] \neq [-\frac{4}{3}], [-\frac{2}{3}]$. [Adamović-Milas '95]

$\exists$ also reducible relaxed modules $\hat{\mathcal{D}}^{\pm}_{[\lambda];-1/3}$ with $[\lambda] = [-\frac{4}{3}], [-\frac{2}{3}]$:

$$0 \longrightarrow \hat{\mathcal{L}}_{-4/3} \longrightarrow \hat{\mathcal{D}}^{+}_{[-4/3];-1/3} \longrightarrow w(\hat{\mathcal{L}}_{-2/3}) \longrightarrow 0,$$

$$0 \longrightarrow \hat{\mathcal{L}}_{-2/3} \longrightarrow \hat{\mathcal{D}}^{+}_{[-2/3];-1/3} \longrightarrow w(\hat{\mathcal{L}}_{-4/3}) \longrightarrow 0,$$

$$\hat{\mathcal{D}}^{-}_{[\lambda];-1/3} = w(\hat{\mathcal{D}}^{+}_{[\lambda];-1/3}).$$

[Creutzig-DR '13], [Adamović '17], [Kawasetsu-DR '18].

Time doesn't permit further discussion but ...

- For $\mathfrak{g}$ simple and k an admissible level, quite a lot is known:
  - highest-weight modules classified. [Arakawa '12]
  - irreducible relaxed hw mods for $L_k(sl_2)$ classified. [Adamović - Milas '95]
  - ————————————— $L_k(sl_3)$ —————. [Kawasetsu - DR - Wood '21]
- Other low-rank cases are known, but combinatorics gets unwieldy.
- $\exists$ algorithm for classifying "parabolic coherent families" of $L_k(\mathfrak{g})$, given a hw classification. [Kawasetsu - DR '19]
- $\Rightarrow$ in-principle classification of irreducible weight $L_k(\mathfrak{g})$-modules, when weight spaces are finite-dimensional. [using Futorny - Tsylke '01]
- However, applications demand also some weight $L_k(\mathfrak{g})$-modules with infinite-dimensional weight spaces! [Grantcharov - Raymond - DR '26?]