Correlation functions for bosonic ghosts

David Ridout

[Joint with Xueting Li and Damodar Rajbhandari]

University of Melbourne

July 14, 2026

[International Conference on Vertex Algebra, Category Theory and Related Topics, Tianjin]

Outline

Logarithmic CFTs

I want to understand vertex operator algebras (VOAs) and conformal field theories (CFTs)...

rational	factorisable	non-rational
	[lattice VOAs] [free fermion] [fermionic ghosts] [minimal models] [compact WZW models] [a few W-algebras]	[Heisenberg] [Liouville/Toda] [non-compact WZW?]
discrete		continuous
	[symplectic fermions] [triplet models] [SLE/CLE?]	[bosonic ghosts] [supergroup WZW] [Nappi–Witten] [most affine VOAs] [most W-algebras]
log-rational	logarithmic	generic

CFTs are built from modules of its VOA, *aka* chiral algebra.

A rational CFT is one whose space of states is

- semisimple (completely reducible),
- finite-length: (finitely many composition factors),

The space of states is then q-finite: its q-character $\operatorname{tr} \mathsf{q}^{L_0 - \mathsf{c}/24}$ exists (this is the partition function).

CFTs are built from modules of its VOA, *aka* chiral algebra.

A rational CFT is one whose space of states is

- semisimple (completely reducible),
- finite-length: (finitely many composition factors),

The space of states is then q-finite: its q-character $\operatorname{tr} \mathsf{q}^{L_0 - \mathsf{c}/24}$ exists (this is the partition function).

Log-rational means non-semisimple but finite-length and q-finite.
[But few accessible examples.]
Non-rational means semisimple but not finite-length (but can be q-finite).
[Usually notoriously difficult.]

Generically, we lose all three conditions. But here we have surprisingly many accessible (and important!) examples... this is log CFT.

Today: the bosonic ghost system (*aka.* symplectic bosons or Weyl VOA). Its state space is a non-semisimple sum of uncountably many composition factors (so it has generic type).

Today: the bosonic ghost system (*aka.* symplectic bosons or Weyl VOA). Its state space is a non-semisimple sum of uncountably many composition factors (so it has generic type).

Its representation theory is pretty well understood, but what about its correlation functions? How do the "standard methods" hold up when there are no q-characters and conformal weights are unbounded below?

Today: the bosonic ghost system (*aka.* symplectic bosons or Weyl VOA). Its state space is a non-semisimple sum of uncountably many composition factors (so it has generic type).

Its representation theory is pretty well understood, but what about its correlation functions? How do the "standard methods" hold up when there are no q-characters and conformal weights are unbounded below?

While logarithmic modules continue to attract lots of interest, work on correlators had languished until relatively recently.

Now, a few groups are using correlators (mostly for q-finite modules) to compute tensor-categorical data.

Today: the bosonic ghost system (*aka.* symplectic bosons or Weyl VOA). Its state space is a non-semisimple sum of uncountably many composition factors (so it has generic type).

Its representation theory is pretty well understood, but what about its correlation functions? How do the "standard methods" hold up when there are no q-characters and conformal weights are unbounded below?

While logarithmic modules continue to attract lots of interest, work on correlators had languished until relatively recently.

Now, a few groups are using correlators (mostly for q-finite modules) to compute tensor-categorical data.

But good news everyone: this means that there's still an awful lot to do...

Bosonic ghosts

The bosonic ghost system was introduced in [Friedan–Martinec–Shenker'86] to study gauge fixing for superstrings.

- It has two (strong and free) generators β and γ that satisfy

$$\beta(z)\beta(w) \sim 0 \sim \gamma(z)\gamma(w), \quad \beta(z)\gamma(w) \sim -\frac{1}{z-w}.$$

Bosonic ghosts

The bosonic ghost system was introduced in [Friedan–Martinec–Shenker'86] to study gauge fixing for superstrings.

- It has two (strong and free) generators β and γ that satisfy

$$\beta(z)\beta(w) \sim 0 \sim \gamma(z)\gamma(w), \quad \beta(z)\gamma(w) \sim -\frac{1}{z-w}.$$

- There is a Heisenberg field $J(z) = :\beta(z)\gamma(z):$ that assigns β and γ the weights $j_\beta = 1$ and $j_\gamma = -1$, respectively.

Bosonic ghosts

The bosonic ghost system was introduced in [Friedan–Martinec–Shenker'86] to study gauge fixing for superstrings.

- It has two (strong and free) generators β and γ that satisfy

$$\beta(z)\beta(w) \sim 0 \sim \gamma(z)\gamma(w), \quad \beta(z)\gamma(w) \sim -\frac{1}{z-w}.$$

- There is a Heisenberg field $J(z) = {:}\beta(z)\gamma(z){:}$ that assigns β and γ the weights $j_\beta = 1$ and $j_\gamma = -1$, respectively.
- There is also a 1-parameter family of Virasoro fields. We choose

$$T(z) = -{:}\beta(z)\partial\gamma(z){:},$$

so that $h_\beta = 1$, $h_\gamma = 0$ and $\mathsf{c} = 2$.

[This is the right setup for, *eg*., Wakimoto free-field realisations.]

The mode algebra has commutation relations

$$[\beta_m, \beta_n] = 0 = [\gamma_m, \gamma_n], \quad [\beta_m, \gamma_n] = -\delta_{m+n=0}\mathbb{1}, \qquad m, n \in \mathbb{Z}.$$

The mode algebra has commutation relations

$$[\beta_m, \beta_n] = 0 = [\gamma_m, \gamma_n], \quad [\beta_m, \gamma_n] = -\delta_{m+n=0}\mathbb{1}, \qquad m, n \in \mathbb{Z}.$$

It therefore admits spectral flow automorphisms

$$\begin{aligned} \sigma^\ell(\beta_n) &= \beta_{n-\ell}, \\ \sigma^\ell(\gamma_n) &= \gamma_{n+\ell}, \end{aligned} \quad \Rightarrow \quad \begin{aligned} \sigma^\ell(J_n) &= J_n + \ell\delta_{n=0}\mathbb{1}, \\ \sigma^\ell(L_n) &= L_n - \ell J_n - \tfrac{1}{2}\ell(\ell-1)\delta_{n=0}\mathbb{1}. \end{aligned}$$

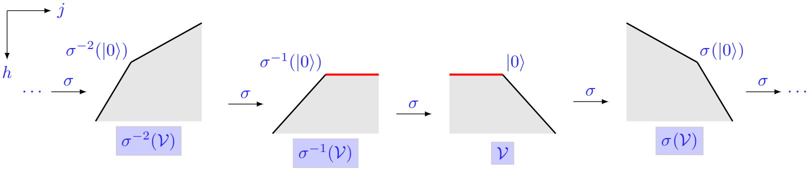
The mode algebra has commutation relations

$$[\beta_m, \beta_n] = 0 = [\gamma_m, \gamma_n], \quad [\beta_m, \gamma_n] = -\delta_{m+n=0}\mathbb{1}, \qquad m, n \in \mathbb{Z}.$$

It therefore admits spectral flow automorphisms

$$\begin{aligned} \sigma^\ell(\beta_n) &= \beta_{n-\ell}, \\ \sigma^\ell(\gamma_n) &= \gamma_{n+\ell}, \end{aligned} \quad \Rightarrow \quad \begin{aligned} \sigma^\ell(J_n) &= J_n + \ell\delta_{n=0}\mathbb{1}, \\ \sigma^\ell(L_n) &= L_n - \ell J_n - \tfrac{1}{2}\ell(\ell-1)\delta_{n=0}\mathbb{1}. \end{aligned}$$

These can be used to twist the action of the ghost algebra on a module $\mathcal{M}$, producing new (non-isomorphic) modules $\sigma^\ell(\mathcal{M})$.



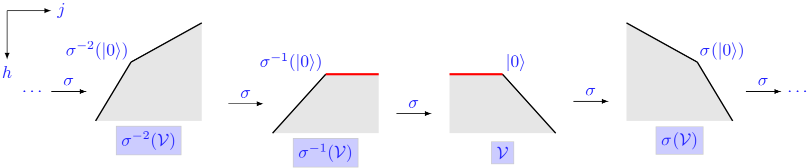
The mode algebra has commutation relations

$$[\beta_m, \beta_n] = 0 = [\gamma_m, \gamma_n], \quad [\beta_m, \gamma_n] = -\delta_{m+n=0}\mathbb{1}, \qquad m, n \in \mathbb{Z}.$$

It therefore admits spectral flow automorphisms

$$\begin{aligned} \sigma^\ell(\beta_n) &= \beta_{n-\ell}, \\ \sigma^\ell(\gamma_n) &= \gamma_{n+\ell}, \end{aligned} \quad \Rightarrow \quad \begin{aligned} \sigma^\ell(J_n) &= J_n + \ell\delta_{n=0}\mathbb{1}, \\ \sigma^\ell(L_n) &= L_n - \ell J_n - \tfrac{1}{2}\ell(\ell-1)\delta_{n=0}\mathbb{1}. \end{aligned}$$

These can be used to twist the action of the ghost algebra on a module $\mathcal{M}$, producing new (non-isomorphic) modules $\sigma^\ell(\mathcal{M})$.



Only two spectral flows of the vacuum module $\mathcal{V}$ are lower bounded (meaning their conformal weights are bounded below).

Irreducibles

The only highest-weight module is the vacuum module $\mathcal{V}$. $\sigma^{-1}(\mathcal{V})$ is also lower bounded.

Irreducibles

The only highest-weight module is the vacuum module $\mathcal{V}$. $\sigma^{-1}(\mathcal{V})$ is also lower bounded.

But, there is another 1-parameter family of lower-bounded modules

$$\mathcal{W}_{[j]}, \quad [j] \in \mathbb{C}/\mathbb{Z}.$$

Here, the top space (space of lowest conformal weight) has a basis $\big\{|\phi_i\rangle \ : \ i \in [j]\big\}$ satisfying

$$\beta_0|\phi_i\rangle = i|\phi_{i+1}\rangle, \quad \gamma_0|\phi_i\rangle = |\phi_{i-1}\rangle, \qquad i \in [j].$$

$|\phi_i\rangle$ has charge $j_i = i$ and conformal weight $h_i = 0$.

Irreducibles

The only highest-weight module is the vacuum module $\mathcal{V}$. $\sigma^{-1}(\mathcal{V})$ is also lower bounded.

But, there is another 1-parameter family of lower-bounded modules

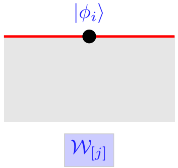
$$\mathcal{W}_{[j]}, \quad [j] \in \mathbb{C}/\mathbb{Z}.$$

Here, the top space (space of lowest conformal weight) has a basis $\{|\phi_i\rangle \,:\, i \in [j]\}$ satisfying

$$\beta_0|\phi_i\rangle = i|\phi_{i+1}\rangle, \quad \gamma_0|\phi_i\rangle = |\phi_{i-1}\rangle, \qquad i \in [j].$$

$|\phi_i\rangle$ has charge $j_i = i$ and conformal weight $h_i = 0$.

The module $\mathcal{W}_{[j]}$, with $[j] \neq [0]$, is irreducible.



When $[j] = [0]$, there are two reducible lower-bounded modules $\mathcal{W}^{\pm}_{[0]}$.

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{W}^{+}_{[0]} \longrightarrow \sigma^{-1}(\mathcal{V}) \to 0,$$
$$0 \longrightarrow \sigma^{-1}(\mathcal{V}) \longrightarrow \mathcal{W}^{-}_{[0]} \longrightarrow \mathcal{V} \to 0.$$

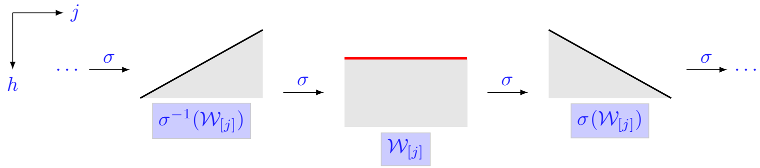
[The action on the previous slide applies only to $\mathcal{W}^{+}_{[0]}$.]

When $[j] = [0]$, there are two reducible lower-bounded modules $\mathcal{W}^{\pm}_{[0]}$.

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{W}^{+}_{[0]} \longrightarrow \sigma^{-1}(\mathcal{V}) \rightarrow 0,$$
$$0 \longrightarrow \sigma^{-1}(\mathcal{V}) \longrightarrow \mathcal{W}^{-}_{[0]} \longrightarrow \mathcal{V} \rightarrow 0.$$

[The action on the previous slide applies only to $\mathcal{W}^{+}_{[0]}$.]

In both the reducible and irreducible cases, we can also apply spectral flow to get non-lower-bounded modules.

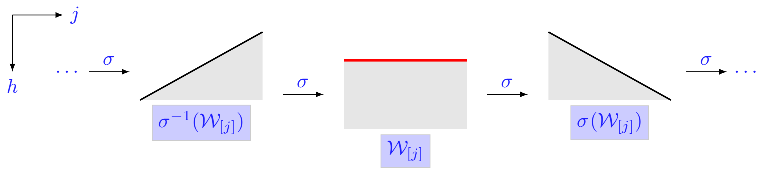


When $[j] = [0]$, there are two reducible lower-bounded modules $\mathcal{W}_{[0]}^{\pm}$.

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{W}_{[0]}^{+} \longrightarrow \sigma^{-1}(\mathcal{V}) \to 0,$$
$$0 \longrightarrow \sigma^{-1}(\mathcal{V}) \longrightarrow \mathcal{W}_{[0]}^{-} \longrightarrow \mathcal{V} \to 0.$$

[The action on the previous slide applies only to $\mathcal{W}_{[0]}^{+}$.]

In both the reducible and irreducible cases, we can also apply spectral flow to get non-lower-bounded modules.



The characters of the $\sigma^{\ell}(\mathcal{W}_{[j]})$ can be computed explicitly. They are modular (transforming under the usual $\mathsf{SL}(2;\mathbb{Z})$-action), so are expected to form a consistent CFT spectrum. [DR–Wood'14]

Fusion

The "standard Verlinde formula" of [Creutzig–DR'13] implies the generic fusion rules [DR–Wood'14, Adamović–Pedić'19]

$$\sigma^{\ell}(\mathcal{W}_{[j]}) \times \sigma^{\ell'}(\mathcal{W}_{[j']}) \cong \sigma^{\ell+\ell'}(\mathcal{W}_{[j+j']}) \oplus \sigma^{\ell+\ell'-1}(\mathcal{W}_{[j+j']}),$$

which hold for all $\ell, \ell' \in \mathbb{Z}$ when $[j], [j'], [j+j'] \neq [0]$.

Fusion

The "standard Verlinde formula" of [Creutzig–DR'13] implies the generic fusion rules [DR–Wood'14, Adamović–Pedić'19]

$$\sigma^{\ell}(\mathcal{W}_{[j]}) \times \sigma^{\ell'}(\mathcal{W}_{[j']}) \cong \sigma^{\ell+\ell'}(\mathcal{W}_{[j+j']}) \oplus \sigma^{\ell+\ell'-1}(\mathcal{W}_{[j+j']}),$$

which hold for all $\ell, \ell' \in \mathbb{Z}$ when $[j], [j'], [j+j'] \neq [0]$.

The logarithmic nature of the bosonic ghost system is manifested by the following non-generic fusion rules [DR–Wood'14, Allen–Wood'20]:

$$\sigma^{\ell}(\mathcal{W}_{[j]}) \times \sigma^{\ell'}(\mathcal{W}_{[-j]}) \cong \sigma^{\ell+\ell'-1}(\mathcal{P}), \quad [j] \neq [0].$$

Fusion

The "standard Verlinde formula" of [Creutzig–DR'13] implies the generic fusion rules [DR–Wood'14, Adamović–Pedić'19]

$$\sigma^{\ell}(\mathcal{W}_{[j]}) \times \sigma^{\ell'}(\mathcal{W}_{[j']}) \cong \sigma^{\ell+\ell'}(\mathcal{W}_{[j+j']}) \oplus \sigma^{\ell+\ell'-1}(\mathcal{W}_{[j+j']}),$$

which hold for all $\ell, \ell' \in \mathbb{Z}$ when $[j], [j'], [j+j'] \neq [0]$.

The logarithmic nature of the bosonic ghost system is manifested by the following non-generic fusion rules [DR–Wood'14, Allen–Wood'20]:

$$\sigma^{\ell}(\mathcal{W}_{[j]}) \times \sigma^{\ell'}(\mathcal{W}_{[-j]}) \cong \sigma^{\ell+\ell'-1}(\mathcal{P}), \quad [j] \neq [0].$$

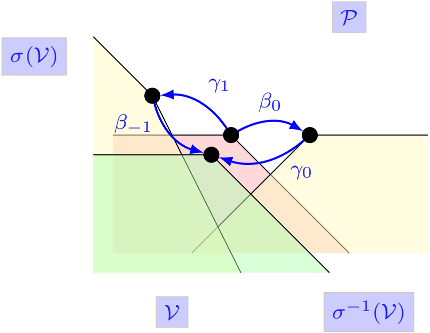
$\mathcal{P}$ is a logarithmic module: reducible but indecomposable with a non-diagonalisable L_0-action [Lesage–Mathieu–Rasmussen–Saleur'03, DR'10].

It follows [Gurarie'93] that the bosonic ghost system has correlation functions with logarithmic singularities.

$\mathcal{P}$ is the projective cover of the vacuum module $\mathcal{V}$ [Allen–Wood'20].

$$0 \longrightarrow \mathcal{W}^+_{[0]} \longrightarrow \mathcal{P} \longrightarrow \sigma(\mathcal{W}^+_{[0]}) \to 0,$$
$$0 \longrightarrow \sigma(\mathcal{W}^-_{[0]}) \longrightarrow \mathcal{P} \longrightarrow \mathcal{W}^-_{[0]} \to 0.$$

$$\begin{tikzcd} & \mathcal{V} \arrow[dl] \arrow[dr] & \\ \sigma^{-1}(\mathcal{V}) \arrow[dr] & \mathcal{P} & \sigma(\mathcal{V}) \arrow[dl] \\ & \mathcal{V} & \end{tikzcd}$$



Bosonic ghost correlators

Because we chose an asymmetric conformal structure ($h_\beta \neq h_\gamma$), the usual Ward identities for primary fields (intertwiners) need not apply.

Bosonic ghost correlators

Because we chose an asymmetric conformal structure ($h_\beta \neq h_\gamma$), the usual Ward identities for primary fields (intertwiners) need not apply.

This is clear from the adjoints

$$\beta_n^\dagger = \gamma_{-n}, \quad \gamma_n^\dagger = \beta_{-n} \quad \Rightarrow \quad J_n^\dagger = J_{-n}, \quad L_n^\dagger = L_{-n} + nJ_{-n}.$$

Bosonic ghost correlators

Because we chose an asymmetric conformal structure ($h_\beta \neq h_\gamma$), the usual Ward identities for primary fields (intertwiners) need not apply.

This is clear from the adjoints

$$\beta_n^\dagger = \gamma_{-n}, \quad \gamma_n^\dagger = \beta_{-n} \quad \Rightarrow \quad J_n^\dagger = J_{-n}, \quad L_n^\dagger = L_{-n} + nJ_{-n}.$$

The Ward identity for L_1 is therefore modified and the solutions are

$$\begin{aligned}
\big\langle \psi_1(w_1) \big\rangle &= \delta_{h_1=0} \delta_{j=0} C_1, \\
\big\langle \psi_1(w_1) \psi_2(w_2) \big\rangle &= \delta_{2h_1 - j_1 = 2h_2 - j_2} \delta_{j=0} C_{12} w_{12}^{-(2h_1 - j_1)}, \\
\big\langle \psi_1(w_1) \psi_2(w_2) \psi_3(w_3) \big\rangle &= \delta_{j=0} C_{123} \prod_{1 \leqslant a < b \leqslant 3} w_{ab}^{h - 2h_{ab} + j_{ab}}, \\
\big\langle \psi_1(w_1) \psi_2(w_2) \psi_3(w_3) \psi_4(w_4) \big\rangle &= \delta_{j=0} H(\eta) \prod_{1 \leqslant a < b \leqslant 4} w_{ab}^{h/3 - h_{ab} + j_{ab}/2}
\end{aligned}$$

($w_{ab} = w_a - w_b$, $\eta = \frac{w_{12} w_{34}}{w_{13} w_{24}}$, $j = \sum_i j_i$, $h = \sum_i h_i$, $h_{ab} = h_a + h_b$, $j_{ab} = j_a + j_b$).

Conjugates

The ghost primaries correspond to the $|\phi_i\rangle \in \mathcal{W}_{[i]}$ and their spectral flows

$$|\phi_i^\ell\rangle = \sigma^\ell(|\phi_i\rangle) \in \sigma^\ell(\mathcal{W}_{[i]}) \quad \Rightarrow \quad j_i^\ell = i - \ell, \quad h_i^\ell = i\ell - \frac{1}{2}\ell(\ell+1).$$

Their 2-point functions have the form

$$\left\langle \phi_i^\ell(w_1)\phi_j^m(w_2)\right\rangle = \delta_{i+j=1}^{\ell+m=1}\left[\begin{smallmatrix}\ell & m\\ i & j\end{smallmatrix}\right] w_{12}^{\ell^2-(2\ell-1)i}.$$

Conjugates

The ghost primaries correspond to the $|\phi_i\rangle \in \mathcal{W}_{[i]}$ and their spectral flows

$$|\phi_i^\ell\rangle = \sigma^\ell(|\phi_i\rangle) \in \sigma^\ell(\mathcal{W}_{[i]}) \quad \Rightarrow \quad j_i^\ell = i - \ell, \quad h_i^\ell = i\ell - \frac{1}{2}\ell(\ell+1).$$

Their 2-point functions have the form

$$\left\langle \phi_i^\ell(w_1)\phi_j^m(w_2)\right\rangle = \delta_{i+j=1}^{\ell+m=1}\left[\begin{smallmatrix} \ell & m \\ i & j \end{smallmatrix}\right] w_{12}^{\ell^2-(2\ell-1)i}.$$

The field conjugate to $\phi_i^\ell(z)$ is thus $\phi_{1-i}^{1-\ell}(z)$ and the module conjugate to $\sigma^\ell(\mathcal{W}_{[i]})$ is $\sigma^{1-\ell}(\mathcal{W}_{[-i]})$.

Conjugates

The ghost primaries correspond to the $|\phi_i\rangle \in \mathcal{W}_{[i]}$ and their spectral flows

$$|\phi_i^\ell\rangle = \sigma^\ell(|\phi_i\rangle) \in \sigma^\ell(\mathcal{W}_{[i]}) \quad \Rightarrow \quad j_i^\ell = i - \ell, \quad h_i^\ell = i\ell - \frac{1}{2}\ell(\ell+1).$$

Their 2-point functions have the form

$$\left\langle \phi_i^\ell(w_1)\phi_j^m(w_2)\right\rangle = \delta_{i+j=1}^{\ell+m=1}\left[\begin{smallmatrix} \ell & m \\ i & j \end{smallmatrix}\right] w_{12}^{\ell^2-(2\ell-1)i}.$$

The field conjugate to $\phi_i^\ell(z)$ is thus $\phi_{1-i}^{1-\ell}(z)$ and the module conjugate to $\sigma^\ell(\mathcal{W}_{[i]})$ is $\sigma^{1-\ell}(\mathcal{W}_{[-i]})$.

This is consistent with the fusion rules because

$$\sigma^\ell(\mathcal{W}_{[i]}) \times \sigma^{1-\ell}(\mathcal{W}_{[-i]}) \cong \mathcal{P}.$$

(It is the log-partner of the vacuum that has the non-zero 1-point function — that of the vacuum vanishes.)

Whither spectral flow?

Note that 2-point functions of lower-bounded primaries ($\ell = 0$) always vanish. This generalises considerably.

Whither spectral flow?

Note that 2-point functions of lower-bounded primaries ($\ell = 0$) always vanish. This generalises considerably.

- Any primary correlator with all spectral flow indices $\leqslant 0$ must vanish.

This follows because $|\phi_i^\ell\rangle = \gamma_\ell |\phi_{i+1}^\ell\rangle$ and $\gamma(z)\phi_j^m(w) \sim 0$ for $m \leqslant 0$.

Similarly (with $\gamma \to \beta$):

- Any primary correlator with all spectral flow indices > 0 must vanish.

Whither spectral flow?

Note that 2-point functions of lower-bounded primaries ($\ell = 0$) always vanish. This generalises considerably.

- Any primary correlator with all spectral flow indices $\leqslant 0$ must vanish.

This follows because $|\phi_i^\ell\rangle = \gamma_\ell |\phi_{i+1}^\ell\rangle$ and $\gamma(z)\phi_j^m(w) \sim 0$ for $m \leqslant 0$.

Similarly (with $\gamma \to \beta$):

- Any primary correlator with all spectral flow indices > 0 must vanish.

Unfortunately, non-primary fields are not always descendants of primaries. *eg.*, $\gamma_n |\phi_i^\ell\rangle$ is not primary nor a descendant for $\ell > 1$ and $0 < n < \ell$.

Correlators with such non-descendant fields are not obviously expressible in terms of primary correlators. This makes life hard...

Knizhnik–Zamolodchikov equation(s)

Our aim is to show that some correlators have logarithmic singularities.
For this, we restrict to the following case:

$$\left\langle \phi_{j_1}(w_1) \cdots \phi_{j_{N-1}}(w_{N-1}) \phi^{\ell}_{j_N}(w_N) \right\rangle, \quad \ell \in \mathbb{Z}_{\geqslant 1}.$$

Knizhnik–Zamolodchikov equation(s)

Our aim is to show that some correlators have logarithmic singularities. For this, we restrict to the following case:

$$\langle \phi_{j_1}(w_1) \cdots \phi_{j_{N-1}}(w_{N-1}) \phi^{\ell}_{j_N}(w_N) \rangle, \quad \ell \in \mathbb{Z}_{\geqslant 1}.$$

Ghost correlators satisfy KZ equations because $T(z)$ is composite.

- Inserting L_{-1} at the i-th coordinate ($i \neq N$) gives

$$\partial_i \langle \phi_{j_1} \cdots \phi_{j_{N-1}} \phi^{\ell}_{j_N} \rangle = j_i \sum_{k=1}^{\ell} k w_{iN}^{-k-1} \langle \phi_{j_1} \cdots \phi_{j_i+1} \cdots \phi_{j_{N-1}} (\gamma_k \phi^{\ell}_{j_N}) \rangle.$$

  This involves non-descendants if $\ell > 1$.

Knizhnik–Zamolodchikov equation(s)

Our aim is to show that some correlators have logarithmic singularities. For this, we restrict to the following case:

$$\langle \phi_{j_1}(w_1) \cdots \phi_{j_{N-1}}(w_{N-1}) \phi^{\ell}_{j_N}(w_N) \rangle, \quad \ell \in \mathbb{Z}_{\geqslant 1}.$$

Ghost correlators satisfy KZ equations because $T(z)$ is composite.

- Inserting L_{-1} at the i-th coordinate ($i \neq N$) gives

$$\partial_i \langle \phi_{j_1} \cdots \phi_{j_{N-1}} \phi^{\ell}_{j_N} \rangle = j_i \sum_{k=1}^{\ell} k w_{iN}^{-k-1} \langle \phi_{j_1} \cdots \phi_{j_i+1} \cdots \phi_{j_{N-1}} (\gamma_k \phi^{\ell}_{j_N}) \rangle.$$

  This involves non-descendants if $\ell > 1$.
- Inserting L_{-1} at the N-th coordinate similarly fails for $\ell > 1$. Instead, we insert $L_{-1} - (\ell - 1) J_{-1}$ to get

$$\Big(\partial_N + (\ell - 1) \sum_{i \neq N} j_i w_{iN}^{-1} \Big) \langle \phi_{j_1} \cdots \phi_{j_{N-1}} \phi^{\ell}_{j_N} \rangle = - \sum_{i \neq N} j_i w_{iN}^{-\ell-1} \langle \phi_{j_1} \cdots \phi_{j_i+1} \cdots \phi_{j_{N-1}} \phi^{\ell}_{j_N - 1} \rangle.$$

Primary 2-point functions

The 2-point functions have the form

$$\langle \phi_{j_1} \phi^{\ell}_{j_2} \rangle = \delta^{\ell=1}_{j_1+j_2=1} \left[\begin{smallmatrix} 0 & 1 \\ j_1 & j_2 \end{smallmatrix}\right] w^{j_1}_{12}.$$

Primary 2-point functions

The 2-point functions have the form

$$\left\langle \phi_{j_1} \phi^{\ell}_{j_2} \right\rangle = \delta^{\ell=1}_{j_1+j_2=1} \left[\begin{smallmatrix} 0 & 1 \\ j_1 & j_2 \end{smallmatrix}\right] w_{12}^{j_1}.$$

Substituting into (either) $\ell = 1$ KZ equation gives a recursion relation for the 2-point constants:

$$\left[\begin{smallmatrix} 0 & 1 \\ j_1 & j_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 1 \\ j_1-1 & j_2+1 \end{smallmatrix}\right], \quad j_1 + j_2 = 1.$$

Since there is no upper or lower bound on the charges, we have no boundary conditions.

Primary 2-point functions

The 2-point functions have the form

$$\langle \phi_{j_1} \phi^{\ell}_{j_2} \rangle = \delta^{\ell=1}_{j_1+j_2=1} \begin{bmatrix} 0 & 1 \\ j_1 & j_2 \end{bmatrix} w_{12}^{j_1}.$$

Substituting into (either) $\ell = 1$ KZ equation gives a recursion relation for the 2-point constants:

$$\begin{bmatrix} 0 & 1 \\ j_1 & j_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ j_1-1 & j_2+1 \end{bmatrix}, \quad j_1 + j_2 = 1.$$

Since there is no upper or lower bound on the charges, we have no boundary conditions.

But, this recursion is consistent with the expectation that we may normalise conjugate fields so that

$$\begin{bmatrix} 0 & 1 \\ j & 1-j \end{bmatrix} = 1.$$

Primary 3-point functions

The 3-point functions have the form

$$\left\langle \phi_{j_1}\phi_{j_2}\phi^{\ell}_{j_3}\right\rangle = \delta_{j_1+j_2+j_3=\ell}\left[\begin{smallmatrix} 0 & 0 & \ell \\ j_1 & j_2 & j_3 \end{smallmatrix}\right] \cdot w_{12}^{(j_3-\ell/2)(\ell-1)} w_{13}^{-j_2-(j_3-(\ell+1)/2)\ell} w_{23}^{-j_1-(j_3-(\ell+1)/2)\ell}.$$

Primary 3-point functions

The 3-point functions have the form

$$\left\langle \phi_{j_1} \phi_{j_2} \phi^{\ell}_{j_3} \right\rangle = \delta_{j_1+j_2+j_3=\ell} \begin{bmatrix} 0 & 0 & \ell \\ j_1 & j_2 & j_3 \end{bmatrix} \cdot w_{12}^{(j_3-\ell/2)(\ell-1)} w_{13}^{-j_2-(j_3-(\ell+1)/2)\ell} w_{23}^{-j_1-(j_3-(\ell+1)/2)\ell}.$$

Now, the second KZ equation gives a polynomial identity in w_{13} and w_{23} relating 3-point constants. Assuming $j_1 + j_2 + j_3 = \ell$, it gives:

$\ell = 1$: $$\begin{bmatrix} 0 & 0 & 1 \\ j_1 & j_2 & j_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ j_1+1 & j_2 & j_3-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ j_1 & j_2+1 & j_3-1 \end{bmatrix}.$$

Primary 3-point functions

The 3-point functions have the form

$$\left\langle \phi_{j_1} \phi_{j_2} \phi^{\ell}_{j_3} \right\rangle = \delta_{j_1+j_2+j_3=\ell} \left[\begin{smallmatrix} 0 & 0 & \ell \\ j_1 & j_2 & j_3 \end{smallmatrix} \right] \cdot w_{12}^{(j_3-\ell/2)(\ell-1)} w_{13}^{-j_2-(j_3-(\ell+1)/2)\ell} w_{23}^{-j_1-(j_3-(\ell+1)/2)\ell}.$$

Now, the second KZ equation gives a polynomial identity in w_{13} and w_{23} relating 3-point constants. Assuming $j_1+j_2+j_3=\ell$, it gives:

$\ell = 1$:

$$\left[\begin{smallmatrix} 0 & 0 & 1 \\ j_1 & j_2 & j_3 \end{smallmatrix} \right] = \left[\begin{smallmatrix} 0 & 0 & 1 \\ j_1+1 & j_2 & j_3-1 \end{smallmatrix} \right] = \left[\begin{smallmatrix} 0 & 0 & 1 \\ j_1 & j_2+1 & j_3-1 \end{smallmatrix} \right].$$

$\ell = 2$:

$$j_1 \left[\begin{smallmatrix} 0 & 0 & 2 \\ j_1+1 & j_2 & j_3-1 \end{smallmatrix} \right] = (j_3-1) \left[\begin{smallmatrix} 0 & 0 & 2 \\ j_1 & j_2 & j_3 \end{smallmatrix} \right],$$
$$-j_2 \left[\begin{smallmatrix} 0 & 0 & 2 \\ j_1 & j_2+1 & j_3-1 \end{smallmatrix} \right] = (j_3-1) \left[\begin{smallmatrix} 0 & 0 & 2 \\ j_1 & j_2 & j_3 \end{smallmatrix} \right].$$

Primary 3-point functions

The 3-point functions have the form

$$\left\langle \phi_{j_1} \phi_{j_2} \phi^{\ell}_{j_3} \right\rangle = \delta_{j_1+j_2+j_3=\ell} \left[\begin{smallmatrix} 0 & 0 & \ell \\ j_1 & j_2 & j_3 \end{smallmatrix} \right] \cdot w_{12}^{(j_3-\ell/2)(\ell-1)} w_{13}^{-j_2-(j_3-(\ell+1)/2)\ell} w_{23}^{-j_1-(j_3-(\ell+1)/2)\ell}.$$

Now, the second KZ equation gives a polynomial identity in w_{13} and w_{23} relating 3-point constants. Assuming $j_1 + j_2 + j_3 = \ell$, it gives:

$\ell = 1$: $$\left[\begin{smallmatrix} 0 & 0 & 1 \\ j_1 & j_2 & j_3 \end{smallmatrix} \right] = \left[\begin{smallmatrix} 0 & 0 & 1 \\ j_1+1 & j_2 & j_3-1 \end{smallmatrix} \right] = \left[\begin{smallmatrix} 0 & 0 & 1 \\ j_1 & j_2+1 & j_3-1 \end{smallmatrix} \right].$$

$\ell = 2$: $$j_1 \left[\begin{smallmatrix} 0 & 0 & 2 \\ j_1+1 & j_2 & j_3-1 \end{smallmatrix} \right] = (j_3 - 1) \left[\begin{smallmatrix} 0 & 0 & 2 \\ j_1 & j_2 & j_3 \end{smallmatrix} \right],$$
$$-j_2 \left[\begin{smallmatrix} 0 & 0 & 2 \\ j_1 & j_2+1 & j_3-1 \end{smallmatrix} \right] = (j_3 - 1) \left[\begin{smallmatrix} 0 & 0 & 2 \\ j_1 & j_2 & j_3 \end{smallmatrix} \right].$$

$\ell \geqslant 3$: $$\left[\begin{smallmatrix} 0 & 0 & \ell \\ j_1 & j_2 & j_3 \end{smallmatrix} \right] = 0.$$

This last result is consistent with the (generic) fusion rules. The $\ell = 2$ result suggests interesting times when a weight is integral.

Primary 4-point functions

The 4-point functions have the specialised form

$$\left\langle \phi_{j_1} \middle| \phi_{j_2}(1) \phi_{j_3}(\eta) \middle| \phi_{j_4}^{\ell} \right\rangle = \delta_{j=\ell} \eta^{-2h/3-(j_1+j_2)/2} (1-\eta)^{h/3+(j_2+j_3)/2} H(\eta),$$

where $j = j_1 + j_2 + j_3 + j_4$ and $h = j_4 \ell - \frac{1}{2}\ell(\ell+1)$.

Primary 4-point functions

The 4-point functions have the specialised form

$$\langle \phi_{j_1} | \phi_{j_2}(1) \phi_{j_3}(\eta) | \phi_{j_4}^{\ell} \rangle = \delta_{j=\ell} \eta^{-2h/3-(j_1+j_2)/2} (1-\eta)^{h/3+(j_2+j_3)/2} H(\eta),$$

where $j = j_1 + j_2 + j_3 + j_4$ and $h = j_4 \ell - \frac{1}{2} \ell(\ell+1)$.

For $\ell = 1$, the KZ equation with $i = 3$ gives an infinite family of order-1 differential recursion relations:

$$\partial_\eta \langle \phi_{j_1} | \phi_{j_2}(1) \phi_{j_3}(\eta) | \phi_{j_4}^{1} \rangle = j_3 \eta^{-2} \langle \phi_{j_1} | \phi_{j_2}(1) \phi_{j_3+1}(\eta) | \phi_{j_4-1}^{1} \rangle.$$

Primary 4-point functions

The 4-point functions have the specialised form

$$\left\langle \phi_{j_1} \middle| \phi_{j_2}(1) \phi_{j_3}(\eta) \middle| \phi_{j_4}^{\ell} \right\rangle = \delta_{j=\ell} \eta^{-2h/3-(j_1+j_2)/2} (1-\eta)^{h/3+(j_2+j_3)/2} H(\eta),$$

where $j = j_1 + j_2 + j_3 + j_4$ and $h = j_4 \ell - \frac{1}{2}\ell(\ell+1)$.

For $\ell = 1$, the KZ equation with $i = 3$ gives an infinite family of order-1 differential recursion relations:

$$\partial_\eta \left\langle \phi_{j_1} \middle| \phi_{j_2}(1) \phi_{j_3}(\eta) \middle| \phi_{j_4}^{1} \right\rangle = j_3 \eta^{-2} \left\langle \phi_{j_1} \middle| \phi_{j_2}(1) \phi_{j_3+1}(\eta) \middle| \phi_{j_4-1}^{1} \right\rangle.$$

To solve this family of coupled ODEs, we derive another such family by inserting J_0 instead of L_{-1}:

$$\left\langle \phi_{j_1} \cdots \phi_{j_{N-1}} \phi_{j_N}^{\ell} \right\rangle = \sum_{k=1}^{\ell} w_{iN}^{-k} \left\langle \phi_{j_1} \cdots \phi_{j_i+1} \cdots \phi_{j_{N-1}} (\gamma_k \phi_{j_N}^{\ell}) \right\rangle.$$

This is thus an algebraic "KZ-like" equation.

Specialising, the J_0 KZ-like equation becomes

$$\left\langle \phi_{j_1} \middle| \phi_{j_2}(1)\phi_{j_3}(\eta) \middle| \phi^1_{j_4} \right\rangle = \eta^{-1} \left\langle \phi_{j_1} \middle| \phi_{j_2}(1)\phi_{j_3+1}(\eta) \middle| \phi^1_{j_4-1} \right\rangle.$$

Specialising, the J_0 KZ-like equation becomes

$$\left\langle \phi_{j_1} \middle| \phi_{j_2}(1)\phi_{j_3}(\eta) \middle| \phi^1_{j_4} \right\rangle = \eta^{-1} \left\langle \phi_{j_1} \middle| \phi_{j_2}(1)\phi_{j_3+1}(\eta) \middle| \phi^1_{j_4-1} \right\rangle.$$

Substituting back into the L_{-1} KZ equation then uncouples the ODEs:

$$\partial_\eta \left\langle \phi_{j_1} \middle| \phi_{j_2}(1)\phi_{j_3}(\eta) \middle| \phi^1_{j_4} \right\rangle = j_3 \eta^{-1} \left\langle \phi_{j_1} \middle| \phi_{j_2}(1)\phi_{j_3}(\eta) \middle| \phi^1_{j_4} \right\rangle.$$

Specialising, the J_0 KZ-like equation becomes

$$\left\langle \phi_{j_1} \middle| \phi_{j_2}(1)\phi_{j_3}(\eta) \middle| \phi^1_{j_4} \right\rangle = \eta^{-1} \left\langle \phi_{j_1} \middle| \phi_{j_2}(1)\phi_{j_3+1}(\eta) \middle| \phi^1_{j_4-1} \right\rangle.$$

Substituting back into the L_{-1} KZ equation then uncouples the ODEs:

$$\partial_\eta \left\langle \phi_{j_1} \middle| \phi_{j_2}(1)\phi_{j_3}(\eta) \middle| \phi^1_{j_4} \right\rangle = j_3 \eta^{-1} \left\langle \phi_{j_1} \middle| \phi_{j_2}(1)\phi_{j_3}(\eta) \middle| \phi^1_{j_4} \right\rangle.$$

We thus arrive at a very simple solution:

$$\left\langle \phi_{j_1} \middle| \phi_{j_2}(1)\phi_{j_3}(\eta) \middle| \phi^1_{j_4} \right\rangle = \delta_{j_1+j_2+j_3+j_4=1} \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ j_1 & j_2 & j_3 & j_4 \end{smallmatrix} \right] \eta^{j_3},$$
$$\left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ j_1 & j_2 & j_3 & j_4 \end{smallmatrix} \right] = \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ j_1+1 & j_2 & j_3 & j_4-1 \end{smallmatrix} \right] = \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ j_1 & j_2+1 & j_3 & j_4-1 \end{smallmatrix} \right] = \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ j_1 & j_2 & j_3+1 & j_4-1 \end{smallmatrix} \right].$$

Unfortunately, there is no logarithmic singularity...

Specialising, the J_0 KZ-like equation becomes

$$\left\langle \phi_{j_1} \middle| \phi_{j_2}(1)\phi_{j_3}(\eta) \middle| \phi^1_{j_4} \right\rangle = \eta^{-1} \left\langle \phi_{j_1} \middle| \phi_{j_2}(1)\phi_{j_3+1}(\eta) \middle| \phi^1_{j_4-1} \right\rangle.$$

Substituting back into the L_{-1} KZ equation then uncouples the ODEs:

$$\partial_\eta \left\langle \phi_{j_1} \middle| \phi_{j_2}(1)\phi_{j_3}(\eta) \middle| \phi^1_{j_4} \right\rangle = j_3 \eta^{-1} \left\langle \phi_{j_1} \middle| \phi_{j_2}(1)\phi_{j_3}(\eta) \middle| \phi^1_{j_4} \right\rangle.$$

We thus arrive at a very simple solution:

$$\left\langle \phi_{j_1} \middle| \phi_{j_2}(1)\phi_{j_3}(\eta) \middle| \phi^1_{j_4} \right\rangle = \delta_{j_1+j_2+j_3+j_4=1} \begin{bmatrix} 0 & 0 & 0 & 1 \\ j_1 & j_2 & j_3 & j_4 \end{bmatrix} \eta^{j_3},$$
$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ j_1 & j_2 & j_3 & j_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ j_1+1 & j_2 & j_3 & j_4-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ j_1 & j_2+1 & j_3 & j_4-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ j_1 & j_2 & j_3+1 & j_4-1 \end{bmatrix}.$$

Unfortunately, there is no logarithmic singularity...

To see such singularities, we will need to solve the $\ell = 2$ case. This will require more than just KZ (and KZ-like) technology.

Fake Belavin–Polyakov–Zamolodchikov equations

The commutant of the Heisenberg subalgebra generated by J in the bosonic ghosts VOA is the singlet algebra of central charge -2 [DR'10].

Fake Belavin–Polyakov–Zamolodchikov equations

The commutant of the Heisenberg subalgebra generated by J in the bosonic ghosts VOA is the singlet algebra of central charge -2 [DR'10].

We can therefore find a Virasoro field with this central charge that has regular OPE with $J(w)$:

$$\widetilde{T}(z) = T(z) + \frac{1}{2} {:}J(z)J(z){:} + \frac{1}{2}\partial J(z).$$

Fake Belavin–Polyakov–Zamolodchikov equations

The commutant of the Heisenberg subalgebra generated by J in the bosonic ghosts VOA is the singlet algebra of central charge -2 [DR'10].

We can therefore find a Virasoro field with this central charge that has regular OPE with $J(w)$:

$$\widetilde{T}(z) = T(z) + \frac{1}{2}{:}J(z)J(z){:} + \frac{1}{2}\partial J(z).$$

The ghost primary $\phi_{1/2}$ has (modified) conformal weight $\widetilde{h} = -\frac{1}{8}$, so

$$\left(\widetilde{L}_{-1}^2 - \frac{1}{2}\widetilde{L}_{-2}\right)|\phi_{1/2}\rangle = 0,$$

$$\textit{ie.} \quad \left(L_{-1}^2 - \frac{1}{2}L_{-2} + J_{-1}L_{-1}\right)|\phi_{1/2}\rangle = 0.$$

This is a "fake" null vector — the LHS vanishes identically when written in terms of modes of β and γ.

This null vector nevertheless gives rise to fake (but still useful) BPZ equations. *eg.*, if $i \neq N$, we have

$$\left[\partial_i^2 + \sum_{k\neq i} \frac{\partial_k - 2j_k\partial_i}{2w_{ki}} - \frac{j_N\ell - \frac{1}{2}\ell(\ell+1)}{2w_{Ni}^2}\right] \cdot \left\langle \phi_{j_1}(w_1)\cdots\phi_{1/2}(w_i)\cdots\phi_{j_{N-1}}(w_{N-1})\phi_{j_N}^{\ell}(w_N)\right\rangle = 0.$$

This null vector nevertheless gives rise to fake (but still useful) BPZ equations. *eg.*, if $i \neq N$, we have

$$\left[\partial_i^2 + \sum_{k\neq i} \frac{\partial_k - 2j_k\partial_i}{2w_{ki}} - \frac{j_N\ell - \frac{1}{2}\ell(\ell+1)}{2w_{Ni}^2}\right] \cdot \left\langle \phi_{j_1}(w_1)\cdots\phi_{1/2}(w_i)\cdots\phi_{j_{N-1}}(w_{N-1})\phi^{\ell}_{j_N}(w_N)\right\rangle = 0.$$

- For $N = 2$, fake BPZ gives nothing new.

This null vector nevertheless gives rise to fake (but still useful) BPZ equations. *eg.*, if $i \neq N$, we have

$$\left[\partial_i^2 + \sum_{k \neq i} \frac{\partial_k - 2j_k \partial_i}{2w_{ki}} - \frac{j_N \ell - \frac{1}{2}\ell(\ell+1)}{2w_{Ni}^2}\right] \cdot \left\langle \phi_{j_1}(w_1) \cdots \phi_{1/2}(w_i) \cdots \phi_{j_{N-1}}(w_{N-1}) \phi_{j_N}^{\ell}(w_N) \right\rangle = 0.$$

- For $N = 2$, fake BPZ gives nothing new.
- For $N = 3$, the solutions of the Ward identities are solutions of fake BPZ if and only if $\ell = 1, 2$.

This null vector nevertheless gives rise to fake (but still useful) BPZ equations. *eg.*, if $i \neq N$, we have

$$\left[\partial_i^2 + \sum_{k \neq i} \frac{\partial_k - 2j_k \partial_i}{2w_{ki}} - \frac{j_N \ell - \frac{1}{2}\ell(\ell+1)}{2w_{Ni}^2}\right] \cdot \left\langle \phi_{j_1}(w_1) \cdots \phi_{1/2}(w_i) \cdots \phi_{j_{N-1}}(w_{N-1}) \phi_{j_N}^{\ell}(w_N) \right\rangle = 0.$$

- For $N = 2$, fake BPZ gives nothing new.
- For $N = 3$, the solutions of the Ward identities are solutions of fake BPZ if and only if $\ell = 1, 2$.
- For $N = 4$ and $\ell = 1$, fake BPZ becomes a second-order ODE with conformal blocks

$$\eta^{1/2} \quad \text{and} \quad \eta^{1/2} \mathsf{B}\big(\tfrac{1}{2} - j_4, \tfrac{1}{2} - j_2; \eta\big).$$

Single-valuedness of the bulk correlator rules out the second block.

- For $N = 4$ and $\ell = 3$, the conformal blocks are

$$\eta^{2-j_4}(1-\eta)^{\frac{1}{2}-j_2} \quad \text{and} \quad \eta^{2-j_4}(1-\eta)^{\frac{1}{2}-j_2}\mathsf{B}\big(j_4-\tfrac{1}{2}, j_2-\tfrac{1}{2}; \eta\big)$$

and again the second block is ruled out.

- For $N = 4$ and $\ell = 3$, the conformal blocks are

$$\eta^{2-j_4}(1-\eta)^{\frac{1}{2}-j_2} \quad \text{and} \quad \eta^{2-j_4}(1-\eta)^{\frac{1}{2}-j_2}\mathsf{B}\big(j_4-\tfrac{1}{2}, j_2-\tfrac{1}{2}; \eta\big)$$

and again the second block is ruled out.

- For $N = 4$ and $\ell = 2$, the conformal blocks are

$$\eta\, {}_2\mathsf{F}_1\left[\begin{matrix} \frac{1}{2} & j_2+j_4-\frac{1}{2} \\ & j_4+\frac{1}{2} \end{matrix}; \eta\right] \quad \text{and} \quad \eta^{\frac{3}{2}-j_4}\, {}_2\mathsf{F}_1\left[\begin{matrix} j_2 & 1-j_4 \\ & \frac{3}{2}-j_4 \end{matrix}; \eta\right].$$

This time, single-valuedness does not rule out either block.

- For $N = 4$ and $\ell = 3$, the conformal blocks are

$$\eta^{2-j_4}(1-\eta)^{\frac{1}{2}-j_2} \quad \text{and} \quad \eta^{2-j_4}(1-\eta)^{\frac{1}{2}-j_2}\mathsf{B}\big(j_4-\tfrac{1}{2}, j_2-\tfrac{1}{2}; \eta\big)$$

  and again the second block is ruled out.

- For $N = 4$ and $\ell = 2$, the conformal blocks are

$$\eta\,{}_2\mathsf{F}_1\left[\begin{matrix}\frac{1}{2} & j_2+j_4-\frac{1}{2} \\ & j_4+\frac{1}{2}\end{matrix};\eta\right] \quad \text{and} \quad \eta^{\frac{3}{2}-j_4}\,{}_2\mathsf{F}_1\left[\begin{matrix}j_2 & 1-j_4 \\ & \frac{3}{2}-j_4\end{matrix};\eta\right].$$

  This time, single-valuedness does not rule out either block.

Suppose now that $j_4 \in \mathbb{Z} + \frac{1}{2}$. Then, one of these hypergeometric functions develops a logarithmic singularity at $\eta = 0$.

- For $N = 4$ and $\ell = 3$, the conformal blocks are

$$\eta^{2-j_4}(1-\eta)^{\frac{1}{2}-j_2} \quad \text{and} \quad \eta^{2-j_4}(1-\eta)^{\frac{1}{2}-j_2}\mathsf{B}\big(j_4-\tfrac{1}{2}, j_2-\tfrac{1}{2}; \eta\big)$$

  and again the second block is ruled out.
- For $N = 4$ and $\ell = 2$, the conformal blocks are

$$\eta\,{}_2\mathsf{F}_1\left[\begin{matrix}\frac{1}{2} & j_2+j_4-\frac{1}{2} \\ & j_4+\frac{1}{2}\end{matrix}; \eta\right] \quad \text{and} \quad \eta^{\frac{3}{2}-j_4}\,{}_2\mathsf{F}_1\left[\begin{matrix}j_2 & 1-j_4 \\ & \frac{3}{2}-j_4\end{matrix}; \eta\right].$$

  This time, single-valuedness does not rule out either block.

Suppose now that $j_4 \in \mathbb{Z} + \frac{1}{2}$. Then, one of these hypergeometric functions develops a logarithmic singularity at $\eta = 0$.

Similarly, expanding in $1 - \eta$ shows that there is a logarithmic singularity at $\eta = 1$ when $j_2 \in \mathbb{Z} + \frac{1}{2}$.

- For $N = 4$ and $\ell = 3$, the conformal blocks are

$$\eta^{2-j_4}(1-\eta)^{\frac{1}{2}-j_2} \quad \text{and} \quad \eta^{2-j_4}(1-\eta)^{\frac{1}{2}-j_2}\mathsf{B}\big(j_4 - \tfrac{1}{2}, j_2 - \tfrac{1}{2}; \eta\big)$$

  and again the second block is ruled out.

- For $N = 4$ and $\ell = 2$, the conformal blocks are

$$\eta\,{}_2\mathsf{F}_1\left[\begin{matrix}\frac{1}{2} & j_2+j_4-\frac{1}{2} \\ & j_4+\frac{1}{2}\end{matrix};\eta\right] \quad \text{and} \quad \eta^{\frac{3}{2}-j_4}\,{}_2\mathsf{F}_1\left[\begin{matrix}j_2 & 1-j_4 \\ \frac{3}{2}-j_4 & \end{matrix};\eta\right].$$

  This time, single-valuedness does not rule out either block.

Suppose now that $j_4 \in \mathbb{Z} + \frac{1}{2}$. Then, one of these hypergeometric functions develops a logarithmic singularity at $\eta = 0$.

Similarly, expanding in $1 - \eta$ shows that there is a logarithmic singularity at $\eta = 1$ when $j_2 \in \mathbb{Z} + \frac{1}{2}$.

This is precisely the behaviour expected given the fusion rules.

BPZ + KZ = ???

The BPZ solutions require $j_3 = \frac{1}{2}$. Using them as initial conditions to the KZ recursion thus gives 4-point functions with $j_3 \in \mathbb{N} + \frac{1}{2}$.

BPZ + KZ = ???

The BPZ solutions require $j_3 = \frac{1}{2}$. Using them as initial conditions to the KZ recursion thus gives 4-point functions with $j_3 \in \mathbb{N} + \frac{1}{2}$.

- For $\ell = 1$, this recovers our previous results.

BPZ + KZ = ???

The BPZ solutions require $j_3 = \frac{1}{2}$. Using them as initial conditions to the KZ recursion thus gives 4-point functions with $j_3 \in \mathbb{N} + \frac{1}{2}$.

- For $\ell = 1$, this recovers our previous results.
- For $\ell = 3$, we get completely new results (for $j_3 \in \mathbb{N} + \frac{1}{2}$). They have the form of a power of η times a power of $1 - \eta$ times a polynomial.

BPZ + KZ = ???

The BPZ solutions require $j_3 = \frac{1}{2}$. Using them as initial conditions to the KZ recursion thus gives 4-point functions with $j_3 \in \mathbb{N} + \frac{1}{2}$.

- For $\ell = 1$, this recovers our previous results.
- For $\ell = 3$, we get completely new results (for $j_3 \in \mathbb{N} + \frac{1}{2}$). They have the form of a power of η times a power of $1 - \eta$ times a polynomial.

However, these polynomials are secretly hypergeometric functions:

$$\langle \phi_{j_1} | \phi_{j_2}(1) \phi_{j_3}(\eta) | \phi^3_{j_4} \rangle = \delta_{j_1+j_2+j_3+j_4=3} \left[\begin{smallmatrix} 0 & 0 & 0 & 3 \\ j_1 & j_2 & \frac{1}{2} & j_3+j_4-\frac{1}{2} \end{smallmatrix} \right]$$

$$\cdot \frac{B(\frac{1}{2}, -j_4+1)}{B(j_3, -j_3-j_4+\frac{3}{2})} \eta^{2j_3-j_4+1} (1-\eta)^{-j_2+\frac{1}{2}} \, {}_2\mathsf{F}_1 \left[\begin{smallmatrix} -j_1+\frac{3}{2} & -j_3+\frac{1}{2} \\ -j_3-j_4+\frac{3}{2} & \end{smallmatrix} ; \eta \right].$$

BPZ + KZ = ???

The BPZ solutions require $j_3 = \frac{1}{2}$. Using them as initial conditions to the KZ recursion thus gives 4-point functions with $j_3 \in \mathbb{N} + \frac{1}{2}$.

- For $\ell = 1$, this recovers our previous results.
- For $\ell = 3$, we get completely new results (for $j_3 \in \mathbb{N} + \frac{1}{2}$). They have the form of a power of η times a power of $1 - \eta$ times a polynomial.

  However, these polynomials are secretly hypergeometric functions:

$$\langle \phi_{j_1} | \phi_{j_2}(1) \phi_{j_3}(\eta) | \phi^3_{j_4} \rangle = \delta_{j_1+j_2+j_3+j_4=3} \left[\begin{smallmatrix} 0 & 0 & 0 & 3 \\ j_1 & j_2 & \frac{1}{2} & j_3+j_4-\frac{1}{2} \end{smallmatrix} \right]$$
$$\cdot \frac{B(\frac{1}{2}, -j_4+1)}{B(j_3, -j_3-j_4+\frac{3}{2})} \eta^{2j_3-j_4+1} (1-\eta)^{-j_2+\frac{1}{2}} \, {}_2\mathsf{F}_1 \left[\begin{smallmatrix} -j_1+\frac{3}{2} & -j_3+\frac{1}{2} \\ -j_3-j_4+\frac{3}{2} & \end{smallmatrix} ; \eta \right].$$

  Is this formula correct for all $j_3 \in \mathbb{R}$?
  It passes the KZ-recursion test when $j_3 \in \mathbb{Z} + \frac{1}{2}$...

- For $\ell = 2$, the results are even more interesting!

- For $\ell = 2$, the results are even more interesting!

  Each conformal block for $j_3 \in \mathbb{N} + \frac{1}{2}$ are a power of η times a finite linear combination of hypergeometric functions.

- For $\ell = 2$, the results are even more interesting!

  Each conformal block for $j_3 \in \mathbb{N} + \frac{1}{2}$ are a power of η times a finite linear combination of hypergeometric functions.

  Amazingly, these finite linear combinations turn out to be generalised hypergeometric functions! With this realisation, the conformal blocks take the form

$$\delta_{j_1+j_2+j_3+j_4=2}\eta^{2j_3}(1-\eta)^{j_1-1}\,{}_3\mathsf{F}_2\left[\begin{matrix} j_3+j_4-\frac{1}{2} & -j_1+1 & j_3 \\ j_3+j_4 & \frac{1}{2} & \end{matrix};-\frac{\eta}{1-\eta}\right],$$

$$\delta_{j_1+j_2+j_3+j_4=2}\frac{\mathsf{B}(\frac{1}{2},j_3-j_4+\frac{1}{2})}{\mathsf{B}(j_3,1-j_4)}\eta^{j_3-j_4+1}(1-\eta)^{-j_2}$$

$${}_3\mathsf{F}_2\left[\begin{matrix} \frac{1}{2} & j_2 & -j_4+1 \\ j_1+j_2 & j_1+j_2-\frac{1}{2} & \end{matrix};-\frac{\eta}{1-\eta}\right].$$

- For $\ell = 2$, the results are even more interesting!

  Each conformal block for $j_3 \in \mathbb{N} + \frac{1}{2}$ are a power of η times a finite linear combination of hypergeometric functions.

  Amazingly, these finite linear combinations turn out to be generalised hypergeometric functions! With this realisation, the conformal blocks take the form

$$\delta_{j_1+j_2+j_3+j_4=2}\eta^{2j_3}(1-\eta)^{j_1-1}\,{}_3\mathsf{F}_2\left[\begin{matrix} j_3+j_4-\frac{1}{2} & -j_1+1 & j_3 \\ j_3+j_4 & \frac{1}{2} & \end{matrix};-\frac{\eta}{1-\eta}\right],$$

$$\delta_{j_1+j_2+j_3+j_4=2}\frac{\mathsf{B}(\frac{1}{2},j_3-j_4+\frac{1}{2})}{\mathsf{B}(j_3,1-j_4)}\eta^{j_3-j_4+1}(1-\eta)^{-j_2}$$
$${}_3\mathsf{F}_2\left[\begin{matrix} \frac{1}{2} & j_2 & -j_4+1 \\ j_1+j_2 & j_1+j_2-\frac{1}{2} & \end{matrix};-\frac{\eta}{1-\eta}\right].$$

  Are these formulae correct for all $j_3 \in \mathbb{R}$?
  They also pass the KZ-recursion test when $j_3 \in \mathbb{Z} + \frac{1}{2}$...

Conclusions

The representation theory of bosonic ghosts is considered to be well understood: we know the irreducibles, projectives and their fusion rules.

However, the primary correlators are perhaps not so easy to compute with standard methods.

KZ allows us to reproduce the generic fusion rules, but only gives recursion relations in general (because the irreducibles have infinitely many independent primaries).

One case ($\ell = 1$) could nevertheless be solved using a second KZ equation. Another case ($j_3 = \frac{1}{2}$) was solved using a "fake" BPZ equation.

The latter manifested the logarithmic nature of the theory, confirming the non-generic fusion rules.

Are similar methods available in more general theories?

Outlook

There are still many avenues to explore:

- Do the $j_3 = \frac{1}{2}$ 4-point functions constrain the 3-point constants?
- Is it possible to interpolate the solutions for N-point constants using, *eg*., twisted localisation [Mathieu'00]?
- Are the remaining primary 4-point functions computable?
  - Try a Dotsenko–Fateev approach using FMS bosonisation.
  - Try a conformal bootstrap approach.

The overarching aim is to determine if, for log CFTs like bosonic ghosts, the correlation functions are entirely fixed by mathematical consistency.

If not, what physical input is required? Either way, there is some surely beautiful new mathematics (and mathematical physics) to explore!

"*Only those who attempt the absurd will achieve the impossible.*"

— Miguel de Unamuno