Wess-Zumino-Witten Models and the Knizhnik-Zamolodchikov Equations

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Abstract

This thesis introduces the Wess-Zumino-Witten models by studying the representation theory of untwisted affine Lie algebras. We compute a class of 4-point functions for the $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ Wess-Zumino-Witten model at level 1 using the Knizhnik-Zamolodchikov equations, the Ward identities, and the affine Ward identities. We then introduce vertex algebras as a method of formalising the notion of a chiral two-dimensional conformal field theory, and use this formalism to give a free field realisation of our model. This free field realisation is used to compute the 4-point functions again using vertex operators. The results from the two methods of calculation are compared. Finally, we describe the Knizhnik-Zamolodchikov equations as a system of differential equations for sections of the conformal block bundle, and then explain how the monodromy of these equations gives a representation of the Artin braid group. An explicit two-dimensional representation of the braid group on 4 strands is found using our solutions to the Knizhnik-Zamolodchikov equations.

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Contents

Introduction

Conformal field theory is a vast and rich area of study with connections to many areas of mathematics and physics. Most notably, the mathematics of conformal field theory encompasses Lie theory, topology, geometry, number theory, and category theory. Perhaps the most striking connection to mathematics is monstrous moonshine, which provides a connection between the monster group and modular functions by the construction of a vertex operator algebra. This connection begins with the construction of the monster vertex algebra in 1989 by Frenkel, Lepowsky, and Meurman [13]. The conjectured relationship between the monster vertex algebra and modular forms was established by Borcherds in 1992 [3].

In physics, conformal field theory finds applications to string theory, condensed matter physics, and statistical mechanics. In string theory, one can describe the propagation of strings on a manifold by a sigma model. Some sigma models can be quantised, giving conformal field theories. In particular, the Wess-Zumino-Witten models are obtained by quantising a nonlinear sigma model whose fields take values in a semisimple Lie group. In condensed matter physics, one can compute critical exponents within the formalism of conformal field theory. Certain models in statistical mechanics such as the Ising model and the 3-state Potts model can be viewed as Virasoro minimal models.

The study of conformal field theory grew largely out of a paper published by Belavin, Polyakov, and Zamolodchikov in 1984 [1]. This paper discusses the Virasoro minimal models, which are unitary theories that can be constructed from certain highest weight representations of the Virasoro algebra. In the same year, the complete action of the Wess-Zumino-Witten models was discovered by Witten [34]. Witten showed that the quantum field theory defined by this action was conformally invariant and that the symmetry algebra of the theory was given by an untwisted affine Lie algebra. Knizhnik and Zamolodchikov extended the work of Witten in [24]. In particular, they derived the Knizhnik-Zamolodchikov equations and used them to compute a collection of four-point functions.

The main goal of this thesis is to investigate the Wess-Zumino-Witten models and the Knizhnik-Zamolodchikov equations. In chapters 1 and 2, we introduce the Lie theory required to understand the Wess-Zumino-Witten models. Chapter 1 focusses primarily on the theory of semisimple Lie algebras and their representations. We introduce Verma modules and their irreducible quotients, which are fundamental to conformal field theory. In chapter 2, we discuss affine Lie algebras and their representations. We mention the classification of affine Lie algebras before providing the loop algebra construction for the untwisted affine Lie algebras. We conclude chapter 2 with a discussion of irreducible, integrable highest weight modules, which are the allowed modules of the Wess-Zumino-Witten models.

In chapter 3, we introduce two-dimensional conformal field theory in the context of the Wess-Zumino-Witten models. Chapter 3 will focus on the basic concepts of two-dimensional conformal field theory such as states, fields, correlation functions, and operator product expansions. It will also provide a brief discussion of the action of the Wess-Zumino-Witten models.

Chapter 4 focusses on the Wess-Zumino-Witten models in more detail and we introduce the Knizhnik-Zamolodchikov equations. However, we will also see some more general concepts in chapter 4 such as the Ward identities, which constrain the correlation functions of a given conformal field theory. Chapter 4 contains one of our main results, which is the calculation of a collection of four-point functions in the $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ model at level 1 using the Ward identities, the affine Ward identities, and the Knizhnik-Zamolodchikov equations. We also provide a derivation of the Sugawara construction using only the state-field correspondence. This differs from the standard derivation, which uses a generalization of Wick's theorem.

In chapter 5, we introduce the formalism of vertex operator algebras. Using this, we give a free field realisation of the $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ Wess-Zumino-Witten model at level 1. This free field realisation is obtained by proving that the Kac-Moody vertex operator algebra is isomorphic to a lattice Heisenberg vertex operator algebra. Through this isomorphism, we can show the Wess-Zumino-Witten fields of our module in chapter 4 can be realised as vertex operators. We conclude this section by computing the four-point functions of chapter 4 using vertex operators and comparing to the result obtained by solving the Knizhnik-Zamolodchikov equations.

Chapter 6 introduces conformal blocks. We give a construction of a flat vector bundle called the conformal block bundle and show that the Knizhnik-Zamolodchikov equations can be realised as a system of differential equations for sections of this bundle. We show that for any flat vector bundle, one can obtain a representation of the fundamental group of the base space called the monodromy representation. In particular, the KZ connection on the conformal block bundle gives a representation of the pure braid group, which can then be used to obtain a representation of the Artin braid group. To conclude this chapter, we study the monodromy of our solutions to the Knizhnik-Zamolodchikov equations and obtain an explicit two-dimensional representation of the braid group on 4 strands.

Chapter 1

Semisimple Lie Algebras

This chapter will mostly concern the theory of (complex, finite-dimensional) semisimple Lie algebras. Our discussion will contain enough theory to introduce affine Lie algebras with the goal of studying the Wess-Zumino-Witten models in mind. We will present a number of important theorems that will not be proven as their proofs can be found in many standard textbooks including those that we cite. In particular, these theorems lead up to the classification of semisimple Lie algebras. For our study of semisimple Lie algebras, we have used [15] and [16]. Here, one can find the main definitions and theorems of this chapter.

1.1 Basic Definitions and Examples

We begin with some elementary definitions.

Definition 1.1.1. A Lie algebra is a vector space $\mathfrak{g}$ with an antisymmetric bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \tag{1.1}$$

for all $X, Y, Z \in \mathfrak{g}$. We call $[\cdot,\cdot]$ the Lie bracket of the Lie algebra.

A vector subspace that is closed under the Lie bracket is called a Lie subalgebra. A vector subspace I of $\mathfrak{g}$ that satisfies $[X,Y] \in I$ for all $X \in I$, $Y \in \mathfrak{g}$ is called an ideal.

Definition 1.1.2. A homomorphism of Lie algebras is a linear map preserving the Lie bracket. That is, $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a homomorphism if

$$f([X,Y]_1) = [f(X), f(Y)]_2 \tag{1.2}$$

for all $X, Y \in \mathfrak{g}_1$. Here, $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$ are the Lie brackets on $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively.

Definition 1.1.3. A representation of a Lie algebra $\mathfrak{g}$ is a vector space V with a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V). \tag{1.3}$$

Here, $\mathfrak{gl}(V)$ is the Lie algebra of endomorphisms on V with bracket given by the commutator.

Definition 1.1.4. Given a Lie algebra $\mathfrak{g}$, a $\mathfrak{g}$-module is a vector space V with a bilinear map $\mathfrak{g} \times V \to V$ satisfying

$$[X, Y] \cdot v = X \cdot (Y \cdot v) - Y \cdot (X \cdot v) \tag{1.4}$$

for all $X, Y \in \mathfrak{g}$ and $v \in V$.

A representation of a Lie algebra $\mathfrak{g}$ is equivalent to a $\mathfrak{g}$-module.

Example 1.1.1. The adjoint representation is defined as

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}) && (1.5)\\ X &\mapsto \mathrm{ad}_X. && (1.6) \end{aligned}$$

Here, ad_X is defined as

$$\begin{aligned} \mathrm{ad}_X : \mathfrak{g} &\to \mathfrak{g} && (1.7)\\ Y &\mapsto [X, Y]. && (1.8) \end{aligned}$$

Example 1.1.2. One of the simplest nontrivial Lie algebras is $\mathfrak{gl}_2(\mathbb{C})$, which consists of 2×2 matrices with coefficients in $\mathbb{C}$. The Lie bracket is given by the commutator. We then have a Lie subalgebra[1] $\mathfrak{sl}_2(\mathbb{C})$ which consists of the traceless elements of $\mathfrak{gl}_2(\mathbb{C})$. A standard choice of basis for $\mathfrak{sl}_2(\mathbb{C})$ is

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.9}$$

with brackets

$$[H, E] = 2E \qquad [H, F] = -2F \qquad [E, F] = H. \tag{1.10}$$

Definition 1.1.5. Given a Lie algebra $\mathfrak{g}$, we define the lower central series $D_n(\mathfrak{g})$ by $D_1(\mathfrak{g}) = \mathfrak{g}$ and $D_n(\mathfrak{g}) = [\mathfrak{g}, D_{n-1}(\mathfrak{g})]$ for $n > 1$. Explicitly, this series is

$$\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]], \ldots. \tag{1.11}$$

A Lie algebra is nilpotent if its lower central series terminates at zero.

Example 1.1.3. The Lie subalgebra $\mathfrak{st}_3(\mathbb{C})$ of $\mathfrak{gl}_3(\mathbb{C})$ consisting of strictly upper triangular matrices is nilpotent. Given $A, B, C \in \mathfrak{st}_3(\mathbb{C})$ we can write

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & d & e \\ 0 & 0 & f \\ 0 & 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & g & h \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.12}$$

[1] In fact, this is an ideal since $tr([X, Y]) = tr(XY) - tr(YX) = 0$ for all $X \in \mathfrak{sl}_2(\mathbb{C})$ and $Y \in \mathfrak{gl}_2(\mathbb{C})$.

Then

$$[A,B] = \begin{pmatrix} 0 & 0 & af-cd \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.13}$$

We now see that $[C,[A,B]] = 0$. We conclude that $D_3(\mathfrak{g}) = \{0\}$ and so $\mathfrak{st}_3(\mathbb{C})$ is nilpotent. One can generalise this argument to show that $\mathfrak{st}_n(\mathbb{C})$ is nilpotent.

Definition 1.1.6. Given a Lie algebra $\mathfrak{g}$, we define the upper central series $D^n(\mathfrak{g})$ by $D^1(\mathfrak{g}) = \mathfrak{g}$ and $D^n(\mathfrak{g}) = [D^{n-1}(\mathfrak{g}), D^{n-1}(\mathfrak{g})]$ for $n > 1$. Explicitly, this series is

$$\mathfrak{g}, [\mathfrak{g},\mathfrak{g}], [[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]], \dots. \tag{1.14}$$

A Lie algebra is solvable if its upper central series terminates at zero.

Clearly nilpotent Lie algebras are solvable. We will now give a counterexample to show that the converse is not necessarily true.

Example 1.1.4. The Lie subalgebra $\mathfrak{t}_3(\mathbb{C})$ of $\mathfrak{gl}_3(\mathbb{C})$ consisting of upper triangular matrices is solvable but not nilpotent. Given $A, B \in \mathfrak{t}_3(\mathbb{C})$, an explicit computation shows that $[A,B] \in \mathfrak{st}_3(\mathbb{C})$. Since $\mathfrak{st}_3(\mathbb{C})$ is nilpotent by Example 1.1.3 and therefore solvable, we see that $\mathfrak{t}_3(\mathbb{C})$ is solvable. Now, let E_{ij} denote the matrix with a 1 at the entry (i,j) and zeros elsewhere. Consider the Lie subalgebra of $\mathfrak{t}_3(\mathbb{C})$ generated by E_{12} and $E_{11} - E_{22}$. We have $[E_{11} - E_{22}, E_{12}] = 2E_{12}$ and so this subalgebra is not nilpotent. Therefore, $\mathfrak{t}_3(\mathbb{C})$ is not nilpotent. This argument can be generalised to show that $\mathfrak{t}_n(\mathbb{C})$ is solvable but not nilpotent.

Definition 1.1.7. A Lie algebra is semisimple if it contains no nonzero solvable ideals.

Semisimple Lie algebras have nice properties that we will see shortly.

Definition 1.1.8. The Killing form κ of a Lie algebra $\mathfrak{g}$ is defined to be

$$\kappa(X,Y) = tr(ad(X)ad(Y)). \tag{1.15}$$

Theorem 1.1.1. *The Killing form is bilinear, symmetric, invariant, and nondegenerate. By invariant, we mean that for all $X, Y, Z \in \mathfrak{g}$, $\kappa([X,Y],Z) = \kappa(X,[Y,Z])$. For finite-dimensional Lie algebras, nondegeneracy means that if $X \in \mathfrak{g}$ satisfies $\kappa(X,Y) = 0$ for all $Y \in \mathfrak{g}$ then $X = 0$.*[2]

Example 1.1.5. The Killing form on $\mathfrak{sl}_n(\mathbb{C})$ is

$$\kappa(X,Y) = 2ntr(XY). \tag{1.16}$$

In this thesis, we will work with the rescaled Killing form, which is obtained by dividing the Killing form by $2h^\vee$ where $h^\vee$ is the dual Coxeter number given by Definition 1.2.11. We use the notation $\kappa(\cdot,\cdot)$ to denote the rescaled Killing form.

[2] If we do not assume that the Lie algebra has finite dimension then nondegeneracy means that the map $X \mapsto (Y \mapsto \kappa(X,Y))$ defines an isomorphism.

Theorem 1.1.2. ***Cartan's Criterion.*** *[16, p. 480] Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field of characteristic* 0*. Then $\mathfrak{g}$ is semisimple if and only if the Killing form on $\mathfrak{g}$ is nondegenerate.*

This criterion provides a useful method of determining whether a Lie algebra is semisimple.

Example 1.1.6. The Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ is semisimple. To prove this, we can use Cartan's criterion. For $1 \leq i \leq n-1$, let $H_i \in \mathfrak{sl}_n(\mathbb{C})$ consist of zeros everywhere except for the entries (i,i) and $(i+1,i+1)$ with values 1 and -1 respectively. For $i \neq j$, let $E_{ij} \in \mathfrak{sl}_n(\mathbb{C})$ consist of zeros everywhere except for the entry (i,j) with a value of 1.[3] We have

$$\kappa(X,Y) = tr(XY) = \sum_{k=1}^{n}\sum_{l=1}^{n} X_{kl}Y_{lk}. \tag{1.17}$$

Assume there exists $X \in \mathfrak{sl}_n(\mathbb{C})$ such that $tr(XY) = 0$ for all $Y \in \mathfrak{sl}_n(\mathbb{C})$. If we choose Y to be E_{ij} for $i \neq j$ then Equation (1.17) implies that $X_{ij} = 0$ for $i \neq j$. If we choose Y to be H_i for $1 \leq i \leq n-1$ then we see that $X_{ii} = X_{i+1,i+1}$ for $1 \leq i \leq n-1$. Since X is traceless, all of these entries must be 0. We conclude that X is the zero matrix and so the Killing form on $\mathfrak{sl}_n(\mathbb{C})$ is nondegenerate.

Definition 1.1.9. An element X of a semisimple Lie algebra is said to be semisimple if ad_X is diagonalizable.

Definition 1.1.10. A Cartan subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$ is a maximal abelian subalgebra consisting of only semisimple elements.

Example 1.1.7. Using our basis for $\mathfrak{sl}_2(\mathbb{C})$ in Example 1.1.2, the subalgebra generated by H is a Cartan subalgebra.

1.2 Root Systems

Definition 1.2.1. A root system Φ in a finite-dimensional real vector space E with inner product $(\cdot,\cdot)$ is a collection of nonzero vectors satisfying:

1. $E = \mathrm{Span}(\Phi)$.

2. If α is a root then $-\alpha$ is. No other scalar multiple of α is a root.

3. Φ is closed under reflection about the hyperplane perpendicular to each $\alpha \in \Phi$.

4. If $\alpha, \beta \in \Phi$ then $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$.

[3] This defines a Cartan-Weyl basis for $\mathfrak{sl}_n(\mathbb{C})$.

We will now show how to construct a root system from a semisimple Lie algebra. Consider a semisimple Lie algebra $\mathfrak{g}$ with a Cartan subalgebra $\mathfrak{h}$. Consider the adjoint representation ad. Since the elements of $\mathfrak{h}$ are semisimple, the operators $\mathrm{ad}(X)$ are diagonalizable for $X \in \mathfrak{h}$. They also commute since $\mathfrak{h}$ is its own centralizer. So we have the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha. \tag{1.18}$$

Here, $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \ \forall H \in \mathfrak{h}\}$.

Definition 1.2.2. The spaces $\mathfrak{g}_\alpha$ above are called root spaces. The nonzero linear functionals α are called roots and we will denote the collection of roots by Φ.

The Killing form allows us to define an inner product on $\mathfrak{h}^*$ with $\mathfrak{h}$ a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$. First, note that the properties of the Killing form imply that the adjoint module $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ are isomorphic as modules. An explicit isomorphism is given by

$$i : \mathfrak{g} \to \mathfrak{g}^* \tag{1.19}$$
$$X \mapsto \kappa(X, \cdot). \tag{1.20}$$

We then define the inner product $(\cdot, \cdot)$ on $\mathfrak{h}^*$ by

$$(\alpha, \beta) = \kappa(i^{-1}(\alpha), i^{-1}(\beta)). \tag{1.21}$$

Definition 1.2.3. Given two roots α and β, we define

$$\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}. \tag{1.22}$$

Definition 1.2.4. Given a root α, the map σ_α defined on Φ by

$$\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \beta \tag{1.23}$$

is called a Weyl reflection.[4] The subgroup of the orthogonal group generated by the Weyl reflections is called the Weyl group.

Theorem 1.2.1. *We now list some properties of roots:*

1. *The roots span* $\mathfrak{h}^*$*.*
2. *If α is a root then $-\alpha$ is. No other scalar multiple of α is a root.*
3. *If α and β are roots then $\sigma_\alpha(\beta)$ is a root.*
4. $\dim(\mathfrak{g}_\alpha)$ *is one-dimensional for all roots α.*

[4] Geometrically, we can view this map as a reflection about the hyperplane perpendicular to α. By property 3 of Definition 1.2.1, σ_α maps Φ to Φ for all $\alpha \in \Phi$.

5. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

Note that the third condition ensures that Φ is closed under reflection of its elements about a perpendicular hyperplane. In fact, we have the following theorem.

Theorem 1.2.2. *The collection of roots of a semisimple Lie algebra equipped with the inner product* $(\cdot,\cdot)$ *defines a root system.*

Definition 1.2.5. We can generalize the concept of a root to an arbitrary representation. Given some finite-dimensional representation (ρ, V) of a semisimple Lie algebra $\mathfrak{g}$, define $V_\lambda = \{X \in V \mid \rho(H)X = \lambda(H)X \ \forall H \in \mathfrak{h}\}$. This is known as a weight space, and we have the weight space decomposition

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda. \tag{1.24}$$

The linear functionals λ are called weights. Note that the roots of a Lie algebra are the nonzero weights of the adjoint representation.

Definition 1.2.6. The decomposition for a semisimple Lie algebra $\mathfrak{g}$ given by Equation (1.18) allows us to choose a basis of the form $\{H_i \mid i \in \{1, \dots, \dim \mathfrak{h}\}\} \cup \{E^\alpha \mid \alpha \in \Phi\}$ where the H_i span the Cartan subalgebra and each E^α spans the corresponding root space. Such a basis is called a Cartan-Weyl basis.

Example 1.2.1. In our study of the Wess-Zumino-Witten models, we will be interested in a Cartan-Weyl basis $\{H_i \mid i \in \{1, \dots, \dim \mathfrak{h}\}\} \cup \{E^\alpha \mid \alpha \in \Phi\}$ chosen such that the following conditions are satisfied for all $i, j \in \{1, \dots, \dim \mathfrak{h}\}$, $\alpha \in \Phi$:

1. $\kappa(E^\alpha, H_i) = 0$.
2. $\kappa(H_i, H_j) = \delta_{i,j}$.
3. $\kappa(E^\alpha, E^\beta) = \delta_{\alpha,-\beta}$.

In the case of $\mathfrak{sl}_2(\mathbb{C})$, the basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{1.25}$$

satisfies these conditions.

Definition 1.2.7. Given a root α, we define the corresponding coroot to be $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$.

Definition 1.2.8. When plotting a root system for a Lie algebra, we can choose a hyperplane that separates the roots into two disjoint sets of the same cardinality. We then call one of these sets the collection of positive roots and the other set the collection of negative roots. If we take some positive root α, then $-\alpha$ is a negative root. If a positive root is not the sum of two positive roots, it is called a simple root.

A useful result is that any positive root $\alpha \in \Phi$ can be written as

$$\alpha = \sum_{i=1}^{n} n_i \alpha_i \tag{1.26}$$

with n_i nonnegative integers and $\{\alpha_i\}$ the collection of simple roots. A consequence of this result is that the collection of simple roots forms a basis for $\mathfrak{h}^*$. Thus, there are dim $\mathfrak{h}$ simple roots.[5]

Example 1.2.2. We will plot the root system for $\mathfrak{sl}_3(\mathbb{C})$. A Cartan-Weyl basis for $\mathfrak{sl}_3(\mathbb{C})$ is given by choosing the generators of the Cartan subalgebra to be

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{1.27}$$

and choosing the generators of the root spaces to be E_{ij} for $i \neq j$ where E_{ij} has the entry 1 at the position (i, j) and zeros elsewhere. It is easy to see that $E_{ij}E_{kl} = \delta_{j,k}E_{il}$ and so $[E_{ij}, E_{kl}] = \delta_{j,k}E_{il} - \delta_{l,i}E_{kj}$. Writing $H_1 = E_{11} - E_{22}$ and $H_2 = E_{22} - E_{33}$, we have

$$[H_1, E_{ij}] = \delta_{1,i}E_{1j} + \delta_{2,j}E_{i2} - (\delta_{1,j}E_{i1} + \delta_{2,i}E_{2j}) = \lambda_{ij}(H_1) \tag{1.28}$$
$$[H_2, E_{ij}] = \delta_{2,i}E_{2j} + \delta_{3,j}E_{i3} - (\delta_{2,j}E_{i2} + \delta_{3,i}E_{3j}) = \lambda_{ij}(H_2) \tag{1.29}$$

where λ_{ij} are the roots. By the above equations, we have

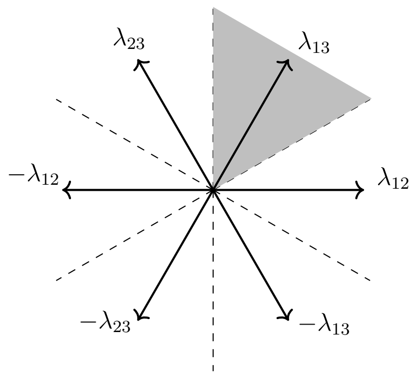
$$\lambda_{12}(H_1) = 2 \qquad \lambda_{12}(H_2) = -1 \tag{1.30}$$
$$\lambda_{13}(H_1) = 1 \qquad \lambda_{13}(H_2) = 1 \tag{1.31}$$
$$\lambda_{23}(H_1) = -1 \qquad \lambda_{23}(H_2) = 2. \tag{1.32}$$

We can choose λ_{12}, λ_{13}, and λ_{23} to be the positive roots. Now, note that $\kappa(H_1, H_1) = 2$ and $\kappa(H_1, H_2) = -1$. This implies that $i(H_1) = \lambda_{12}$ and $i(H_2) = \lambda_{23}$. We now know that λ_{12} and λ_{23} span our root system. In fact, we can immediately see that $\lambda_{13} = \lambda_{12} + \lambda_{23}$. To plot our root system, it remains to compute the following inner products.

$$(\lambda_{12}, \lambda_{23}) = \kappa(i^{-1}(\lambda_{12}), i^{-1}(\lambda_{23})) = \kappa(H_1, H_2) = -1 \tag{1.33}$$
$$(\lambda_{12}, \lambda_{12}) = \kappa(H_1, H_1) = 2 \qquad (\lambda_{23}, \lambda_{23}) = \kappa(H_2, H_2) = 2. \tag{1.34}$$

We then deduce that $\cos(\theta) = -0.5$ where θ is the angle between λ_{12} and λ_{23}. A plot of the root system is given below.

[5] We call dim $\mathfrak{h}$ the rank of the Lie algebra. It can be shown that any two Cartan subalgebras of a semisimple Lie algebra have the same dimension and so the rank is well-defined.



The dashed lines in our plot above indicate the hyperplanes perpendicular to some root. Our choice of hyperplane separating the positive and negative roots is the hyperplane perpendicular to λ_{13}. Each connected component of the complement of the union of these hyperplanes is called a Weyl chamber. The shaded Weyl chamber above is called the fundamental Weyl chamber, which we will now define.[6]

Theorem 1.2.3. *[15, p. 213] Each Weyl chamber can be expressed uniquely in the form*

$$\{\lambda \in \mathfrak{h}^* \mid (\sigma(\lambda), \alpha_i) > 0 \textit{ for all simple roots } \alpha_i\} \tag{1.35}$$

for some element σ of the Weyl group. The Weyl chamber corresponding to σ being the identity is called the fundamental Weyl chamber.

We now return briefly to our discussion of weights as there is a particular class of weights that we will be interested in when studying the Wess-Zumino-Witten models. Recall that the roots of $\mathfrak{g}$ span $\mathfrak{h}^*$. As a consequence, we can write any weight as a linear combination of roots. However, we are more interested in the basis $\{\omega_i\}$ of weights satisfying $(\omega_i, \alpha_j^\vee) = \delta_{ij}$ where $\{\alpha_i^\vee\}$ is a simple coroot basis. Elements of this basis are called fundamental weights.

Definition 1.2.9. Given a weight λ and its expansion

$$\lambda = \sum_i \lambda_i \omega_i, \tag{1.36}$$

we call the coefficients λ_i the Dynkin labels of the weight. A weight whose Dynkin labels are all integers is called an integral weight. If the Dynkin labels are also non-negative, the weight is called dominant (or dominant integral).[7]

[6] Notice that the highest root (see Theorem 1.2.4) λ_{13} lies in the fundamental Weyl chamber. This turns out to always be the case since the expansion of a highest root as a linear combination of simple roots has strictly positive coefficients.

[7] Equivalently, an integral weight is dominant if it lies in the fundamental Weyl chamber.

Example 1.2.3. Given our simple roots $\{\lambda_{12}, \lambda_{23}\}$ for $\mathfrak{sl}_3(\mathbb{C})$, we see that $\omega_1 = \frac{2}{3}\lambda_{12} + \frac{1}{3}\lambda_{23}$ and $\omega_2 = \frac{1}{3}\lambda_{12} + \frac{2}{3}\lambda_{23}$ are our fundamental weights. To show this, just note that $\lambda_{12}^\vee = \lambda_{12}$ and $\lambda_{23}^\vee = \lambda_{23}$. Furthermore, $(\omega_1, \lambda_{12}) = \frac{4}{3} - \frac{1}{3} = 1$, $(\omega_1, \lambda_{13}) = -\frac{2}{3} + \frac{2}{3} = 0$, $(\omega_2, \lambda_{12}) = \frac{4}{3} - \frac{1}{3} = \frac{2}{3} - \frac{2}{3} = 0$, and $(\omega_2, \lambda_{13}) = -\frac{1}{3} + \frac{4}{3} = 1$.

Definition 1.2.10. The height of a root α is $ht(\alpha) = \sum_{i=1}^n k_i$ where $\alpha = \sum_{i=1}^n k_i\alpha_i$ and $\{\alpha_i\}$ is the basis of simple roots.

Let $\mathfrak{g}$ be a semisimple Lie algebra. We can introduce the root space gradation on $\mathfrak{g}$ [10, p. 35]

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i. \tag{1.37}$$

Here, $\mathfrak{g}_i = \mathrm{Span}\{X \in \mathfrak{g} \mid ht(\alpha) = i,\ X \in \mathfrak{g}_\alpha\}$.

Theorem 1.2.4. *[10, p. 35] If $\mathfrak{g}$ is simple then there is a highest root θ satisfying $ht(\theta) > ht(\alpha)$ for all $\alpha \in \Phi\backslash\{\theta\}$.*

Definition 1.2.11. Let $\mathfrak{g}$ be a simple Lie algebra. For the highest root θ of $\mathfrak{g}$, write $\theta = \sum_{i=1}^n \theta_i\alpha_i$ and $\theta^\vee = \sum_{i=1}^n \theta_i^\vee\alpha_i^\vee$ (with $\vee$ denoting the corresponding coroots). We can then define the Coxeter number h and dual Coxeter number $h^\vee$ by [10, p. 36]

$$h = 1 + \sum_{i=1}^n \theta_i \qquad h^\vee = 1 + \sum_{i=1}^n \theta_i^\vee. \tag{1.38}$$

Example 1.2.4. We will now compute the Coxeter and dual Coxeter numbers for $\mathfrak{sl}_3(\mathbb{C})$. From the previous example, we see that a basis of simple roots is given by $\{\lambda_{12}, \lambda_{23}\}$ and so the highest root is λ_{13}. Since $\lambda_{13} = \lambda_{12} + \lambda_{23}$, we have $h = 3$. Furthermore, $\lambda_{13}^\vee = \frac{2\lambda_{13}}{(\lambda_{13},\lambda_{13})} = \frac{2\lambda_{12}}{(\lambda_{12},\lambda_{12})} + \frac{2\lambda_{23}}{(\lambda_{23},\lambda_{23})} = \lambda_{12}^\vee + \lambda_{23}^\vee$ where we used the fact that $(\lambda_{12}, \lambda_{12}) = (\lambda_{13}, \lambda_{13}) = (\lambda_{23}, \lambda_{23})$. We conclude that $h^\vee = 3$.

The above example shows that the rescaled Killing form on $\mathfrak{sl}_3(\mathbb{C})$ is $\kappa(X, Y) = tr(XY)$. More generally, the dual Coxeter number of $\mathfrak{sl}_n(\mathbb{C})$ is n, and so the rescaled Killing form on $\mathfrak{sl}_n(\mathbb{C})$ is $\kappa(X, Y) = tr(XY)$.

To conclude this section, we introduce the notion of a triangular decomposition. Given some Lie algebra $\mathfrak{g}$ with Cartan-Weyl basis $\{H_i \mid i \in \{1, \ldots, \dim\mathfrak{h}\}\} \cup \{E^\alpha \mid \alpha \in \Phi\}$, we have a canonical isomorphism

$$\mathfrak{g} \cong \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+ \tag{1.39}$$

where $\mathfrak{g}_+$ is generated by $\{E^\alpha \mid \alpha \in \Phi_+\}$ and $\mathfrak{g}_-$ is generated by $\{E^{-\alpha} \mid \alpha \in \Phi_+\}$. We call this decomposition of $\mathfrak{g}$ a triangular decomposition. We wish to generalise this notion to Lie algebras that are not necessarily semisimple such as the Virasoro algebra, and so we have the following definition.

Definition 1.2.12. A triangular decomposition of a Lie algebra is a decomposition into subalgebras of the form

$$\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+ \tag{1.40}$$

such that $\mathfrak{g}_0$ is abelian and we have

$$[\mathfrak{g}_\pm, \mathfrak{g}_0] \subseteq \mathfrak{g}_\pm. \tag{1.41}$$

Example 1.2.5. One of the most fundamental Lie algebras[8] in the study of conformal field theory is the Virasoro algebra. The Virasoro algebra is the unique central extension of the Witt algebra. The generators of the infinitesimal conformal transformations of a pseudo-Riemannian manifold form a Lie algebra called the conformal algebra. The Witt algebra is precisely the conformal algebra for a two-dimensional pseudo-Riemannian manifold with constant metric of Euclidean signature. For a proof of this, see [32, p. 16].

The Witt algebra $\mathcal{D}$ is defined as [18, pp. 3-4]

$$\mathcal{D} = \mathbb{C}[t, t^{-1}]\frac{d}{dt}. \tag{1.42}$$

The space $\mathbb{C}[t, t^{-1}]$ consists of polynomials in t and t^{-1} over $\mathbb{C}$. Elements of the Witt algebra are viewed as derivations on $\mathbb{C}[t, t^{-1}]$. The operations are pointwise addition and composition. We have generators given by $d_n = -t^{n+1}\frac{d}{dt}$. We will now compute the commutation relations of these generators. We have

$$\begin{aligned}
[d_n, d_m] &= t^{n+1}\frac{d}{dt}\left(t^{m+1}\frac{d}{dt}\right) - t^{m+1}\left(\frac{d}{dt}t^{n+1}\frac{d}{dt}\right) && (1.43)\\
&= (m+1)t^{n+1}t^{m}\frac{d}{dt} + t^{n+1}t^{m+1}\frac{d^2}{dt^2} - (n+1)t^{m+1}t^{n}\frac{d}{dt} + t^{m+1}t^{n+1}\frac{d^2}{dt^2} && (1.44)\\
&= (m-n)t^{m+n+1}\frac{d}{dt} && (1.45)\\
&= (n-m)d_{n+m}. && (1.46)
\end{aligned}$$

Next, consider the central extension of $\mathcal{D}$ given by the short exact sequence

$$0 \to \mathbb{C}c \to \mathrm{Vir} \to \mathcal{D} \to 0. \tag{1.47}$$

We define a bracket on Vir by the cocycle $c(d_n, d_m) = \frac{1}{12}(n^3 - n)\delta_{n+m,0}$. That is, the bracket on the generators L_n of Vir is given by

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}(n^3 - n)\delta_{n+m,0}c. \tag{1.48}$$

The element c is called the central charge and Vir is called the Virasoro algebra. We will see in Section 3.5 that the Virasoro algebra appears in two-dimensional conformal field theory as the Lie algebra generated by the modes of the energy-momentum tensor.

The Virasoro algebra has a triangular decomposition given by

$$\left(\bigoplus_{n\leq -1}\mathbb{C}L_n\right) \oplus (\mathbb{C}L_0 \oplus \mathbb{C}c) \oplus \left(\bigoplus_{n\geq 1}\mathbb{C}L_n\right). \tag{1.49}$$

The notion of a triangular decomposition is important as it allows us to define highest weight representations for Lie algebras that do not necessarily have root systems.[9]

[8]Although not in fact semisimple

[9]Recall that our construction of a root system from a Lie algebra assumed semisimplicity.

1.3 Dynkin Diagrams

Definition 1.3.1. Given a root system with a choice of positive roots, let $\{\alpha_1, \ldots, \alpha_n\}$ be an ordered collection of its simple roots. The Cartan integers are defined to be $a_{ij} = \left(\alpha_i, \alpha_j^\vee\right)$ and the corresponding matrix a_{ij} is called the Cartan matrix.

The Cartan matrix depends on the ordering of the simple roots but not the choice of positive roots. A Cartan matrix of a root system satisfies the following properties.

Theorem 1.3.1. *Let $A = (a_{ij})$ be a Cartan matrix of a root system. Then*

1. $a_{ii} = 2$.
2. $a_{ij} \leq 0$ *for* $i \neq j$.
3. $a_{ij} = 0$ *if and only if* $a_{ji} = 0$.
4. $a_{ij} \in \mathbb{Z}$.
5. $det(A) > 0$.

The first and second properties allow us to define the Dynkin diagram of a root system in Definition 1.3.2. The importance of Cartan matrices is summarised by the following theorem.

Theorem 1.3.2. *The Cartan matrix of a root system determines the root system up to isomorphism.*

For a proof of this, see [17, p. 55].

Example 1.3.1. The calculations of Example 1.2.2 tell us that the Cartan matrix of $\mathfrak{sl}_3(\mathbb{C})$ is

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{1.50}$$

Here, we have chosen our ordered positive roots to be $\{\lambda_{12}, \lambda_{23}\}$.

Dynkin diagrams give us a way to represent the Cartan matrix of a root system visually. They also help us to classify all root systems. This is done by noting first that it suffices to classify the root systems that are not the disjoint union of proper orthogonal subsets. Such root systems are called irreducible and correspond to connected Dynkin diagrams. The connected Dynkin diagrams can then be classified.

Definition 1.3.2. Given a root system Φ with positive roots $\{\alpha_1, \ldots, \alpha_n\}$, we define its corresponding Dynkin diagram as follows. Consider the graph with n vertices with the ith and jth vertices joined by $a_{ij}a_{ji}$ edges. If there is a double or triple edge between two roots, add an arrow pointing to the shorter root.

Example 1.3.2. Our calculations from Example 1.2.2 show that the Dynkin diagram of $\mathfrak{sl}_3(\mathbb{C})$ is $\bullet\!\!-\!\!\bullet$.

We conclude by noting that the classification of finite-dimensional semisimple Lie algebras follows from the classification of root systems. This is a consequence of a theorem of Serre, which we now present.

Theorem 1.3.3. ***Serre's Theorem.*** *[17, p. 99] Let Φ be a root system with simple roots $\{\alpha_1, \ldots, \alpha_n\}$. Let $\mathfrak{g}$ be the Lie algebra spanned by the elements $\{E^i, E^{-i}, H^i\}_{1 \leq i \leq n}$ satisfying the relations[10]:*

1. $[E^i, E^{-j}] = \delta_{ij} H^i$.
2. $[H^i, H^j] = 0$.
3. $[H^i, E^j] = a_{ji} E^j$.
4. $[H^i, E^{-j}] = -a_{ji} E^{-j}$.
5. $(ad(E^i))^{1-a_{ji}}(E^j) = 0 \quad$ *for all* $i \neq j$.
6. $(ad(E^{-i}))^{1-a_{ji}}(E^{-j}) = 0 \quad$ *for all* $i \neq j$.

Then $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra with root system Φ and Cartan subalgebra spanned by $\{H^i\}_{1 \leq i \leq n}$.

Given a Cartan Weyl basis $\{E^\alpha\} \cup \{H^i\}$ of a semisimple Lie algebra, we can rescale the elements E^α and discard those not corresponding to simple roots to obtain a collection of elements $\{E^{\pm i}, H^i\}_{1 \leq i \leq n}$ satisfying the relations of Theorem 1.3.3. We call these elements Chevalley-Serre generators of the Lie algebra. They generate the Lie algebra in the sense that any element of the Lie algebra is given by taking linear combinations and Lie brackets of the Chevalley-Serre generators.

1.4 Verma Modules

In this section, we will give a construction of a particular kind of module called a Verma module. These modules help to classify the finite-dimensional irreducible representations of a semisimple Lie algebra. They also have the important universal property that every highest weight module over a Lie algebra is the quotient of a Verma module. Instead of defining Verma modules by their universal property, we give an explicit construction.

Definition 1.4.1. Given a Lie algebra $\mathfrak{g}$, its universal enveloping algebra $U(\mathfrak{g})$ is defined to be

$$U(\mathfrak{g}) = \left[\mathbb{C} \oplus \bigoplus_{i=1}^{\infty} \mathfrak{g}^{\otimes i} \right] \Big/ \langle X \otimes Y - Y \otimes X - [X, Y] \mid X, Y \in \mathfrak{g} \rangle . \tag{1.51}$$

[10] These relations are called the Chevalley-Serre relations.

The universal enveloping algebra is an associative algebra with product $\cdot$ defined by setting

$$[X_1 \otimes X_2 \otimes \cdots \otimes X_n] \cdot [Y_1 \otimes Y_2 \otimes \cdots \otimes Y_m] = [X_1 \otimes \cdots \otimes X_n \otimes Y_1 \otimes \cdots \otimes Y_m] \quad (1.52)$$

and extending by linearity.

Theorem 1.4.1. ***Poincaré-Birkoff-Witt (PBW) Theorem.*** *Let $\{x_1, x_2, \dots\}$ be an ordered basis for $\mathfrak{g}$. A basis for $U(\mathfrak{g})$ is given by 1 and the elements $x_{i_1} \cdot x_{i_2} \cdot \dots \cdot x_{i_n}$ with $i_1 \leq i_2 \leq \cdots \leq i_n$.*

For a proof of this theorem, one may consult [17, pp. 93-94].

Definition 1.4.2. A subspace of $U(\mathfrak{g})$ that is closed under left multiplication of $U(\mathfrak{g})$ is called a left ideal. Let $\lambda \in \mathfrak{h}^*$ and consider the left ideal I_λ generated by the positive roots of $\mathfrak{g}$ and elements of the form $H - \lambda(H)1$ for $H \in \mathfrak{h}$. Then $W_\lambda := U(\mathfrak{g})/I_\lambda$ is a left $\mathfrak{g}$-module called a Verma module [15, p. 256].[11]

The $\mathfrak{g}$-module structure on a Verma module is given by restricting the $U(\mathfrak{g})$-module structure defined by $X[v] = [Xv]$.[12] Clearly this is well-defined and it being a module structure follows from associativity of $U(\mathfrak{g})$.[13]

Definition 1.4.3. Given a Lie algebra $\mathfrak{g}$ with a triangular decomposition $\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$, we say a representation V of $\mathfrak{g}$ is a highest weight representation if there exists some $v_\lambda \in V$ and a map $\lambda : \mathfrak{g}_0 \to \mathbb{C}$ satisfying:

1. $Xv_\lambda = \lambda(X)v_\lambda$ for all $X \in \mathfrak{g}_0$.
2. $\mathfrak{g}_+ v_\lambda = 0$.
3. Any element of V is obtained by acting on v_λ with elements of $\mathfrak{g}_-$.

We call λ a highest weight and v_λ a highest weight vector.

Theorem 1.4.2. *Let $\mathfrak{g}$ be a Lie algebra with triangular decomposition $\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$. Let λ be a weight of $\mathfrak{g}$. Then we have*

$$U(\mathfrak{g}_-) \cong W_\lambda \quad (1.53)$$

as vector spaces.

One can prove this using the PBW theorem. An important corollary is the following.

[11] Equivalently, one can define a Verma module by the induced representation $\mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{g}_+}^{\mathfrak{g}} V$ where V is a one-dimensional representation of $\mathfrak{h} \oplus \mathfrak{g}_+$ such that $\mathfrak{h}$ acts by scalar multiplication and $\mathfrak{g}_+$ acts trivially.

[12] The PBW theorem gives us an inclusion map $\mathfrak{g} \hookrightarrow U(\mathfrak{g})$.

[13] Note that the reason we quotient by a **left** ideal is to ensure well-definedness of the left action of $\mathfrak{g}$.

Corollary 1.4.3. *Let $\mathfrak{g}$ be a Lie algebra with triangular decomposition $\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$. Let V be the $\mathfrak{g}$-module $U(\mathfrak{g}_-)v_\lambda$ with v_λ a highest weight vector with highest weight λ such that no nonzero element of $U(\mathfrak{g}_-)$ annihilates v_λ.*[14] *Then $V \cong W_\lambda$ as $\mathfrak{g}$-modules.*

This corollary gives us an alternative method of constructing Verma modules, which we will use extensively.

1.5 Irreducible Representations

In this section we will present a characterisation of the finite-dimensional, irreducible representations of a semisimple Lie algebra. We begin with a proof for the case of $\mathfrak{sl}_2(\mathbb{C})$ and then state the general result.

Theorem 1.5.1. *Any nonzero finite-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ is a highest weight representation with highest weight n and lowest weight $-n$ for some $n \in \mathbb{N}$.*

Proof. Let V be a finite-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. We know there exists an eigenvector v of H with eigenvalue λ. Let $V_\lambda \subseteq V$ denote the collection of all such elements of V. By irreducibility of V, the submodule generated by V_λ is V. Now, by induction we can show that for all $v \in V_\lambda$ and $n \in \mathbb{N}$ we have $F^n v \in V_{\lambda-2n}$ and $E^n v \in V_{\lambda+2n}$. The base case of the first claim holds since $HFv = [H,F]v + FHv = (\lambda-2)Fv$. Furthermore, if $F^n v \in V_{\lambda-2n}$ then $HF^{n+1}v = [H,F]F^n v + FHF^n v = (\lambda - 2(n+1))F^{n+1}v$ as required. In a similar manner, one can show that the second claim holds.

The above argument shows that we have the decomposition

$$V = \bigoplus_{n\in\mathbb{Z}} V_{\lambda+2n}. \tag{1.54}$$

Since V has finite dimension, there is some minimal $N \geq 0$ such that $V_{\lambda+2(N+1)} = 0$. Take $w \in V_{\lambda+2N}$. We claim that

$$V = \operatorname{span}\{w, Fw, F^2w, \dots\}. \tag{1.55}$$

By irreducibility and noting that $\operatorname{span}\{w, Fw, F^2w, \dots\}$ is closed under the action of H and F, it suffices to show that the space is closed under the action of E. First, we have $Ew = 0 \in \operatorname{span}\{w, Fw, F^2w, \dots\}$. By induction, we will now show that

$$EF^n w = n(\lambda + 2N - n + 1)F^{n-1}w. \tag{1.56}$$

For the $n = 1$ case, we have

$$EFw = [E,F]w + FEw = Hw = (\lambda + 2N)w \tag{1.57}$$

[14] In other words, we do not impose any relations.

as required. Now, assume $EF^n w = n(\lambda + 2N - n + 1)F^{n-1}w$. We have

$$\begin{aligned}
EF^{n+1}w &= [E, F]F^n w + FEF^n w && (1.58)\\
&= HF^n w + n(\lambda + 2N - n + 1)F^n w && (1.59)\\
&= (\lambda + 2N - 2n)F^n w + n(\lambda + 2N - n + 1)F^n w && (1.60)\\
&= (n + 1)(\lambda + 2N - (n + 1) + 1)F^n w && (1.61)
\end{aligned}$$

as required.

Given our expression for $EF^n w$, we will again invoke the fact that V is finite-dimensional. This implies there is some minimal M such that $F^M w = 0$. We thus have

$$M(\lambda + 2N - M + 1) = 0 \tag{1.62}$$

and so $\lambda + 2N = M - 1$ is a nonnegative integer. This then implies $F^{M-1}w$ has eigenvalue $-(M - 1)$ and so the possible eigenvalues of H in V are $M - 1, M - 2, \ldots, 0, \ldots, -(M - 2), -(M - 1)$. Since $V = \operatorname{span}\{w, Fw, F^2 w, \ldots, F^{M-1}w\}$, each eigenspace is one-dimensional. □

For a general semisimple Lie algebra, we have the following.

Theorem 1.5.2. *[15, p. 243] Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. There is a one-to-one correspondence between dominant integral elements of $\mathfrak{h}^*$ and irreducible finite-dimensional representations of $\mathfrak{g}$.*[15]

[15] As one might expect, the proof of this remarkable theorem is quite long and so it has been omitted. However, it is interesting to note that the proof relies heavily on the use of Verma modules.

Chapter 2

Affine Lie Algebras

Affine Lie algebras are an important class of Lie algebras that we are interested in. A collection of these called the untwisted affine Lie algebras are used to study the Wess-Zumino-Witten models. In this section, we will give a well-known construction of the untwisted affine Lie algebras. We also briefly discuss the classification of affine Lie algebras. Our primary resources for studying affine Lie algebras are the books [10], [11], and [4].

2.1 Classification

Recall that in Section 1.3 we outlined how to classify the finite-dimensional semisimple Lie algebras using Dynkin diagrams. One proceeds in a similar manner to classify the affine Lie algebras. The definitions and theorems of this section are obtained from chapters 14 and 15 of [4].

Definition 2.1.1. A generalised Cartan matrix is a square matrix $A = (a_{ij})$ satisfying:

1. $a_{ii} = 2$.
2. $a_{ij} \leq 0$ for $i \neq j$.
3. $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.
4. $a_{ij} \in \mathbb{Z}$.

Notice from Theorem 1.3.1 that these are the defining properties of a Cartan matrix, except that we have omitted the condition $\det(A) > 0$. We say that two generalised Cartan matrices are equivalent if they differ only by a permutation of the rows and columns. We call a generalised Cartan matrix indecomposable if it is not equivalent to a block diagonal matrix where the blocks are generalised Cartan matrices.

Definition 2.1.2. Let A be an indecomposable generalised Cartan matrix. Let $A_{\{i\}}$ denote the matrix obtained by deleting the ith row and column from A. We say that[1]

1. A has finite type if $\det(A) > 0$.
2. A has affine type if $\det(A) = 0$ and $\det(A_{\{i\}}) > 0$ for all i.
3. A has indefinite type if it is not of finite or affine type.

The Lie algebras corresponding to the indecomposable generalised Cartan matrices of affine type are called affine Lie algebras. We conclude this section by constructing some of the indecomposable generalised Cartan matrices of finite and affine type as examples, following [10, pp. 92-93]. For the complete list of the Dynkin diagrams corresponding to the affine Lie algebras, see [4, pp. 354-355].

Now, it is clear that the only rank 0 affine Lie algebra has Cartan matrix

$$A = (2). \tag{2.1}$$

This Lie algebra is denoted by A_1. The Cartan matrices of the rank 1 simple Lie algebras are

$$A_2 : \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{2.2}$$

$$C_2 : \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \tag{2.3}$$

$$G_2 : \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}. \tag{2.4}$$

These are the only 2×2 indecomposable generalised Cartan matrices of finite type since we require that their determinant be positive. There are also two non-simple rank 1 Lie algebras with Cartan matrices

$$A_1^{(1)} : \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \tag{2.5}$$

$$A_1^{(2)} : \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}. \tag{2.6}$$

These are the 2×2 indecomposable generalised Cartan matrices of affine type. We can see that there are no others due to the requirement that the determinant be zero. To obtain the 3×3 matrices of affine type, notice that removing the ith row and column gives one of the matrices corresponding to A_2, C_2, or G_2. If we consider the case of G_2, the matrix takes the form

$$\begin{pmatrix} 2 & a & b \\ c & 2 & -3 \\ d & -1 & 2 \end{pmatrix}. \tag{2.7}$$

[1] Although a Corollary in [4], we take this as a definition for simplicity.

The determinant of this matrix is $2 - a(2c + 3d) - b(c + 2d)$, which must be zero. This gives the following two possibilities.

$$G_2^{(1)} : \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix} \tag{2.8}$$

$$G_2^{(3)} : \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -3 \\ -1 & -1 & 2 \end{pmatrix}. \tag{2.9}$$

One can repeat these arguments to obtain the remaining indecomposable generalised Cartan matrices of finite and affine type.

2.2 Untwisted Construction

Definition 2.2.1. [10, p. 103] Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra. The loop algebra of $\mathfrak{g}$ is defined as

$$\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]. \tag{2.10}$$

Here, $\mathbb{C}[t, t^{-1}]$ is the space of polynomials in t and t^{-1} over $\mathbb{C}$. The bracket on $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ is given by

$$[X \otimes P(t, t^{-1}), Y \otimes Q(t, t^{-1})] = [X, Y] \otimes P(t, t^{-1})Q(t, t^{-1}). \tag{2.11}$$

Definition 2.2.2. [10, p. 103] Given a complex finite-dimensional simple Lie algebra $\mathfrak{g}$ with Killing form κ, we can define a Lie algebra $\hat{\mathfrak{g}}'$ by the central extension

$$0 \to \mathbb{C}K \to \hat{\mathfrak{g}}' \to \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \to 0 \tag{2.12}$$

with bracket[2]

$$\begin{aligned} [X \otimes P(t, t^{-1}) + \alpha K, Y \otimes Q(t, t^{-1}) + \beta K] = [X, Y] \otimes P(t, t^{-1})Q(t, t^{-1}) - \\ \left[\operatorname*{Res}_{t=0} \left(P(t, t^{-1}) \frac{dQ(t, t^{-1})}{dt} \right) \right] \kappa(X, Y)K. \end{aligned} \tag{2.13}$$

Let $\{J^a\}$ be a basis for $\mathfrak{g}$. Define $J_n^a = J^a \otimes t^n$. The untwisted affine Lie algebra $\hat{\mathfrak{g}}$ is defined as

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}' \oplus \mathbb{C}d \tag{2.14}$$

where we impose [10, p. 103]

$$[d, K] = 0 \tag{2.15}$$

$$[d, J_n^a] = -[J_n^a, d] = nJ_n^a. \tag{2.16}$$

The element d is usually called the derivation of $\hat{\mathfrak{g}}$. The Lie algebra $\hat{\mathfrak{g}}$ is sometimes referred to as an affine Kac-Moody algebra.

[2]We note that there is a unique choice of non-trivial central extension up to a rescaling of the Killing form. For an explanation of this, see [10, p. 102].

We can now compute the Lie bracket on the elements $\{J^a_n\}$. Noting that

$$\operatorname*{Res}_{t=0}\left(t^n \frac{d}{dt}(t^m)\right) = m\delta_{n+m,0}, \tag{2.17}$$

we see that

$$[J^a_n, J^b_m] = [J^a, J^b]_{n+m} + n\delta_{n+m,0}\kappa(J^a, J^b)K. \tag{2.18}$$

Definition 2.2.3. We have a canonical inclusion map

$$i : \mathfrak{g} \hookrightarrow \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d \cong \hat{\mathfrak{g}}. \tag{2.19}$$

We call $\overline{\mathfrak{g}} := i(\mathfrak{g})$ the horizontal subalgebra of $\hat{\mathfrak{g}}$.

To construct a Cartan subalgebra for $\hat{\mathfrak{g}}$, we simply need find a maximal abelian subalgebra consisting of only semisimple elements. Let $\{H^i\}$ be a basis for the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. It is easy to see that the set

$$\{H^i_0, K, d \mid i \in \{1, 2, ..., \dim \mathfrak{h}\}\} \tag{2.20}$$

generates a maximal abelian subalgebra. Finally, we note that $\hat{\mathfrak{h}}$ consists only of semisimple elements since ad_X is clearly diagonal and therefore diagonalizable for all $X \in \hat{\mathfrak{h}}$. Thus, we have a Cartan subalgebra given by

$$\hat{\mathfrak{h}} = \mathrm{span}\{H^i_0, K, d \mid i \in \{1, 2, ..., \dim \mathfrak{h}\}\}. \tag{2.21}$$

Definition 2.2.4. For elements of $\hat{\mathfrak{h}}^*$, we can define an inner product by

$$(\lambda, \mu) = (\overline{\lambda}, \overline{\mu}) + \lambda(K)\mu(d) + \lambda(d)\mu(K) \tag{2.22}$$

where $\overline{\lambda}$ and $\overline{\mu}$ are the restrictions of λ and μ to the horizontal subalgebra respectively.

Since K is a central element, we have $\alpha(K) = 0$ for any root α. As a consequence, for any roots α and β we have

$$(\alpha, \beta) = (\overline{\alpha}, \overline{\beta}). \tag{2.23}$$

From now on, we may write an affine root α in the form

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) \tag{2.24}$$

where the restriction of α to the horizontal subalgebra, $\mathbb{C}K$, and $\mathbb{C}d$ is α_1, α_2, and α_3 respectively.

Theorem 2.2.1. *[10, p. 108] Let δ be the linear functional satisfying*

$$\delta(H^i_0) = 0 \qquad \delta(K) = 0 \qquad \delta(d) = 1. \tag{2.25}$$

Let Φ be the root system for the horizontal subalgebra $\overline{\mathfrak{g}}$. The root system for an untwisted affine Lie algebra $\hat{\mathfrak{g}}$ constructed from a Lie algebra $\mathfrak{g}$ is

$$\hat{\Phi} = \{n\delta \mid n \in \mathbb{Z}\backslash\{0\}\} \;\cup\; \{\alpha + n\delta \mid \alpha \in \Phi, n \in \mathbb{Z}\}. \tag{2.26}$$

Here, we can take $\alpha \in \Phi$ to be a root of $\hat{\mathfrak{g}}$ by defining $\alpha(K) = \alpha(d) = 0$.

Proof. We begin by noting that $\operatorname*{Res}_{t=0}[nt^{n-1}] = 0$ for all n and so

$$[H_0^i, E_n^{\overline{\alpha}}] = [H^i, E^{\overline{\alpha}}] \otimes t^n = \overline{\alpha}^i(H^i)E_n^{\overline{\alpha}}. \tag{2.27}$$

Furthermore, it is clear that we have

$$\begin{aligned} [K, E_n^{\overline{\alpha}}] = [H_0^i, H_n^j] = [K, H_n^j] = 0 \\ [D, E_n^{\overline{\alpha}}] = nE_n^{\overline{\alpha}} \quad [D, H_n^j] = nH_n^j. \end{aligned} \tag{2.28}$$

We conclude that the affine roots are of the form

$$(\overline{\alpha}, 0, n) \quad \text{for } n \in \mathbb{Z}, \overline{\alpha} \in \Phi(\overline{\mathfrak{g}}) \tag{2.29}$$

$$\text{and } (0, 0, n) \quad \text{for nonzero } n \in \mathbb{Z}. \tag{2.30}$$

□

It can be shown that the Cartan matrix of this root system is an irreducible affine Cartan matrix so that the loop algebra construction does in fact give an affine Lie algebra [10, p. 113]. We will only show this for some simple examples. Note that this construction only gives some of the affine Lie algebras. These are called the untwisted affine Lie algebras. The rest of the affine Lie algebras are called twisted and can be constructed from the untwisted affine Lie algebras.[3]

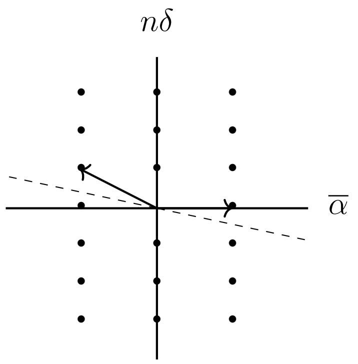
Example 2.2.1. As an example, we will plot the root system of $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ and compute the corresponding Cartan matrix. Let $\{E, F, H\}$ be the standard Cartan-Weyl basis for $\mathfrak{sl}_2(\mathbb{C})$. We know that the affine roots are of the form

$$(\pm\overline{\alpha}, 0, n) \text{ for } n \in \mathbb{Z} \tag{2.31}$$

and

$$(0, 0, n) \text{ for nonzero } n \in \mathbb{Z}. \tag{2.32}$$

Here, $\overline{\alpha}(H) = 2$. A plot of the root system is now given below with the dashed line indicating the choice of hyperplane. The arrows indicate the basis of positive simple roots.



[3]In particular, one imposes the condition on the loop algebra that $X \otimes f(e^{2\pi i}z) = \sigma(X) \otimes f(z)$ for some outer automorphism σ of $\mathfrak{g}$.

A choice of basis of positive simple roots is $\{\alpha_0 := (-\overline{\alpha}, 0, 1),\ \alpha_1 := (\overline{\alpha}, 0, 0)\}$. We then have $(\alpha_0, \alpha_0) = (\alpha_1, \alpha_1) = (\overline{\alpha}, \overline{\alpha}) = \kappa(H, H) = 2$. Furthermore, $(\alpha_0, \alpha_1) = (\alpha_1, \alpha_0) = -(\overline{\alpha}, \overline{\alpha}) = -2$. We deduce that the Cartan matrix for $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ is

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{2.33}$$

It is clear that this matrix does in fact satisfy the conditions required to be an irreducible affine Cartan matrix.

More generally, we can define the positive roots of an affine Lie algebra to be

$$\Phi_+ = \{(\overline{\alpha}, 0, n) \in \Phi \mid n > 0\} \cup \{(\overline{\alpha}, 0, 0) \in \Phi \mid \overline{\alpha} \in \overline{\Phi}_+\}. \tag{2.34}$$

The positive simple roots are then of the form

$$(\overline{\alpha}_i, 0, 0) \tag{2.35}$$

and

$$(-\overline{\theta}, 0, 1) \tag{2.36}$$

where $\overline{\alpha}_i$ are the positive simple roots of $\overline{\mathfrak{g}}$ and $\overline{\theta}$ is the highest root of $\overline{\mathfrak{g}}$ [10, p. 110]. Note that the generators of the corresponding root spaces satisfy the Chevalley-Serre relations. From now, we will write $\{E^i_\pm\} \cup \{H^i\}$ for a Chevalley-Serre basis.

Example 2.2.2. Given our characterisation of the positive simple roots, we can easily compute the Cartan matrix for $\widehat{\mathfrak{sl}_3(\mathbb{C})}$. Recall the positive simple roots of $\mathfrak{sl}_3(\mathbb{C})$ from Section 1.2. Our positive simple roots for $\widehat{\mathfrak{sl}_3(\mathbb{C})}$ are

$$\alpha_0 = (-\lambda_{13}, 0, 1) \tag{2.37}$$
$$\alpha_1 = (\lambda_{12}, 0, 0) \tag{2.38}$$
$$\alpha_2 = (\lambda_{23}, 0, 0). \tag{2.39}$$

We clearly have $(\alpha_0, \alpha_0) = 2$ as we expect. Noting that $(\lambda_{13}, \lambda_{12}) = (\lambda_{13}, \lambda_{23}) = 1$ and recalling the Cartan matrix for $\mathfrak{sl}_3(\mathbb{C})$, we see that the Cartan matrix for $\widehat{\mathfrak{sl}_3(\mathbb{C})}$ is

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \tag{2.40}$$

We can immediately see that this is an irreducible affine Cartan matrix.

Definition 2.2.5. For affine Lie algebras, the basis of weights dual to the simple coroots are called the fundamental weights.[4]

[4]Notice that some roots such as δ have zero norm and so they do not have coroots. However, simple roots always do.

Example 2.2.3. For $\widehat{\mathfrak{sl}_2(\mathbb{C})}$, our simple coroot basis is given by

$$\alpha_0^\vee = (-\overline{\alpha}, 0, 1) \qquad \alpha_1^\vee = (\overline{\alpha}, 0, 0). \tag{2.41}$$

To compute the fundamental weights, we have to solve

$$(\omega_i, \alpha_0^\vee) = (\overline{\omega_i}, \overline{\alpha}) = \delta_{i,1} \tag{2.42}$$
$$(\omega_i, \alpha_1^\vee) = -(\overline{\omega_i}, \overline{\alpha}) + \omega_i(K) = \delta_{i,2}. \tag{2.43}$$

By inspection, we see that we have

$$\omega_0 = (0, 1, 0) \tag{2.44}$$
$$\omega_1 = \left(\frac{\overline{\alpha}}{2}, 1, 0\right). \tag{2.45}$$

Note that we can expand any affine weight λ as [8, p. 565]

$$\lambda = \sum_i \lambda_i \omega_i + a\delta \tag{2.46}$$

for some $a \in \mathbb{R}$. Similar to the non-affine case, we call λ_i the Dynkin labels of the weight λ. We conclude this section with a theorem that will be useful in Section 2.5.

Theorem 2.2.2. *[10, p. 110] Given Chevalley-Serre generators $\{E^{\pm i}, H^i\}_{1\le i\le n}$ for a semisimple Lie algebra with simple roots $\{\alpha_i\}_{1\le i\le n}$, we obtain Chevalley-Serre generators of the untwisted affine Lie algebra $\hat{\mathfrak{g}}$*

$$H^0 := -H_0^\theta + K \qquad H^i := H_0^i$$
$$E_\pm^0 := E_{\pm 1}^{\mp\theta} \qquad E_\pm^i := E_0^{\pm i}.$$

We can therefore write a collection of affine Chevalley-Serre generators as $\{E_\pm^i, H^i\}$ where the index i starts at 0.

2.3 Highest Weight Modules

Recall Definition 1.4.3 where we defined the notion of a highest weight representation for any Lie algebra with a triangular decomposition. An affine Lie algebra $\hat{\mathfrak{g}}$ constructed from a Lie algebra $\mathfrak{g}$ has a canonical triangular decomposition, which allows us to talk about highest weight modules of affine Lie algebras. These are the modules that we are interested in when studying the WZW models.

Example 2.3.1. In Section 2.2, we constructed the root system for an affinised Lie algebra $\hat{\mathfrak{g}}$. We also gave a construction of a Cartan subalgebra $\hat{\mathfrak{h}}$. This gives us a triangular decomposition $\hat{\mathfrak{g}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{g}}_+$ where $\hat{\mathfrak{g}}_+$ and $\hat{\mathfrak{g}}_-$ are generated by the elements of the positive and negative affine root spaces respectively. From this, we see that a $\hat{\mathfrak{g}}$-module V is a highest weight module if there exists a highest weight vector $v_\lambda \in V$ with highest weight $\lambda : \hat{\mathfrak{h}} \to \mathbb{C}$ satisfying

1. $H_0^i v_\lambda = \lambda(H_0^i)v_\lambda \quad \text{for all } H^i \in \mathfrak{h}$.
2. $E_0^\alpha v_\lambda = 0 \quad \text{for all } \alpha \in \Phi_+$.
3. $E_n^{\pm\alpha} v_\lambda = H_n v_\lambda = 0 \quad \text{for all } \alpha \in \Phi_+, H \in \mathfrak{h}, n \in \mathbb{N}$.
4. $Kv_\lambda = kv_\lambda, \quad dv_\lambda = \gamma v_\lambda \quad \text{for some } k, \gamma \in \mathbb{C}$.
5. Any element of V is obtained by acting on v_λ with elements of $\hat{\mathfrak{g}}_-$.

Note that for $k \neq 0$, we can shift the eigenvalue of d to be any element of $\mathbb{C}$. This is because the bracket of d with any element is invariant under the transformation $d \mapsto d + \alpha K$ for $\alpha \in \mathbb{C}$ and d cannot be obtained as the Lie bracket of any two elements. We call k the level of the module.

Example 2.3.2. The Virasoro algebra has a triangular decomposition $\mathrm{Vir}_- \oplus \mathrm{Vir}_0 \oplus \mathrm{Vir}_+$ where we define

$$\mathrm{Vir}_- = \bigoplus_{n \leq -1} \mathbb{C}L_n \qquad \mathrm{Vir}_0 = \mathbb{C}L_0 \oplus \mathbb{C}c \qquad \mathrm{Vir}_+ = \bigoplus_{n \geq 1} \mathbb{C}L_n. \tag{2.47}$$

Let $U(\mathrm{Vir}_-)v_\lambda$ be the free module consisting of the vectors given by acting on v_λ with elements of $U(\mathrm{Vir}_-)$ where v_λ is a highest weight vector of highest weight λ. This is in fact a Verma module by Corollary 1.4.3. The Poincaré-Birkhoff-Witt theorem tells us that a basis is given by elements of the form

$$1 \cdot v_\lambda, \qquad L_{-n_1} \cdots L_{-n_k} v_\lambda \tag{2.48}$$

with $0 < n_1 \leq n_2 \cdots \leq n_k$. All of these elements are eigenvectors of L_0 and so the Verma module is graded by the corresponding eigenvalues (called conformal weights).[5] The Lie bracket of the Virasoro algebra allows us to show for example that $L_{-2}v_\lambda$ and $(L_{-1})^2 v_\lambda$ have the same conformal weight. We have

$$\begin{aligned} L_0 L_{-2} v_\lambda &= [L_0, L_{-2}]v_\lambda + L_{-2}L_0 v_\lambda && (2.49)\\ &= 2L_{-2}v_\lambda + \lambda L_{-2} v_\lambda && (2.50)\\ &= (\lambda + 2)L_{-2}v_\lambda. && (2.51) \end{aligned}$$

Similarly,

$$\begin{aligned} L_0 (L_{-1})^2 v_\lambda &= [L_0, L_{-1}]L_{-1}v_\lambda + L_{-1}[L_0, L_{-1}]v_\lambda + (L_{-1})^2 L_0 v_\lambda && (2.52)\\ &= L_{-1}^2 v_\lambda + L_{-1}^2 v_\lambda + \lambda L_{-1}^2 v_\lambda && (2.53)\\ &= (\lambda + 2)L_{-1}^2 v_\lambda. && (2.54) \end{aligned}$$

More generally, acting on some vector $L_{-n_1} \cdots L_{-n_k} v_\lambda$ with L_{-m} for $m > 0$ will increase its conformal weight by m. Below is a diagram of the module.[6]

[5] Here we see some motivation for the definition of a vertex algebra, which we discuss later in the thesis. The highest weight modules of certain other Lie algebras are graded in a similar way and correspond to the states of chiral CFTs.

[6] We say that the states with conformal weight $\lambda + i$ have level i.

$$\begin{array}{ccc}
 & \vdots & \\
\hdashline
L_{-2}v_\lambda & (L_{-1})^2 v_\lambda & \lambda+2 \\
\hdashline
 & L_{-1}v_\lambda & \lambda+1 \\
\hdashline
 & v_\lambda & \lambda
\end{array}$$

Because of this grading, a highest weight module of the Virasoro algebra is sometimes called a lowest weight module. Note that we have identified the map λ with the eigenvalue $\lambda(L_0)$. In Section 2.5, we show that we can constrain the central charge and conformal weight of the highest weight vector by imposing that these Verma modules be unitarizable.

2.4 Singular Vectors

In this section we introduce singular vectors. These are elements of a highest weight representation satisfying the following important property for the case of the Virasoro algebra and untwisted affine Lie algebras; if we quotient by the submodule generated by the collection of singular vectors then we obtain an irreducible module.

From now on, we are interested in the free modules defined by $U(\mathfrak{g}_-)v_\lambda$ for some highest weight vector v_λ with highest weight λ. By Corollary 1.4.3, these are all of the Verma modules up to isomorphism. These are the modules we have in mind when speaking of Verma modules.

Definition 2.4.1. Given a highest weight module V with a bilinear form $(\cdot,\cdot)$, a null vector is an element of V orthogonal to the entire module with respect to $(\cdot,\cdot)$.

Definition 2.4.2. Let V be a highest weight module with highest weight vector v_λ. A nonzero vector v that is not proportional to v_λ is called a singular vector if it satisfies the defining conditions of a highest weight vector.[7]

Before discussing the singular vectors of the Virasoro Verma modules, we would like to state an important result. For this, we must consider the Gram matrix M_{ij}, which we define to be the matrix consisting of all inner products of the basis elements for a given Verma module $U(\mathrm{Vir}_-)v_\lambda$ of the Virasoro algebra. Since L_0 is a Hermitian operator, eigenspaces of L_0 with distinct eigenvectors are orthogonal. This implies that M_{ij} is block diagonal with each block matrix containing all inner products of the basis elements for all vectors in the Verma module of a given conformal weight. We denote by $M^{(i)}$ the block diagonal matrix containing all inner products of the basis elements for the vector space of elements at level i. We have the following important result.

[7] Any singular vector is a null vector with respect to the Shapovalov form. This follows directly from the PBW theorem and the definition of the Shapovalov form.

Theorem 2.4.1. ***(Kac determinant formula)*** *[19] We have*

$$det(M^{(i)}) = a_i \prod_{pq \leq i} (\lambda - \lambda_{p,q}(c))^{P(i-pq)}. \tag{2.55}$$

Here, $a_i > 0$ is independent of λ and c, $P(N)$ denotes the number of partitions of some integer N, and we can write

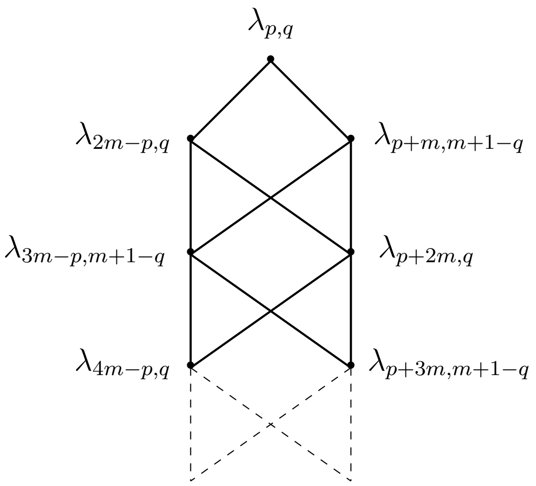
$$\lambda_{p,q}(m) = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)} \tag{2.56}$$

with

$$m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}}. \tag{2.57}$$

Note that the determinant is independent of the choice of branch for m. The Kac determinant formula can be used to prove that the Virasoro highest weight modules for $c \geq 1$ and $\lambda \geq 0$ are irreducible. It can also be shown that the Virasoro highest weight modules for $0 < c < 1$ are not unitarizable except possibly for certain values of c and λ in some discrete set. For proofs of both claims, see [8, pp. 209-211].

Example 2.4.1. We can compute the conformal weights of the singular vectors of the Verma module $U(\mathrm{Vir}_-)v_\lambda$ where $\lambda = \lambda_{p,q}$ for some positive integers p and q.[8] For this, we can use the Kac determinant formula (see Theorem 2.4.1). From now on, we will denote by pq the minimal product over the allowed p and q. The Kac determinant formula tells us that the determinant first vanishes at level $i = pq$. This implies that the first singular vector occurs at this level and therefore has conformal weight $\lambda_{pq} + pq = \lambda_{p+m,m+1-q}$. The invariance of the determinant formula under $p \mapsto m-p$, $q \mapsto m+1-q$ implies that there is another singular vector at level $(m-p)(m+1-q)$ of conformal weight $\lambda_{p,q} + (m-p)(m+1-q) = \lambda_{2m-p,q}$. We can repeat this argument for each of these two singular vectors to obtain the conformal weights of the singular vectors of the submodules generated by these singular vectors. The singular vector structure of our original module is shown below.



[8]For a fixed λ, the integers p and q are not unique.

Example 2.4.2. [10, pp. 126-127] Given affine Chevalley-Serre generators $\{E^i_\pm, H^i\}_{0\leq i\leq r}$ and a Verma module of an untwisted affine Lie algebra $\hat{\mathfrak{g}}$ with highest weight vector v_λ and dominant integral highest weight λ, the vectors

$$(E^i_-)^{\lambda_i+1}v_\lambda \qquad i\in\{0,\dots,r\} \tag{2.58}$$

are singular vectors, where λ_i are the Dynkin labels of λ. Recall in Section 1.5, we showed that

$$EF^n w = n(\lambda+2N-(n-1))F^{n-1}w \tag{2.59}$$

for all $n\in\mathbb{N}$, where w is a highest weight vector of $\mathfrak{sl}_2(\mathbb{C})$ of highest weight $\lambda+2N$. Since $\lambda_i=\lambda(H^i)$, we deduce

$$E^i_+(E^i_-)^n v_\lambda = n(\lambda_i-(n-1))(E^i_-)^{n-1}v_\lambda. \tag{2.60}$$

Since λ is dominant integral, we have $\lambda_i\in\mathbb{Z}_{\geq 0}$. For $n=\lambda_i+1$, we therefore have

$$E^i_+(E^i_-)^{\lambda_i+1}v_\lambda = 0. \tag{2.61}$$

Since $[E^j_+, E^i_-]=0$ for $j\neq i$, we conclude that

$$E^j_+(E^i_-)^{\lambda_i+1}v_\lambda = 0 \tag{2.62}$$

for $j\in\{0,\dots,r\}$. Since the elements E^j_+ generate the positive subalgebra $\hat{\mathfrak{g}}_+$, we see that $(E^i_-)^{\lambda_i+1}v_\lambda$ are singular vectors. It turns out that these singular vectors generate all of the singular vectors.

2.5 Unitarizable and Integrable Modules

In this section we introduce integrable modules. These are the modules that we are interested in when studying the Wess-Zumino-Witten models. More specifically, they give the allowed states of these conformal field theories. We also discuss unitarizable modules and their implications for the case of the Virasoro algebra.

Definition 2.5.1. A complex representation V of a group G is called unitary if it admits a non-degenerate Hermitian inner product such that

$$(Uv, Uw) = (v, w) \tag{2.63}$$

for all $v,w\in V$ and $U\in G$.

For physical reasons, we want our irreducible highest weight modules in conformal field theory to also define unitary representations of a Lie group.[9] Using

[9]The situation is much more subtle than this. For example, there is no known Lie group whose corresponding Lie algebra is the Virasoro algebra. There are also purely mathematical reasons (from the perspective of vertex algebras) to be interested in the irreducible, integrable, and therefore unitarizable highest weight modules in the study of the Wess-Zumino-Witten models. Furthermore, there are interesting conformal field theories (such as those describing the Yang-Lee edge singularity) that are not unitary.

this requirement and the exponential map, we can see how this restricts our possible choice of adjoint on the Lie algebra. First note that Equation (2.63) implies that U is a unitary operator. In particular, $U^{-1} = U^\dagger$. Writing U in terms of the exponential map gives

$$U = \exp\left(\sum_a \alpha_a J^a\right) \qquad U^\dagger = U^{-1} = \exp\left(-\sum_a \alpha_a J^a\right) \tag{2.64}$$

where $\{J^a\}$ is a basis for the Lie algebra. For a real Lie algebra, we can also write $U^\dagger = \exp\left(\sum_a \alpha_a (J^a)^\dagger\right)$ and so we can conclude that $(J^a)^\dagger = -J^a$. For complex Lie algebras, one can define the adjoint on the Chevalley-Serre generators by $(E^i_\pm)^\dagger = E^i_\mp$, $(H^i)^\dagger = H^i$ and extend by anti-linearity [11].[10] We call this the Chevalley involution and it corresponds to the compact real form of the complexification of the Lie group.

Given our choice of adjoint, there is a canonical choice of inner product on a given Verma module module called the Shapovalov form [22]. In practice, this inner product is defined on two states of the same level[11] by applying the adjoint to obtain an expression of the form (v_λ, Xv_λ) for some X that is a product of modes. For example, with the Virasoro algebra X might look like $L_1L_1L_1L_{-2}L_{-1}$. It is a general fact that we will not prove, that we can always commute the positive modes through the negative modes to obtain a linear combination of L_0 and c. Fixing the central charge and setting $(v_\lambda, v_\lambda) = 1$ then gives the value of the inner product. Similarly, one can define the Shapovalov form for other Lie algebras with a triangular decomposition. Our modules in conformal field theory should be constrained so that the Shapovalov form defines an inner product.

Definition 2.5.2. [10, p. 55] Let $\mathfrak{g}$ be a Lie algebra with Chevalley-Serre generators $\{E^i_\pm\} \cup \{H^i\}$. A $\mathfrak{g}$-module V is called unitarizable if it admits a Hermitian inner product such that the Chevalley involution defines an adjoint.

We will shortly give a characterisation of the unitarizable highest weight modules of any Lie algebra with Chevalley-Serre generators.

Example 2.5.1. Ginsparg [14, p. 37] shows how imposing the requirement that our highest weight modules be unitarizable and irreducible for the Virasoro algebra constrains the central charge and conformal weight of the highest weight vector. Consider a highest weight representation of the Virasoro algebra with highest weight vector v_λ. For elements of the Virasoro algebra, we define an adjoint by[12]

$$L_n^\dagger = L_{-n}, \qquad c^\dagger = c. \tag{2.65}$$

For $n > 0$, we now compute

$$(L_{-n}v_\lambda, L_{-n}v_\lambda) = (v_\lambda, L_nL_{-n}v_\lambda) \tag{2.66}$$

[10] This is not the only possibility.

[11] Recall that states of different levels are orthogonal.

[12] This choice of adjoint is justified by a physical argument in [14, p. 34].

$$= (v_\lambda, [L_n, L_{-n}]v_\lambda) \tag{2.67}$$

$$= 2n(v_\lambda, L_0 v_\lambda) + \frac{c}{12}(n^3 - n)(v_\lambda, v_\lambda) \tag{2.68}$$

$$= \left(2n\lambda + \frac{c}{12}(n^3 - n)\right)(v_\lambda, v_\lambda). \tag{2.69}$$

If we consider sufficiently large n then we see that $c \geq 0$ for our Hermitian inner product to be positive-definite. If we consider $n = 1$ then we also see that $\lambda \geq 0$.

We can also show that a unitarizable highest weight module of the Virasoro algebra with central charge 0 is trivial. To do this, consider the collection of states spanned by $L_{-2n}v_\lambda$ and $L_{-n}^2 v_\lambda$. We then consider the Hermitian matrix

$$M = \begin{pmatrix} (L_{-2n}v_\lambda, L_{-2n}v_\lambda) & (L_{-2n}v_\lambda, L_{-n}^2 v_\lambda) \\ (L_{-n}^2 v_\lambda, L_{-2n}v_\lambda) & (L_{-n}^2 v_\lambda, L_{-n}^2 v_\lambda) \end{pmatrix}. \tag{2.70}$$

Since M is Hermitian, it can be diagonalised

$$M = UDU^\dagger \tag{2.71}$$

with U unitary. Given a state v that is a linear combination of $L_{-2n}v_\lambda$ and $L_{-n}^2 v_\lambda$, we can define $w = U^\dagger v$ and so

$$(v, v) = \sum_i D_{ii}|w_i|^2 \tag{2.72}$$

where w_i is the ith component of the vector w with respect to the basis $\{L_{-2n}v_\lambda, L_{-n}^2 v_\lambda\}$ for the vector space spanned by $L_{-2n}v_\lambda$ and $L_{-n}^2 v_\lambda$. We now see that the vector space spanned by $L_{-2n}v_\lambda$ and $L_{-n}^2 v_\lambda$ contains a state of negative norm if one of the eigenvalues of D is negative. To show that one of the eigenvalues of D is negative, it suffices to show that $\det(M) < 0$. For $c = 0$, we have

$$\det(M) = 4n^3\lambda^2(8\lambda - 5n). \tag{2.73}$$

Thus, for $\lambda \neq 0$ and $c = 0$ there are negative norm states in the highest weight module.[13] But if $\lambda = c = 0$ then $(L_{-n}v_0, L_{-n}v_0) = 0$, implying $L_{-n}v_0 = 0$ for all n. We conclude that the only nonzero elements of the highest weight module are multiples of v_0 and every element of Vir acts trivially.

Let us now turn our attention to Lie algebras admitting Chevalley-Serre generators.

Definition 2.5.3. [10, p. 119] A highest weight module is integrable if its highest weight is dominant integral.

Theorem 2.5.1. *[10, p. 128] An irreducible integrable highest weight module of a Lie algebra with Chevalley-Serre generators* $\{E_\pm^i\} \cup \{H^i\}$ *is unitarizable.*

[13]For a given λ, consider n sufficiently large so that $\det(M) < 0$.

Proof. Given an irreducible, integrable module we use the Shapovalov form for the inner product. To show that this defines a unitarizable module note that irreducibility implies that there are no nonzero vectors with zero norm. If there was a nonzero vector with zero norm then the submodule generated by that vector would only contain vectors with zero norm. Irreducibility would then imply that every element of the module has zero norm, which is a contradiction. Next, positive definiteness follows from

$$(E^i_- v_\lambda, E^i_- v_\lambda) = (v_\lambda, E^i_+ E^i_- v_\lambda) \tag{2.74}$$
$$= (v_\lambda, [E^i_+, E^i_-] v_\lambda) \tag{2.75}$$
$$= (v_\lambda, H^i v_\lambda) \tag{2.76}$$
$$= \lambda(H^i)(v_\lambda, v_\lambda) \tag{2.77}$$

and noting that λ being dominant integral implies that $\lambda(H^i) \in \mathbb{Z}_{\geq 0}$. □

The converse of this theorem is also true but the proof is quite difficult and so it has been omitted.

Example 2.5.2. We will now give an example of an irreducible integrable module that we will be interested in when studying the Wess-Zumino-Witten models. Consider the Lie algebra $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ with the fundamental weights ω_0 and ω_1 given by Example 2.2.3. Define

$$\lambda = 0 \cdot \omega_0 + \omega_1 - \frac{1}{4}\delta. \tag{2.78}$$

Consider the Verma module $U(\widehat{\mathfrak{sl}_2(\mathbb{C})}_-)v_\lambda$ where v_λ is a highest weight vector of highest weight λ. The weight λ has Dynkin labels $\lambda_0 = 0$ and $\lambda_1 = 1$. By Example 2.4.2, the singular vectors of our Verma module are generated by

$$\left\{ F_0^2 v_\lambda, E_{-1} v_\lambda \right\}. \tag{2.79}$$

If we quotient by the submodules generated by these vectors, we obtain an irreducible module. Note also that the Dynkin labels of the highest weight are nonnegative integers and so the module we obtain is both irreducible and integrable.

Chapter 3

Wess-Zumino-Witten Models I

The Wess-Zumino-Witten models are a class of two-dimensional conformal field theories whose symmetry algebras are affine Lie algebras. The reader who is unfamiliar with conformal field theory would find it easier to first learn about the free boson and the free fermion. We have decided to introduce the Wess-Zumino-Witten models as our first example of a conformal field theory since they have a much richer algebraic structure.

In this chapter, we emphasise the more formal aspects of conformal field theory: namely that states correspond to elements of certain representations, that fields can be viewed as formal power series, and that there is a correspondence between states and fields. The notion of an operator product expansion helps us to connect the algebraic aspects of conformal field theory (specifically the Lie bracket of the Lie algebra of interest) and the analytic interpretation of a field. In the last section, we introduce conformal invariance in two dimensions, which allows one to study conformal field theory without much mention of Lie algebras or representations.

3.1 The Wess-Zumino-Witten Action

The aim of this section is to motivate the choice of action for the Wess-Zumino-Witten models. In particular, we outline how one can obtain a constraint on a constant called the level of the Wess-Zumino-Witten model. We avoid some of the technical details as they are quite cumbersome. The interested reader may consult the relevant sources cited in this section.

Conformal field theories can be constructed by specifying a functional called the action.[1] This functional is defined on some space of functions. If we introduce some infinitesimal variation of such a function, we obtain a variation of the action. The functional can be extremised by solving for the functions for which this variation is 0. This is known as the principle of least action, and its mathematical formalism is known as the calculus of variations. The corresponding functions which extremise

[1]Although most conformal field theories are not constructed in this manner. This construction is common for more general quantum field theories.

the action under infinitesimal variations are called the fields of the conformal field theory.

A precise geometric description of this process is a bit subtle and so we will avoid discussing it.[2] Instead, we outline how one obtains the equations of motion for the fields of the Wess-Zumino-Witten models. Historically, the Wess-Zumino-Witten models were obtained from a quantum field theory called a non-linear sigma model. This model is an example of a string theory. Such a theory requires the existence of a smooth map

$$g : \Sigma \to M \tag{3.1}$$

where Σ is a Riemann surface (called the worldsheet) and M is some smooth manifold. For us, we choose Σ to be the Riemann sphere.[3] The defining feature of this model (and the Wess-Zumino-Witten models) is that the target manifold M is a (compact, connected, simply-connected, simple) Lie group G with corresponding Lie algebra $\mathfrak{g}$.

We are now interested in the Lie-algebra valued differential form $g^{-1}dg$ defined as follows. Given $p \in \Sigma$, consider $dg_p : T_p\Sigma \to T_{g(p)}G$. Then $g(p)^{-1}$ acts on elements of $T_{g(p)}G$ in the obvious way. That is, the one-form $g^{-1}dg$ maps a point $p \in \Sigma$ to the alternating multilinear map $d(f_{g(p)^{-1}})_{g(p)} \circ dg_p : T_p\Sigma \to \mathfrak{g}$ where $f_{g(p)^{-1}}$ is the smooth map defined by Equation (C.11). In fact, this is the pullback of the Maurer-Cartan form (see Definition C.0.2) under g. Now, note that we can extend the Killing form $\kappa(\cdot,\cdot)$ on the Lie algebra to a Lie algebra-valued differential form by

$$\kappa(X_i \otimes \omega_i \overset{\wedge}{,} Y_j \otimes \eta_j) = \kappa(X_i, Y_j) \otimes (\omega_i \wedge \omega_j). \tag{3.2}$$

See Appendix B for a discussion of Lie algebra-valued differential forms. The extended Killing form allows us to define the action of the non-linear sigma model. The action is

$$S_0(f) = a \int_\Sigma \kappa(g^{-1}dg \overset{\wedge}{,} \star (g^{-1}dg)) \tag{3.3}$$

where $\star$ denotes the Hodge dual. Extremising S_0 under an infinitesimal variation in f, one obtains the equations of motion [33, p. 107]

$$\partial_z(g^{-1}\partial_{\bar{z}}g) + \partial_{\bar{z}}(g^{-1}\partial_z g) = 0 \tag{3.4}$$

where we are working in local coordinates. From this equation, we run into an issue; the nonlinear sigma model does not factorise into holomorphic and antiholomorphic sectors. This is in contrast to the free boson (see Appendix E) where we obtain both a holomorphic and an antiholomorphic field from the equations of motion. To obtain these holomorphic and antiholomorphic fields, we can modify the action of the non-linear sigma model by adding an extra term called the Wess-Zumino term. The addition of this term also ensures that the theory is conformally invariant. Let

[2] For the case of the Wess-Zumino-Witten action, this is discussed in [33, pp. 104-109].

[3] One can generalise the theory to other Riemann surfaces.

B be a compact, oriented smooth 3-manifold whose boundary is Σ. The action of the Wess-Zumino-Witten models is [33, p. 109]

$$S_\Sigma(g) = a \int_\Sigma \kappa(g^{-1}dg \overset{\wedge}{,} \star (g^{-1}dg)) + \frac{2ia}{3} \int_B \kappa(\widetilde{g}^{-1}d\widetilde{g} \overset{\wedge}{,} d(\widetilde{g}^{-1}d\widetilde{g})) \tag{3.5}$$

where $\widetilde{g} : B \to G$ is some extension of g and we identity the second term as the Wess-Zumino term. We will shortly address the ambiguity in this definition. Note first that we can write

$$\kappa(\widetilde{g}^{-1}d\widetilde{g} \overset{\wedge}{,} d(\widetilde{g}^{-1}d\widetilde{g})) = \widetilde{g}^*\kappa(\omega \overset{\wedge}{,} d\omega) \tag{3.6}$$

where ω is the Maurer-Cartan form. We have the following important result about the differential form $\kappa(\omega \overset{\wedge}{,} d\omega)$.

Lemma 3.1.1. *[33, pp. 111-112] Let G be a compact, connected, simply-connected, simple Lie group. If we fix $a = k/8\pi$ for some positive integer k then*

$$\frac{2ia}{3} \int_G \kappa(\omega \overset{\wedge}{,} d\omega) = 2\pi ik. \tag{3.7}$$

Now, we would like the Wess-Zumino-Witten action to be independent of the choice of manifold B and extension $\widetilde{g}$. It turns out that this is not quite true. However, we have the following theorem.

Theorem 3.1.2. *[25, p. 30] Given a compact Riemann surface Σ and a smooth map $g : \Sigma \to G$, the expression $\exp S_\Sigma(g)$ is well-defined.*

Proof. Let $\widetilde{g}$ and $\widetilde{g}'$ be extensions of g to the compact, oriented smooth 3-manifolds B and B' respectively. Let M be the manifold obtained by gluing $-B$ and B' along the boundary Σ where $-B$ denotes the manifold B with orientation reversed. Let G be the map on M that restricts to $\widetilde{g}$ and $\widetilde{g}'$ on B and B' respectively. Then

$$\begin{aligned} S_\Sigma(g') - S_\Sigma(g) &= \frac{2ia}{3} \int_{B'} \kappa(\widetilde{g}'^{-1}d\widetilde{g}' \overset{\wedge}{,} d(\widetilde{g}'^{-1}d\widetilde{g}')) - \frac{2ia}{3} \int_B \kappa(\widetilde{g}^{-1}d\widetilde{g} \overset{\wedge}{,} d(\widetilde{g}^{-1}d\widetilde{g})) && (3.8) \\ &= \frac{2ia}{3} \int_{B'} \kappa(\widetilde{g}'^{-1}d\widetilde{g}' \overset{\wedge}{,} d(\widetilde{g}'^{-1}d\widetilde{g}')) + \frac{2ia}{3} \int_{-B} \kappa(\widetilde{g}^{-1}d\widetilde{g} \overset{\wedge}{,} d(\widetilde{g}^{-1}d\widetilde{g})) && (3.9) \\ &= \frac{2ia}{3} \int_M \kappa(G^{-1}dG \overset{\wedge}{,} d(G^{-1}dG)) && (3.10) \\ &= \frac{2ia}{3} \int_M G^*\kappa(\omega \overset{\wedge}{,} d\omega). && (3.11) \end{aligned}$$

By Lemma 3.1.1, we therefore have

$$S_\Sigma(g') - S_\Sigma(g) = 2\pi ik \tag{3.12}$$

and so $\exp S_\Sigma(g)$ is well-defined. $\square$

The positive integer k is called the level of the theory. Since $\exp S_\Sigma(g)$ is well-defined, the theory obtained from the choice of action for the Wess-Zumino-Witten models is not ambiguous. The equations of motion of the Wess-Zumino-Witten models can be obtained by extremising this action. In local coordinates, one obtains [33, pp. 108-109]

$$\partial_{\overline{z}}(g^{-1}\partial_z g)dz \wedge d\overline{z} = 0, \tag{3.13}$$

which gives rise to the fields

$$J(z) = kg^{-1}\partial_z g \tag{3.14}$$

$$\overline{J}(\overline{z}) = -k\partial_{\overline{z}} g g^{-1} \tag{3.15}$$

that are holomorphic and antiholomorphic respectively.

For the rest of this chapter and the next, we choose Σ to be the Riemann sphere $\mathbb{C}P^1$, which we identify as the one-point compactification of $\mathbb{C}$. This gives us the simplest choice of conformal structure, which will later significantly simplify our calculations of correlation functions.

3.2 States and Fields

In the previous section, we considered certain functions called fields, which extremise the action of a quantum field theory under infinitesimal variations. The notion of a field can be formalised, and this formal definition is used extensively in the theory of vertex algebras. Let V be a complex vector space. A field is a formal power series of the form

$$A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-h_A}. \tag{3.16}$$

Here, $A_n \in \operatorname{End} V$ and h_A is a constant called the conformal weight of $A(z)$. We require that for all $v \in V$, there is some $N \in \mathbb{N}$ such that $A_n v = 0$ for all $n > N$. The elements of V are called states. We assume that there is a one-to-one correspondence between states and fields. We call this the state-field correspondence and in one direction the map takes the explicit form

$$A(z) \mapsto \lim_{z\to 0} A(z) \left|0\right\rangle. \tag{3.17}$$

In general, it is difficult to find an explicit expression for the inverse of this map.[4] The state corresponding to the field $A(z)$ is denoted $\left|A\right\rangle$.

Example 3.2.1. Recall that for the Wess-Zumino-Witten models, we have the Lie algebra-valued fields (in local coordinates)

$$J(z) = kg^{-1}\partial_z g \tag{3.18}$$

[4]In the language of vertex algebras, such an inverse is called a vertex operator and we write down the vertex operators for some simple examples in our discussion of vertex algebras in Chapter 5.

$$\overline{J}(\overline{z}) = -k\partial_{\overline{z}} g g^{-1} \tag{3.19}$$

Given a basis $\{J^a\}$ for the Lie algebra $\mathfrak{g}$, we can write

$$J(z) = \sum_a J^a(z) J^a \tag{3.20}$$

for some scalar fields $J^a(z)$. It is important to note that the fields $J^a(z)$ are only $\mathbb{C}$-valued in the classical theory. In the classical theory, we obtain a Laurent series

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}. \tag{3.21}$$

To obtain a quantum field theory, we promote the elements J^a_n to operators on some vector space. We will see in Section 3.3 that the modes J^a_n are elements of an untwisted affine Lie algebra $\hat{\mathfrak{g}}$. In the quantum field theory, we view $J^a(z)$ as a formal power series. We obtain an analytic interpretation of $J^a(z)$ if we are given a highest weight $\hat{\mathfrak{g}}$-module. Given some state $|\phi\rangle$ in such a module, we have a new Laurent series defined by

$$\langle\phi| J^a(z) |\phi\rangle = \sum_{n\in\mathbb{Z}} \langle\phi| J^a_n |\phi\rangle z^{-n-1}. \tag{3.22}$$

Here, we have used the notation

$$\langle\phi| J^a_n |\phi\rangle = (|\phi\rangle, J^a_n |\phi\rangle) \tag{3.23}$$

where $(\cdot,\cdot)$ denotes the Shapovalov form defined on our highest weight module. When performing computations, we can work with Laurent series of this form so that we can rigorously apply the methods of complex analysis. Note that this is a Laurent series by the definition of a field.

Recall that we defined the concept of a highest weight module for Lie algebras with a triangular decomposition. In the WZW models, we assume that an arbitrary quantum state is an element of some irreducible and integrable highest weight module.[5] Given an untwisted affine Lie algebra, let $\{H^i\} \cup \{E^{\pm\alpha}\}$ be a Cartan-Weyl basis for the horizontal subalgebra. A highest weight vector $|0\rangle$ in a highest weight module with weight 0 is called a vacuum vector if it satisfies

$$E_0^{-\alpha} |0\rangle = 0 \quad \text{for all } \alpha \in \Phi_+. \tag{3.24}$$

Note that a given highest weight vector with weight 0 does not have to satisfy this. If this assumption were not made, the state-field correspondence would fail. For example, consider the $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ level k WZW model and the field $F(z) = \sum_{n\in\mathbb{Z}} F_n z^{-n-1}$. Then $\lim_{z\to 0} F(z)|0\rangle$ is not well-defined unless $F_0|0\rangle = 0$. A fundamental assumption of conformal field theory is the existence of a vacuum vector.

[5] In other conformal field theories such as the free boson, free fermion, and the Virasoro minimal models, the states are also elements of highest weight modules.

We conclude this section by discussing the spectrum of the WZW models. A WZW model is characterised by a finite-dimensional Lie algebra $\mathfrak{g}$ and a level k. The spectrum is the collection of states in a given WZW model. Let $\hat{\mathfrak{g}}$ be the affine Lie algebra corresponding to $\mathfrak{g}$. We can model the collection of states in the $\hat{\mathfrak{g}}$ WZW model at level k by the space[6]

$$\bigoplus_{\lambda \in X_{\hat{\mathfrak{g}}_k}} (V_\lambda \otimes V_\lambda) \tag{3.25}$$

where V_λ is the irreducible highest weight $\hat{\mathfrak{g}}$-module with weight λ. Here, $X_{\hat{\mathfrak{g}}_k}$ is the collection of dominant integral weights of $\hat{\mathfrak{g}}$ at level k.[7] Note that the tensor product accounts for both the holomorphic and antiholomorphic sectors, which arise from the equations of motion (3.13). One obtains similar equations of motion for the free boson, which also give rise to holomorphic and antiholomorphic sectors. This is briefly discussed in Appendix E. Essentially, these sectors allow us to study holomorphic fields and antiholomorphic fields independently for certain conformal field theories. We can then model the collection of all states of a particular irreducible integrable module (which can correspond to both holomorphic and antiholomorphic fields) as a tensor product of the collection of states corresponding to the holomorphic fields and the collection of states corresponding to the antiholomorphic fields.

3.3 Operator Product Expansions

We define the radially-ordered product of two fields $\phi(z)$ and $\psi(w)$ as

$$\mathcal{R}\left\{\phi(z)\psi(w)\right\} = \begin{cases} \phi(z)\psi(w) & |w| < |z| \\ \psi(w)\phi(z) & |z| < |w| \end{cases} \tag{3.26}$$

where we leave the radially-ordered product undefined for the case $|z| = |w|$. Unless stated otherwise, products of fields are assumed to be radially-ordered. Denote the collection of fields in a conformal field theory by $\{V_\alpha\}$. An operator product expansion is a sum of the form [31, p. 13]

$$V_\alpha(z)V_\beta(w) = \sum_\gamma f_\gamma(z,w)V_\gamma(w), \tag{3.27}$$

where the f_γ are meromorphic functions with poles only at $z = w$. We assume that the radially-ordered product of any two fields in a conformal field theory has an

[6] In fact, one can show that the commutator $[J_n^a, \overline{J}_m^b]$ is zero for all modes J_n^a in the holomorphic sector and all modes $\overline{J}_m^b$ in the antiholomorphic sector. Furthermore, our states should not be linear combinations of both holomorphic and antiholomorphic states. For this reason, we can choose decomposition to be in terms of $V_\lambda \otimes V_\lambda$ but not $V_\lambda \oplus V_\lambda$.

[7] It turns out that there are only a finite number of allowed modules for any given level. For this reason, we say that the Wess-Zumino-Witten models of any given level are rational conformal field theories.

OPE. We use the notation

$$V_\alpha(z)V_\beta(w) \sim \sum_\gamma f_\gamma(z,w)V_\gamma(w) \tag{3.28}$$

to indicate the γ for which there is a pole. Note that using Laurent's theorem, we can write

$$V_\alpha(z)V_\beta(w) = \sum_{n\in\mathbb{Z}} \psi_n(w)(z-w)^{-n-2} = \sum_{n\in\mathbb{Z}} \psi'_n(z)(w-z)^{-n-2} \tag{3.29}$$

for some fields $\psi_n(w)$ and $\psi'_n(z)$.

Example 3.3.1. In the case of the WZW models, the OPE of $J^a(z)J^b(w)$ allows us to compute the commutator $[J^a_n, J^b_m]$. This is

$$J^a(z)J^b(w) \sim \frac{k\kappa(J^a, J^b)}{(z-w)^2} + \sum_c f_{abc}\frac{J^c(w)}{z-w} \tag{3.30}$$

where $f_{abc} \in \mathbb{C}$ are the structure constants with respect to the basis $\{J^a\}$ [8, p. 623]. In this context, k is a non-negative integer multiple of the identity field. Furthermore, $\kappa = \frac{1}{2h^\vee}\kappa_{ad}$ where κ_{ad} is the Killing form on $\mathfrak{g}$ and $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. In Appendix D, we discuss how one computes the OPE $J^a(z)J^b(w)$.

We will now compute $[J^a_n, J^b_m]$. Let $|\phi\rangle$ be a state. The first step is to write

$$\langle\phi| J^a_n J^b_m |\phi\rangle = \frac{1}{(2\pi i)^2}\oint_0\oint_0 \langle\phi| J^a(z)J^b(w)|\phi\rangle z^n w^m dzdw. \tag{3.31}$$

Now,

$$\langle\phi| [J^a_n, J^b_m] |\phi\rangle = \frac{1}{(2\pi i)^2}\oint_0\oint_0 \left[\langle\phi| J^a(z)J^b(w)|\phi\rangle - \langle\phi| J^b(w)J^a(z)|\phi\rangle\right] w^m z^n dzdw \tag{3.32}$$

$$= \frac{1}{(2\pi i)^2}\left[\underset{|z|>|w|}{\oint_0\oint_0} - \underset{|z|<|w|}{\oint_0\oint_0}\right]\langle\phi| \mathcal{R}\left\{J^a(z)J^b(w)\right\}|\phi\rangle w^m z^n dzdw \tag{3.33}$$

$$= \frac{1}{(2\pi i)^2}\oint_0\oint_w \langle\phi| \mathcal{R}\left\{J^a(z)J^b(w)\right\}|\phi\rangle w^m z^n dzdw \tag{3.34}$$

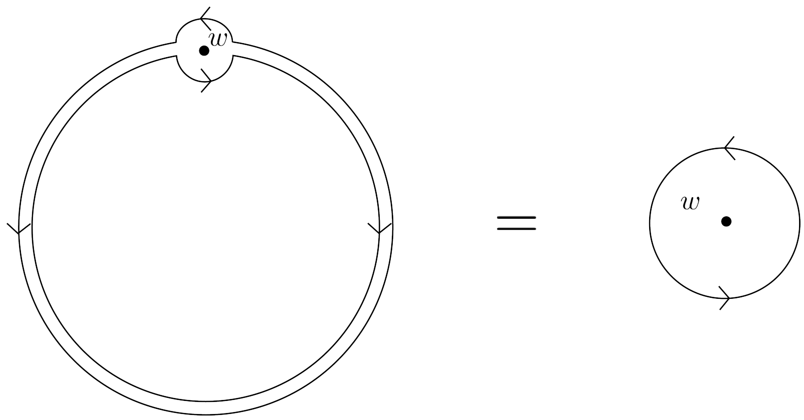
$$= \frac{1}{2\pi i}\oint_0 \left[\sum_c f_{abc}\langle\phi| J^c(w)|\phi\rangle w^{m+n} + nk\kappa(J^a, J^b)w^{m+n-1}\langle\phi|\phi\rangle\right] dw \tag{3.35}$$

$$= \sum_c f_{abc}\langle\phi| J^c_{n+m}|\phi\rangle + nk\kappa(J^a, J^b)\delta_{m+n,0}\langle\phi|\phi\rangle\,. \tag{3.36}$$

Note that we replaced the radially-ordered product with the singular terms of the OPE since the regular terms do not contribute. Since we can repeat the same argument for any state $|\phi\rangle$, we have

$$[J_n^a, J_m^b] = \sum_c f_{abc} J_{n+m}^c + nk\kappa(J^a, J^b)\delta_{m+n,0}. \tag{3.37}$$

In Equation (3.34), we manipulated the contours as shown in the diagram below.



This is a standard trick one commonly uses in conformal field theory to evaluate contour integrals. From Equation (3.37), we can now see that our OPE above is consistent with $\hat{\mathfrak{g}}$ being an untwisted affine Lie algebra.[8] Here, f_{abc} are the structure constants of $\mathfrak{g}$ and the element K is our central element. We call k the level of the WZW model where k is the eigenvalue of K. Note that we will see later how to construct the derivation of $\hat{\mathfrak{g}}$.

3.4 Normal Ordering

We now introduce a new type of product, which allows us to multiply fields that are functions of the same coordinate. This is known as the normally-ordered product and it allows us to construct new fields. One field of particular importance is the energy-momentum tensor, which can be written as a normally ordered product in the Wess-Zumino-Witten models. Let $A(z)$ and $B(w)$ be fields and assume the OPE has the form

$$A(z)B(w) = \sum_{i=-\infty}^{N} \frac{\{AB\}_i(w)}{(z-w)^i}. \tag{3.38}$$

Then we define the normally-ordered product as [8, p. 174]

$$: A(w)B(w) := \{AB\}_0(w). \tag{3.39}$$

[8]More specifically, the modes J_n^a are elements of $U(\hat{\mathfrak{g}})$ and so we can identify the commutator we computed with the Lie bracket defined by our construction of the untwisted affine Lie algebras.

We would like to obtain an expression for the normally-ordered product $: A(w)B(w) :$ as a field in the variable w. This can be done, and we will present the argument given in [8, pp. 174-176]. First, write

$$: A(w)B(w) : = \frac{1}{2\pi i} \oint_w \frac{A(z)B(w)}{z-w} dz \tag{3.40}$$

$$= \frac{1}{2\pi i} \oint_{\substack{0 \\ |z|>|w|}} \frac{A(z)B(w)}{z-w} dz - \frac{1}{2\pi i} \oint_{\substack{0 \\ |z|<|w|}} \frac{B(w)A(z)}{z-w} dz. \tag{3.41}$$

This expression follows directly from the definition of normal ordering. Now, for $|w| < |x| < |z|$, we have

$$A(z) = \sum_{n\in\mathbb{Z}} A_n(x)(z-x)^{-n-h_A} \tag{3.42}$$

$$B(w) = \sum_{n\in\mathbb{Z}} B_n(x)(w-x)^{-n-h_B} \tag{3.43}$$

where h_A and h_B are the conformal weights of $A(z)$ and $B(w)$ respectively. Using a geometric series, we can write

$$\frac{1}{z-w} = \sum_{i=0}^{\infty} \frac{(w-x)^i}{(z-x)^{i+1}} \tag{3.44}$$

for $|w-x| < |z-x|$. Substituting this into the first term of Equation (3.41) gives

$$\frac{1}{2\pi i} \oint_{\substack{0 \\ |z|>|w|}} \frac{A(z)B(w)}{z-w} dz$$

$$= \frac{1}{2\pi i} \sum_{i=0}^{\infty} \sum_{m,n} \oint_{\substack{0 \\ |z|>|w|}} A_n(x)B_m(x)(z-x)^{-n-i-h_A-1}(w-x)^{i-m-h_B} dz \tag{3.45}$$

$$= \sum_{i=0}^{\infty} \sum_{m,n} A_n(x)B_m(x)(w-x)^{i-m-h_B} \delta_{i+n+h_A,0} \tag{3.46}$$

$$= \sum_{m} \sum_{n\leq -h_A} A_n(x)B_m(x)(w-x)^{-n-m-h_A-h_B} \tag{3.47}$$

Similarly, one can show

$$\frac{1}{2\pi i} \oint_{\substack{0 \\ |z|<|w|}} \frac{B(w)A(z)}{z-w} dz = -\sum_{m} \sum_{n>-h_A} B_m(x)A_n(x)(w-x)^{-n-m-h_A-h_B}. \tag{3.48}$$

Therefore, we can write

$$: A(w)B(w) := \sum_{n\in\mathbb{Z}} : AB :_n w^{-n-h_A-h_B} \tag{3.49}$$

where

$$: AB :_n = \sum_{m \leq -h_A} A_m B_{n-m} + \sum_{m > -h_A} B_{n-m} A_m. \tag{3.50}$$

An important result on the normally ordered product of a collection of fields is the generalised Wick theorem. For completeness, we will present this but note that we will avoid using this in our calculations and will instead use the state-field correspondence.

Theorem 3.4.1. ***Generalised Wick Theorem.*** *[8, p. 188] Let $A(z), B(w), C(w)$ be fields. Then*

$$A(z) : B(w)C(w) := \frac{1}{2\pi i} \oint_w \frac{\overbrace{A(z)B(x)}C(w) + B(x)\overbrace{A(z)C(w)}}{x - w} dx \tag{3.51}$$

where $\overbrace{A(z)B(x)}$ denotes the contraction of the two fields $A(z)$ and $B(x)$. The contraction of the fields $A(z)$ and $B(x)$ is the singular part of the OPE $A(z)B(x)$.

3.5 Conformal Invariance

One of the main goals of conformal field theory is computing correlation functions. In trying to compute the correlation functions of a certain class of fields in conformal field theory, we see just how powerful conformal invariance is. In fact, such correlation functions are fixed up to a constant (for 2 and 3 fields) and are highly constrained for 4 fields. This is a consequence of conformal invariance.

Recall that we assumed our space of states was equipped with an inner product $(\cdot, \cdot)$. We can then introduce correlation functions, which are of the form

$$\langle 0| \phi_1(z_1) \cdots \phi_n(z_n) |0\rangle := (|0\rangle, \phi_1(z_1) \cdots \phi_n(z_n) |0\rangle) \tag{3.52}$$

where $\phi_i(z_i)$ are fields. For primary fields, we have a notion of adjoint that allows us to write

$$(|0\rangle, \phi_1(z_1) \cdots \phi_n(z_n) |0\rangle) = (\phi_1(z_1)^\dagger |0\rangle, \phi_2(z_2) \cdots \phi_n(z_n) |0\rangle) \tag{3.53}$$

$$= (\phi_2(z_2)^\dagger \phi_1(z_1)^\dagger |0\rangle, \phi_3(z_3) \cdots \phi_n(z_n) |0\rangle) \tag{3.54}$$

where $\phi^\dagger$ denotes the adjoint. The adjoint of a primary field $\phi(z, \overline{z})$ of conformal weight $(h, \overline{h})$ is defined to be [14, p. 32]

$$\phi(z, \overline{z})^\dagger = \phi\left(\frac{1}{\overline{z}}, \frac{1}{z}\right) \frac{1}{\overline{z}^{2h}} \frac{1}{z^{2\overline{h}}}. \tag{3.55}$$

We sometimes omit the state $|0\rangle$ in the correlator and simply write

$$\langle\, \phi_1(z_1) \cdots \phi_n(z_n)\, \rangle\,. \tag{3.56}$$

Conformal field theories are distinguished from other quantum field theories by their invariance under conformal transformations. This invariance leads to some nice properties. In particular, we will see that the forms of correlation functions of primary fields for $n \leq 4$ are quite simple. In this section, we will mostly restrict our attention to conformal field theories in two dimensions and we will follow Chapters 1 and 2 of [14].

Recall that we can view the generators of the Witt algebra as the generators of the infinitesimal conformal transformations of a Euclidean CFT in two dimensions. More precisely, the generators of the infinitesimal conformal transformations give two copies of the Witt algebra to account for both the holomorphic and antiholomorphic sectors. Denoting the generators by l_n and $\bar{l}_n$, we can write

$$l_n = -z^{n+1}\partial_z \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{3.57}$$

Note that we can exponentiate these generators so that they span the conformal transformations on $\mathbb{C}$. However, we would like to work on the Riemann sphere and for this reason we call the algebra generated by the l_n the local conformal algebra. Given a holomorphic conformal transformation $f(z)$ on $\mathbb{C}$, we can write

$$f(z) = \sum_{n\in\mathbb{Z}} a_n l_n = \sum_{n\in\mathbb{Z}} -a_n z^{n+1}\partial_z. \tag{3.58}$$

Changing to the local coordinate at infinity, we can see that $f(z)$ is only holomorphic on $\mathbb{C}P^1$ if $a_n = 0$ for $n \notin \{0, -1, 1\}$. The same holds true for antiholomorphic conformal transformations. For this reason, we say that $\{l_{-1}, l_0, l_1\} \cup \{\bar{l}_{-1}, \bar{l}_0, \bar{l}_1\}$ spans the global conformal algebra. But we know that the holomorphic conformal transformations on $\mathbb{C}P^1$ are precisely the Möbius transformations and so we call $PSL(2, \mathbb{C})$ the global conformal group.

Let $z \mapsto f(z)$ and $\bar{z} \mapsto \bar{f}(\bar{z})$ be holomorphic transformations. We call a field ϕ primary of conformal weight $(h, \bar{h})$ if it transforms as[9]

$$\phi(z, \bar{z}) \mapsto \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z), \bar{f}(\bar{z})). \tag{3.59}$$

Note that we can work in the chiral sector by omitting any dependence on $\bar{z}$. It can be shown that conformal invariance fixes the form of the n-point functions for primary fields for $n \leq 4$.[10] In fact, for primary fields ϕ_i of conformal weight h_i we have

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\rangle = \frac{C_{12}}{z_{12}^{2h}\bar{z}_{12}^{2\bar{h}}} \tag{3.60}$$

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\phi_3(z_3, \bar{z}_3)\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_3+h_1-h_2}} \times \tag{3.61}$$

[9]Notice that the field is unchanged for the case $h = \bar{h} = 0$. For this reason, we assume that the conformal weight of the vacuum state is 0.

[10]One can either use Equation (3.59) directly, or use the Ward identities of Section 4.2. In Section 4.2, we show how one obtains the two-point functions using the Ward identities.

$$\frac{1}{\overline{z}_{12}^{\overline{h}_1+\overline{h}_2-\overline{h}_3}\overline{z}_{23}^{\overline{h}_2+\overline{h}_3-\overline{h}_1}\overline{z}_{13}^{\overline{h}_3+\overline{h}_1-\overline{h}_2}} \tag{3.62}$$

$$\langle \phi_1(z_1,\overline{z}_1)\phi_2(z_2,\overline{z}_2)\phi_3(z_3,\overline{z}_3)\phi_4(z_4,\overline{z}_4)\rangle = f(\eta,\overline{\eta})\prod_{i<j} z_{ij}^{-(h_i+h_j)+(\sum_i h_i)/3}\prod_{i<j}\overline{z}_{ij}^{-(\overline{h}_i+\overline{h}_j)+(\sum_i \overline{h}_i)/3} \tag{3.63}$$

for some constants C_{12} and C_{123}. Here, $h := h_1 = h_2$ and $z_{ij} := z_i - z_j$. The function $f(\eta, \overline{\eta})$ is some function of the cross ratio

$$\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}} \tag{3.64}$$

and its conjugate $\overline{\eta}$.

One of the most important fields that we are interested in is the energy-momentum tensor $T_{\mu\nu}$. In two-dimensional conformal field theory, a chiral field $T(z)$ is called an energy-momentum tensor if it satisfies the OPE

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} \tag{3.65}$$

where c is a constant called the central charge. If we write

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \tag{3.66}$$

then we can compute $[L_n, L_m]$ using the contour integral argument that was used to compute $[J^a_n, J^b_m]$ in Section 3.3. The result is

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}(n^3-n)\delta_{n+m,0}c. \tag{3.67}$$

and so the modes L_n generate the Virasoro algebra.

In a classical field theory with translation invariance defined by an action S, the energy-momentum tensor is defined by

$$\delta S = -\frac{1}{2\pi}\int T_{\mu\nu}\partial^\mu \epsilon^\nu \tag{3.68}$$

under an infinitesimal differentiable transformation $x^\mu \mapsto x^\mu + \epsilon^\mu$. On-shell (when $\delta S = 0$), we have $\partial^\mu T_{\mu\nu} = 0$. Furthermore, $\delta S = 0$ when ϵ is a constant, which corresponds to translations. The energy-momentum tensor is the conserved current associated with translational symmetry.[11] In two dimensional conformal field theory,

[11] Any differential symmetry of the action S gives a "conserved current" by Noether's theorem. One can integrate this current to obtain a "conserved charge". The details of this are not important for our discussion. We are only interested in the implication of this, which takes the form of Equation (3.71). More details on Noether's theorem can be found in any standard introductory text on quantum field theory.

our coordinates are on the cylinder, and we map to the Riemann sphere. In Appendix E, we explain how this is done for the free boson. On the Riemann sphere, the corresponding conserved charge can be written as

$$Q = \frac{1}{2\pi i}\left(\oint_0 T(z)\epsilon(z)dz + \oint_0 \overline{T}(\overline{z})\overline{\epsilon}(\overline{z})d\overline{z}\right). \tag{3.69}$$

The variation of any field ϕ under our infinitesimal conformal transformation is

$$\delta_{\epsilon,\overline{\epsilon}}\phi(w,\overline{w}) = [Q,\phi(w,\overline{w})] \tag{3.70}$$

$$= \frac{1}{2\pi i}\left(\oint_0 [T(z)\epsilon(z),\phi(w,\overline{w})]dz + \oint_0 [\overline{T}(\overline{z})\overline{\epsilon}(\overline{z}),\phi(w,\overline{w})]d\overline{z}\right). \tag{3.71}$$

Given an infinitesimal conformal transformation defined by $\epsilon(z)$, a primary field $\phi(z,\overline{z})$ transforms as

$$\phi(z,\overline{z}) \mapsto (1+\partial\epsilon(z))^h \left(1+\overline{\partial}\overline{\epsilon}(\overline{z})\right)^{\overline{h}} \phi(z+\epsilon(z),\overline{z}+\overline{\epsilon}(\overline{z})) \tag{3.72}$$

$$= (1+\partial\epsilon(z)h+\cdots)\left(1+\overline{\partial}\overline{\epsilon}(\overline{z})\overline{h}+\cdots\right)\times \left(\phi(z,\overline{z}) + \partial\phi(z,\overline{z})\epsilon(z) + \overline{\partial}\phi(z,\overline{z})\overline{\epsilon}(\overline{z}) + \cdots\right) \tag{3.73}$$

To first order in ϵ and $\partial\epsilon$, the variation of $\phi(z,\overline{z})$ is

$$\delta_{\epsilon,\overline{\epsilon}}\phi(z,\overline{z}) = (h\partial\epsilon(z) + \epsilon(z)\partial + \overline{h}\overline{\partial}\overline{\epsilon}(\overline{z}) + \overline{\epsilon}(\overline{z})\overline{\partial})\phi(z,\overline{z}). \tag{3.74}$$

Now, we can evaluate Equation (3.71) by contour manipulation

$$\delta_{\epsilon,\overline{\epsilon}}\phi(w,\overline{w}) = \frac{1}{2\pi i}\left(\oint_{\substack{0\\|z|>|w|}} - \oint_{\substack{0\\|z|<|w|}}\right)\left(\epsilon(z)\mathcal{R}(T(z)\phi(w,\overline{w}))dz + \overline{\epsilon}(\overline{z})\mathcal{R}(\overline{T}(\overline{z})\phi(w,\overline{w}))dz\right) \tag{3.75}$$

$$= \frac{1}{2\pi i}\oint_0 \left(\epsilon(z)\mathcal{R}(T(z)\phi(w,\overline{w}))dz + \overline{\epsilon}(\overline{z})\mathcal{R}(\overline{T}(\overline{z})\phi(w,\overline{w}))dz\right). \tag{3.76}$$

Comparing with Equation (3.74), we see that we have the OPEs

$$T(z)\phi(w,\overline{w}) \sim \frac{h}{(z-w)^2}\phi(w,\overline{w}) + \frac{1}{z-w}\partial_w\phi(w,\overline{w}) \tag{3.77}$$

$$\overline{T}(\overline{z})\phi(w,\overline{w}) \sim \frac{\overline{h}}{(\overline{z}-\overline{w})^2}\phi(w,\overline{w}) + \frac{1}{\overline{z}-\overline{w}}\partial_{\overline{w}}\phi(w,\overline{w}). \tag{3.78}$$

Note that applying the state-field correspondence to the first of these OPEs gives

$$T(z)\left|\phi\right\rangle = \sum_{n\in\mathbb{Z}} L_n \left|\phi\right\rangle z^{-n-2} = \frac{h}{z^2}\left|\phi\right\rangle + \frac{1}{z}\left|\partial\phi\right\rangle. \tag{3.79}$$

This gives $L_{-1}|\phi\rangle = |\partial\phi\rangle$ and so we can identify L_{-1} with ∂. This will be useful in Section 4.2.

Our expression for the variation of a primary field under an infinitesimal conformal transformation can also be used to derive the conformal Ward identities, which constrain the functional form of correlation functions. To derive the conformal Ward identities, consider a finite collection of coordinates w_i in some bounded region of the complex plane. Let $\epsilon(z)$ be an infinitesimal conformal transformation in this region. Consider a contour C enclosing this bounded region and the origin. This contour can be written as a sum of contours around each of the coordinates. Substituting $\oint_C \epsilon(z)T(z)$ into the correlator of primary fields $\langle\phi_1(w_1,\overline{w}_1)\cdots\phi_n(w_n,\overline{w}_n)\rangle$ gives

$$\frac{1}{2\pi i}\left\langle\oint_C \epsilon(z)T(z)\phi_1(w_1,\overline{w}_1)\cdots\phi_n(w_n,\overline{w}_n)dz\right\rangle \tag{3.80}$$

$$= \frac{1}{2\pi i}\sum_{i=1}^{n}\left\langle\phi_1(w_1,\overline{w}_1)\cdots\oint_{w_i}\epsilon(z)T(z)\phi_i(w_i,\overline{w}_i)dz\cdots\phi_n(w_n,\overline{w}_n)\right\rangle \tag{3.81}$$

$$= \frac{1}{2\pi i}\sum_{i=1}^{n}\langle\phi_1(w_1,\overline{w}_1)\cdots\delta_\epsilon\phi_i(w_i,\overline{w}_i)\cdots\phi_n(w_n,\overline{w}_n)\rangle\,. \tag{3.82}$$

If we substitute Equation (3.74), we obtain

$$\frac{1}{2\pi i}\left\langle\oint_C \epsilon(z)T(z)\phi_1(w_1,\overline{w}_1)\cdots\phi_n(w_n,\overline{w}_n)dz\right\rangle \tag{3.83}$$

$$= \frac{1}{2\pi i}\sum_{i=1}^{n}\langle\phi_1(w_1,\overline{w}_1)\cdots(h_i\partial\epsilon(w_i)+\epsilon(w_i)\partial_{w_i})\phi_i(w_i,\overline{w}_i)\cdots\phi_n(w_n,\overline{w}_n)\rangle \tag{3.84}$$

$$= \frac{1}{2\pi i}\sum_{i=1}^{n}\Bigg\langle\phi_1(w_1,\overline{w}_1)\cdots \oint_{w_i}\epsilon(z)\left(\frac{h_i}{(z-w_i)^2}+\frac{\partial_{w_i}}{z-w_i}\right)\phi_i(w_i,\overline{w}_i)dz\cdots\phi_n(w_n,\overline{w}_n)\Bigg\rangle. \tag{3.85}$$

Since $\epsilon(z)$ is arbitrary, we have the conformal Ward identities

$$\langle T(z)\phi_1(w_1,\overline{w}_n)\cdots\ \phi_n(w_n,\overline{w}_n)\rangle = \sum_{i=1}^{n}\left(\frac{h_i}{(z-w_i)^2}+\frac{\partial_{w_i}}{z-w_i}\right)\langle\phi_1(w_1,\overline{w}_n)\cdots\phi_n(w_n,\overline{w}_n)\rangle\,. \tag{3.86}$$

Chapter 4

Wess-Zumino-Witten Models II

In this section, we study the Wess-Zumino-Witten models in more depth. We begin with the Sugawara construction, which gives a canonical embedding of the associative algebra $U(\mathrm{Vir})$ into $U(\widehat{\mathfrak{g}})$ for any affine Lie algebra $\widehat{\mathfrak{g}}$. We introduce the Ward identities, which constrain the form of correlation functions. We then introduce the Knizhnik-Zamolodchikov equations, which we will use to solve for a class of four-point correlators. We conclude this section with an example of a free field realisation. Such a realisation allows one to compute certain WZW correlators in the framework of the free boson.

4.1 The Sugawara Construction

We begin this section by assuming that the generators J^a of our Lie algebra are chosen to be orthonormal with respect to the Killing form. Note that this is not necessarily a Cartan-Weyl basis. Once our calculations have been done in this basis, the results can easily be rewritten with respect to any basis. Now, we begin by writing down the (quantum) energy-momentum tensor[1]

$$T(z) = \gamma \sum_{a} : J^a(z) J^a(z) : . \tag{4.1}$$

Here, γ is a constant that can be uniquely fixed by requiring that the fields $J^a(z)$ be primary of conformal weight 1 [8, p. 624]. Given an arbitrary basis $\{J^a\}$ of $\mathfrak{g}$, the energy-momentum tensor is

$$T(z) = \gamma \sum_{a} : J^a(z) J_a(z) : \tag{4.2}$$

where $J_a(z) = \sum_{n\in\mathbb{Z}} (J_a)_n z^{-n-1}$ and $\{J_a\}$ is the basis dual to $\{J^a\}$ with respect to $\kappa(\cdot,\cdot)$.

[1] This is an educated guess and the goal of the Sugawara construction is to show that this field satisfies the defining property of an energy-momentum tensor given by Equation (3.65)

We will now compute γ, working in the orthonormal basis. In our discussion below, we assume that $k \neq -h^\vee$. Writing

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \tag{4.3}$$

we can use Equation (3.50) to write

$$L_n = \gamma \sum_b \left\{ \sum_{m\leq -1} J^b_m J^b_{n-m} + \sum_{m>-1} J^b_{n-m} J^b_m \right\}. \tag{4.4}$$

Our expression for L_n follows from our assumption that the fields $J^a(z)$ have conformal weight 1. We now wish to compute $[L_n, J^a_{-1}]\,|0\rangle$ for $n \geq -2$. This will allow us to compute the OPE $T(z)J^a(w)$ using the state-field correspondence. We have

$$\begin{aligned}
[L_n, J^a_{-1}]\,|0\rangle &= \gamma \sum_b \left\{ \sum_{m\leq -1} [J^b_m J^b_{n-m}, J^a_{-1}] + \sum_{m>-1} [J^b_{n-m} J^b_m, J^a_{-1}] \right\} |0\rangle && (4.5)\\
&= \gamma \sum_b \left\{ \sum_{m\leq -1} \left\{ J^b_m [J^b_{n-m}, J^a_{-1}] + [J^b_m, J^a_{-1}] J^b_{n-m} \right\} \right.\\
&\quad \left. + \sum_{m>-1} \left\{ J^b_{n-m} [J^b_m, J^a_{-1}] + [J^b_{n-m}, J^a_{-1}] J^b_m \right\} \right\} |0\rangle\,. && (4.6)
\end{aligned}$$

Now consider the first term

$$\sum_{m\leq -1} \left\{ J^b_m [J^b_{n-m}, J^a_{-1}]\,|0\rangle + [J^b_m, J^a_{-1}] J^b_{n-m}\,|0\rangle \right\}. \tag{4.7}$$

If $n \geq -1$, it is equal to

$$\begin{aligned}
&\sum_{m\leq -1} J^b_m [J^b_{n-m}, J^a_{-1}]\,|0\rangle\\
&\qquad = \sum_{m\leq -1} \left\{ J^b_m \sum_c f_{bac} J^c_{n-m-1}\,|0\rangle + (n-m)k\delta_{n-m-1,0}\delta_{a,b} J^b_m\,|0\rangle \right\}. && (4.8)
\end{aligned}$$

For $n \geq 1$, this is zero. For $n = 0$, we have

$$\sum_{m\leq -1} J^b_m [J^b_{n-m}, J^a_{-1}]\,|0\rangle = k\delta_{a,b} J^b_{-1}\,|0\rangle\,. \tag{4.9}$$

For $n = -1$, we have

$$\sum_{m\leq -1} J^b_m [J^b_{n-m}, J^a_{-1}]\,|0\rangle = \sum_c f_{bac} J^b_{-1} J^c_{-1}\,|0\rangle + k\delta_{a,b} J^b_{-2}\,|0\rangle\,. \tag{4.10}$$

Now consider the second term

$$\sum_{m>-1} \left\{ J^b_{n-m}[J^b_m, J^a_{-1}] + [J^b_{n-m}, J^a_{-1}]J^b_m \right\} |0\rangle \, . \tag{4.11}$$

This is equal to

$$\sum_{m>-1} J^b_{n-m}[J^b_m, J^a_{-1}] = \sum_{m>-1} \left\{ J^b_{n-m} \sum_c f_{bac} J^c_{m-1} \right\} |0\rangle + k\delta_{a,b} J^b_{n-1} |0\rangle \tag{4.12}$$

$$= \sum_c f_{bac} J^b_n J^c_{-1} |0\rangle + k\delta_{a,b} J^b_{n-1} |0\rangle \tag{4.13}$$

So we conclude that for $n \geq 1$, we have

$$[L_n, J^a_{-1}] |0\rangle = \gamma \sum_b \left\{ \sum_c f_{bac} J^b_n J^c_{-1} |0\rangle + k\delta_{a,b} J^b_{n-1} |0\rangle \right\} \tag{4.14}$$

$$= \gamma \sum_{b,c} f_{bac} J^b_n J^c_{-1} |0\rangle = \gamma \sum_{b,c} f_{bac} [J^b_n, J^c_{-1}] |0\rangle \tag{4.15}$$

$$= \gamma \sum_{b,c,d} f_{bac} f_{bcd} J^d_{n-1} |0\rangle = 2\gamma h^\vee J^a_{n-1} |0\rangle = 0. \tag{4.16}$$

Here, we have used the identity

$$\sum_{b,c} f_{abc} f_{cbd} = 2h^\vee \delta_{a,d} \tag{4.17}$$

and the antisymmetry of the structure constants, which we prove in Appendix A. For the $n = 0$ case, we have

$$[L_n, J^a_{-1}] |0\rangle = \gamma \sum_b \left\{ k\delta_{a,b} J^b_{-1} |0\rangle + \sum_c f_{bac} J^b_0 J^c_{-1} |0\rangle + k\delta_{a,b} J^b_{-1} |0\rangle \right\} \tag{4.18}$$

$$= 2\gamma k J^a_{-1} |0\rangle + \gamma \sum_{b,c} f_{bac} [J^b_0, J^c_{-1}] |0\rangle \tag{4.19}$$

$$= 2\gamma k J^a_{-1} |0\rangle + \gamma \sum_{b,c,d} f_{bac} f_{bcd} J^d_{-1} |0\rangle \tag{4.20}$$

$$= 2\gamma (k + h^\vee) J^a_{-1} |0\rangle \, . \tag{4.21}$$

Finally, for the $n = -1$ case, we have

$$[L_n, J^a_{-1}] |0\rangle = \gamma \sum_b \left\{ \sum_c f_{bac} J^b_{-1} J^c_{-1} |0\rangle + k\delta_{a,b} J^b_{-2} |0\rangle \right.$$

$$\left. + \sum_c f_{bac} J^b_{-1} J^c_{-1} |0\rangle + k\delta_{a,b} J^b_{-2} |0\rangle \right\} \tag{4.22}$$

$$= 2\gamma k J^a_{-2} |0\rangle + 2\gamma \sum_{b,c} f_{bac} J^b_{-1} J^c_{-1} |0\rangle \tag{4.23}$$
$$= 2\gamma k J^a_{-2} |0\rangle + \gamma \sum_{b,c} f_{bac} [J^b_{-1}, J^c_{-1}] |0\rangle \tag{4.24}$$
$$= 2\gamma k J^a_{-2} |0\rangle + \gamma \sum_{b,c,d} f_{bac} f_{bcd} J^d_{-2} |0\rangle \tag{4.25}$$
$$= 2\gamma (k + h^\vee) J^a_{-2} |0\rangle \tag{4.26}$$

We will now compute the OPE $T(z)J^a(w)$. We write

$$T(z)J^a(w) = \sum_n \psi_n(w)(z-w)^{-n-2}. \tag{4.27}$$

Applying the state-field correspondence gives

$$T(z) |J^a\rangle = \sum_n |\psi_n\rangle z^{-n-2} \tag{4.28}$$
$$= \sum_{n\in\mathbb{Z}} L_n |J^a\rangle z^{-n-2}. \tag{4.29}$$

This tells us

$$|\psi_n\rangle = L_n J^a_{-1} |0\rangle = [L_n, J^a_{-1}] |0\rangle + J^a_{-1} L_n |0\rangle . \tag{4.30}$$

Using our previous calculations, for $n \geq -1$ we have

$$|\psi_n\rangle = 2\gamma(k + h^\vee) J^a_{n-1} |0\rangle + J^a_{-1} L_n |0\rangle . \tag{4.31}$$

For $n \geq 1$, this is 0. Furthermore,

$$|\psi_0\rangle = 2\gamma(k + h^\vee) J^a_{-1} |0\rangle = 2\gamma(k + h^\vee) |J^a\rangle \tag{4.32}$$
$$|\psi_{-1}\rangle = 2\gamma(k + h^\vee) J^a_{-2} |0\rangle = 2\gamma(k + h^\vee) |\partial J^a\rangle . \tag{4.33}$$

Therefore, we have the OPE

$$T(z)J^a(w) \sim 2\gamma(k + h^\vee) \left\{ \frac{J^a(w)}{(z-w)^2} + \frac{\partial_w J^a(w)}{z-w} \right\}. \tag{4.34}$$

However, note that we have also shown

$$|\psi_0\rangle = L_0 |J^a\rangle = 2\gamma(k + h^\vee) |J^a\rangle . \tag{4.35}$$

Since we required that the fields $J^a(z)$ have conformal weight 1, we obtain the expression

$$\gamma = \frac{1}{2(k + h^\vee)}. \tag{4.36}$$

Note that we did not need to find the OPE $T(z)J^a(w)$ to obtain the expression for γ. We only needed to know $[L_0, J^a_{-1}]$.

We now turn our attention to computing $[L_n, J^a_m]$. To do this, we begin by writing

$$L_n = \frac{1}{2\pi i}\oint_0 T(z)z^{n+1}dz \qquad J^a_m = \frac{1}{2\pi i}\oint_0 J^a(w)w^m dw. \tag{4.37}$$

Here, we assume implicitly that each side of the equation is inside a correlation function. Note that we did this explicitly when computing $\left[J^a_n, J^b_m\right]$. From now on, when we perform similar calculations, we will assume this without stating it. Performing the contour manipulation we used earlier, we obtain

$$[L_n, J^a_m] = \frac{1}{(2\pi i)^2}\oint_0\oint_w \mathcal{R}\{T(z)J^a(w)\}z^{n+1}w^m dzdw. \tag{4.38}$$

Substituting in the OPE $T(z)J^a(w)$ gives

$$[L_n, J^a_m] = \frac{1}{(2\pi i)^2}\oint_0\oint_w \left[\frac{J^a(w)}{(z-w)^2} + \frac{\partial_w J^a(w)}{z-w}\right] z^{n+1}w^m dzdw \tag{4.39}$$

$$= \frac{1}{2\pi i}\oint_0 \partial_w J^a(w)w^{m+n+1}dw + \frac{1}{2\pi i}\oint_0 (n+1)J^a(w)w^{m+n}dw \tag{4.40}$$

$$= \frac{1}{2\pi i}\oint_0 \sum_{k\in\mathbb{Z}} -(k+1)J^a_k w^{m+n-k-1}dw + \frac{1}{2\pi i}\oint_0 (n+1)\sum_{k\in\mathbb{Z}} J^a_k w^{m+n-k-1}dw \tag{4.41}$$

$$= -(m+n+1)J^a_{m+n} + (n+1)J^a_{m+n} = -mJ^a_{n+m} \tag{4.42}$$

This bracket will now be used to determine the OPE of $T(z)T(w)$. We begin by writing the OPE in the form

$$T(z)T(w) = \sum_n \psi_n(w)(z-w)^{-n-2}. \tag{4.43}$$

Applying the state-field correspondence gives

$$T(z)\left|T\right\rangle = \sum_n \left|\psi\right\rangle z^{-n-2} = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}\left|T\right\rangle \tag{4.44}$$

$$\left|\psi_n\right\rangle = L_n\left|T\right\rangle, \quad n\in\mathbb{Z}. \tag{4.45}$$

Now,

$$\left|T\right\rangle = \lim_{w\to 0} T(w)\left|0\right\rangle = \lim_{w\to 0}\sum_{n\in\mathbb{Z}} L_n w^{-n-2}\left|0\right\rangle = L_{-2}\left|0\right\rangle. \tag{4.46}$$

So,

$$\left|\psi_n\right\rangle = L_n L_{-2}\left|0\right\rangle = \gamma\sum_b L_n J^b_{-1}J^b_{-1}\left|0\right\rangle \tag{4.47}$$

$$= \gamma\sum_b [L_n, J^b_{-1}]J^b_{-1}\left|0\right\rangle + \gamma\sum_b J^b_{-1}[L_n, J^b_{-1}]\left|0\right\rangle + \gamma\sum_b J^b_{-1}J^b_{-1}L_n\left|0\right\rangle. \tag{4.48}$$

We now use the fact that $L_n |0\rangle = 0$ for $n \geq -1$, which follows immediately from the state-field correspondence. This gives

$$|\psi_n\rangle = \gamma \sum_b [L_n, J^b_{-1}] J^b_{-1} |0\rangle + \gamma \sum_b J^b_{-1} [L_n, J^b_{-1}] |0\rangle \tag{4.49}$$

for $n \geq -1$. For $n \geq 1$, we have

$$|\psi_n\rangle = \gamma \sum_b J^b_{n-1} J^b_{-1} |0\rangle + \gamma \sum_b J^b_{-1} J^b_{n-1} |0\rangle \tag{4.50}$$

$$= \gamma \sum_b [J^b_{n-1}, J^b_{-1}] |0\rangle \tag{4.51}$$

$$= \gamma \sum_b (n-1) k \delta_{n-2,0} |0\rangle . \tag{4.52}$$

So, $|\psi_n\rangle = 0$ for $n \geq 1$ except when $n = 2$ where we have $|\psi_2\rangle = \gamma k \dim \mathfrak{g} |0\rangle$. There are now two cases to check.

$$|\psi_0\rangle = 2\gamma \sum_b J^b_{-1} J^b_{-1} |0\rangle = 2L_{-1} |0\rangle = 2 |T\rangle . \tag{4.53}$$

$$|\psi_{-1}\rangle = \gamma \sum_b \left\{ J^b_{-2} J^b_{-1} + J^b_{-1} J^b_{-2} \right\} |0\rangle = L_{-3} |0\rangle . \tag{4.54}$$

Using

$$\partial_z T(z) = \sum_{n \in \mathbb{Z}} -(n+1) L_{n-1} z^{-n-2} \tag{4.55}$$

and the state-field correspondence gives

$$L_{-3} |0\rangle = |\partial T\rangle . \tag{4.56}$$

Therefore, we have

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} \tag{4.57}$$

where

$$c = \frac{k \dim \mathfrak{g}}{k + h^\vee}. \tag{4.58}$$

Now that we have computed the OPE $T(z)T(w)$, we can compute $[L_n, L_m]$ using the same method we used to compute $[L_n, J^a_m]$. The result is

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{1}{12}(n^3 - n)\delta_{n+m,0} c. \tag{4.59}$$

This demonstrates that the modes of the energy-momentum tensor can be viewed as elements of the Virasoro algebra with central charge given by Equation (4.58). Since the energy-momentum tensor is defined in terms of the modes of our affine

Lie algebra, we have demonstrated that for each $k \neq -h^\vee$ there exists an injective associative algebra homomorphism

$$i_k : U(\mathrm{Vir}) \to \widetilde{U}(\hat{\mathfrak{g}}) \tag{4.60}$$

$$L_n \mapsto \frac{1}{2(k+h^\vee)} \sum_b \left\{ \sum_{m \leq -1} J^b_m J^b_{n-m} + \sum_{m > -1} J^b_{n-m} J^b_m \right\} \tag{4.61}$$

$$c \mapsto \frac{\dim \mathfrak{g}}{k+h^\vee} K \tag{4.62}$$

where we extend i by linearity. Here, $\widetilde{U}(\hat{\mathfrak{g}})$ denotes a completion of $U(\hat{\mathfrak{g}})$, which allows for the infinite sums in the image of i_k. The calculation of the OPEs and brackets above is known as the Sugawara construction. For an alternative derivation of the Sugawara construction that avoids the use of the state-field correspondence, and instead uses the generalised Wick theorem, see [8, pp. 624-626].

Now that we have constructed the energy-momentum tensor, we can see how the derivation d is realised in $U(\hat{\mathfrak{g}})$. Given the relation $[L_0, J^a_m] = -mJ^a_m$ derived earlier, we can see that $-L_0$ satisfies the relations of the derivation. Note that since the modes J^a_n commute with the central element, $-L_0$ also does.

4.2 The Ward Identities

Let $\phi(z)$ be a Virasoro primary of conformal weight h. We write the OPE of $T(z)\phi(w)$ in the form

$$T(z)\phi(w) = \sum_n \psi_n(w)(z-w)^{-n-2}. \tag{4.63}$$

Applying the state-field correspondence gives

$$T(z) |\phi\rangle = \sum_n |\psi_n\rangle z^{-n-2} \tag{4.64}$$

$$= \sum_n L_n |\phi\rangle z^{-n-2}. \tag{4.65}$$

So for $n \in \mathbb{Z}$,

$$|\psi_n\rangle = L_n |\phi\rangle . \tag{4.66}$$

Since $\phi(w)$ is a Virasoro primary, this is zero for $n > 0$. We also have $|\psi_0\rangle = h|\phi\rangle$. Recall from Section 3.5 that $L_{-1}|\phi\rangle = |\partial\phi\rangle$. So, we have

$$T(z)\phi(w) \sim \frac{h\phi(w)}{(z-w)^2} + \frac{\partial_w \phi(w)}{z-w}. \tag{4.67}$$

Now,

$$[L_n, \phi(w)] = \frac{1}{2\pi i} \oint_0 T(z)\phi(w) z^{n+1} dz - \frac{1}{2\pi i} \oint_0 \phi(w) T(z) z^{n+1} dz. \tag{4.68}$$

Here, the product is not radially-ordered. For the moment we will denote the radially-ordered product by $\mathcal{R}(T(z)\phi(w))$. We can then write

$$[L_n, \phi(w)] = \frac{1}{2\pi i}\left[\oint\limits_{|z|>|w|}_0 - \oint\limits_{|z|<|w|}_0\right]\mathcal{R}(T(z)\phi(w))z^{n+1}dz \tag{4.69}$$

$$= \frac{1}{2\pi i}\oint_w \mathcal{R}(T(z)\phi(w))z^{n+1}dz. \tag{4.70}$$

Inserting the OPE from earlier gives

$$[L_n, \phi(w)] = \frac{1}{2\pi i}\oint_w \left[\frac{h\phi(w)}{(z-w)^2} + \frac{\partial_w\phi(w)}{z-w}\right] z^{n+1}dz \tag{4.71}$$

$$= h(n+1)w^n\phi(w) + w^{n+1}\partial_w\phi(w). \tag{4.72}$$

Now, for $m \leq 1$ consider the following correlation function

$$0 = \langle L_m\phi_1(w_1)\phi_2(w_2)\cdots\phi_n(w_n)\rangle \tag{4.73}$$

$$= \sum_{i=1}^{n}\langle\phi_1(w_1)\cdots[L_m, \phi_i(w_i)]\cdots\phi_n(w_n)\rangle + \langle\phi_1(w_1)\phi_2(w_2)\cdots\phi_n(w_n)L_m\rangle. \tag{4.74}$$

Substituting the commutation relations above then gives the Ward identities, which hold for $m = -1, 0, 1$.

$$\sum_{i=1}^{n} w_i^m(w_i\partial_{w_i} + (m+1)h_i)\langle\phi_1(w_1)\phi_2(w_2)\cdots\phi_n(w_n)\rangle = 0. \tag{4.75}$$

Note that these identities can be obtained as a consequence of Equation (3.86). Note also that we have derived the Ward identities for Virasoro primary fields. However, the result also holds for WZW primaries since these are also Virasoro primaries. To prove this, we will use Equation (4.2).

Consider the Cartan-Weyl basis $\{H^i\} \cup \{E^{\pm\alpha}\}$ satisfying the conditions of Example 1.2.1. That is,

1. $\kappa(E^\alpha, H^i) = 0$.
2. $\kappa(H^i, H^j) = \delta_{i,j}$.
3. $\kappa(E^\alpha, E^\beta) = \delta_{\alpha,-\beta}$.

Let $|\phi\rangle$ be a WZW primary with weight λ. Then for $n > 0$, the Sugawara construction gives

$$L_n|\phi\rangle = \gamma\sum_i H_n^i H_0^i|\phi\rangle + \gamma\sum_{\alpha\in\Phi_+}E_n^\alpha E_0^{-\alpha}|\phi\rangle + \gamma\sum_{\alpha\in\Phi_+}E_n^{-\alpha}E_0^\alpha|\phi\rangle \tag{4.76}$$

$$= \gamma\sum_i \lambda(H_0^i)H_n^i|\phi\rangle + \gamma\sum_{\alpha\in\Phi_+}[E_n^\alpha, E_0^{-\alpha}]|\phi\rangle = 0. \tag{4.77}$$

We know this is zero since the summands of the second term are all proportional to $H^i_n \left|\phi\right\rangle = 0$ for some H^i in the Cartan subalgebra basis. We also have

$$L_0 \left|\phi\right\rangle = \gamma \sum_i H^i_0 H^i_0 \left|\phi\right\rangle + \gamma \sum_{\alpha\in\Phi_+} E^\alpha_0 E^{-\alpha}_0 \left|\phi\right\rangle + \gamma \sum_{\alpha\in\Phi_+} E^{-\alpha}_0 E^\alpha_0 \left|\phi\right\rangle \tag{4.78}$$

$$= \gamma \sum_i \lambda(H^i_0)^2 \left|\phi\right\rangle + \gamma \sum_{\alpha\in\Phi_+} [E^\alpha_0, E^{-\alpha}_0] \left|\phi\right\rangle . \tag{4.79}$$

The summands of the second term are all proportional to $H^i_0 \left|\phi\right\rangle$. So, we can see that $L_0 \left|\phi\right\rangle$ is proportional to $\left|\phi\right\rangle$. This constant of proportionality depends on $\left|\phi\right\rangle$ and on the Lie algebra of interest. We conclude that WZW primaries are Virasoro primaries.

As an application of the Ward identities, we will now see how to compute the 2-point function $\langle\phi_1(z_1)\phi_2(z_2)\rangle$ where ϕ_1 and ϕ_2 are fields of conformal weights h_1 and h_2 respectively. Our solution follows [32, p. 47]. The $m = -1$ Ward identity gives

$$(\partial_{z_1} + \partial_{z_2}) \langle\phi_1(z_1)\phi_2(z_2)\rangle = 0. \tag{4.80}$$

Now make the change of variables $z = z_1 + z_2$ and $z_{12} = z_1 - z_2$. The equation above tells us that $\partial_z \langle\phi_1(z, z_{12})\phi_2(z, z_{12})\rangle = 0$. The $m = 0$ Ward identity then gives us

$$(z\partial_z + z_{12}\partial_{z_{12}} + h_1 + h_2) \langle\phi_1(z_{12})\phi_2(z_{12})\rangle = 0 \tag{4.81}$$

and so

$$(z_{12}\partial_{z_{12}} + h_1 + h_2) \langle\phi_1(z_{12})\phi_2(z_{12})\rangle = 0. \tag{4.82}$$

This is easily solved to give the solution

$$\langle\phi_1(z_{12})\phi_2(z_{12})\rangle = \frac{C_{12}}{z_{12}^{h_1+h_2}}. \tag{4.83}$$

Now, consider the $m = 1$ Ward identity

$$0 = (z_1^2\partial_{z_1} + 2h_1 z_1 + z_2^2\partial_{z_2} + 2h_2 z_2) \langle\phi_1(z_{12})\phi_2(z_{12})\rangle \tag{4.84}$$

$$= ((z - z_2)^2(\partial_z + \partial_{z_{12}}) + z_2^2(\partial_z - \partial_{z_{12}}) + 2h_1 z_1 + 2h_2 z_2) \langle\phi_1(z_{12})\phi_2(z_{12})\rangle . \tag{4.85}$$

Expanding the right hand side and noting that $\partial_z \langle\phi_1(z_{12})\phi_2(z_{12})\rangle = 0$ gives

$$0 = (z^2\partial_{z_{12}} - 2zz_2\partial_{z_{12}} + 2h_1 z_1 + 2h_2 z_2) \langle\phi_1(z_{12})\phi_2(z_{12})\rangle . \tag{4.86}$$

Using

$$z^2 - 2z_2 = zz_{12}, \qquad 2h_1 z_1 + 2h_2 z_2 = (h_1 + h_2)z + (h_1 - h_2)z_{12} \tag{4.87}$$

we arrive at

$$(zz_{12}\partial_{z_{12}} + (h_1 + h_2)z + (h_1 - h_2)z_{12}) \langle\phi_1(z_{12})\phi_2(z_{12})\rangle = 0. \tag{4.88}$$

Substituting our expression for $\langle \phi_1(z_{12})\phi_2(z_{12})\rangle$ into the above equation gives us

$$(h_1 - h_2)\frac{C_{12}}{z_{12}^{h_1+h_2-1}} = 0. \tag{4.89}$$

and so we conclude

$$\langle \phi_1(z_{12})\phi_2(z_{12})\rangle = \frac{C_{12}\delta_{h_1,h_2}}{z_{12}^{h_1+h_2}}. \tag{4.90}$$

Compare this with Equation (3.60). Similar to the above solution, one can use the Ward identities (with the method of characteristics) to solve 3-point functions of primary fields up to a constant factor. Solving 4-point functions is significantly more involved. However, the next section will give a system of PDEs that will allow us to do this computation.

4.3 The $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ Knizhnik-Zamolodchikov Equations

In this section, we will derive the Knizhnik-Zamolodchikov Equations for the $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ WZW model at level k. These equations are a system of PDEs for the correlators of certain fields. Let $\{J^a\}$ denote the basis of $\mathfrak{sl}_2(\mathbb{C})$ given in Example 1.2.1.[2] The fields that we are interested in are those annihilated by J^a_n for $n > 0$.[3] For $1 \leq i \leq n$, let ϕ_i be such a field whose state is in a representation (specifically an irreducible highest weight module) (ρ_i, V_i) of $\widehat{\mathfrak{sl}_2(\mathbb{C})}$. Through the state-field correspondence, we can define an action of $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ on the fields. The notation $\rho_i(X) \otimes \rho_j(Y) \langle \psi_1 \cdots \psi_n \rangle$ denotes the correlator given by first acting on the ith field with X and the jth field with Y. The energy-momentum tensor in our basis is

$$T(z) = \frac{1}{2(k+h^\vee)}\left(: E(z)F(z) : + : F(z)E(z) : + : H(z)H(z) :\right) \tag{4.91}$$

and so we have

$$L_{-1}|\phi_i\rangle = \frac{1}{k+h^\vee}\left(E_{-1}F_0 + F_{-1}E_0 + H_{-1}H_0\right)|\phi_i\rangle. \tag{4.92}$$

Since $L_{-1}|\phi_i\rangle = |\partial\phi_i\rangle$, we deduce the equation

$$\begin{aligned} 0 = &\partial_{w_i}\langle \phi_1(w_1)\cdots\phi_n(w_n)\rangle - \\ &\frac{\rho_i(E_{-1}F_0) + \rho_i(F_{-1}E_0) + \rho_i(H_{-1}H_0)}{k+h^\vee}\langle \phi_1(w_1)\cdots\phi_n(w_n)\rangle. \end{aligned} \tag{4.93}$$

Now, define the states $|\phi^a\rangle = (J_a)_0|\phi_i\rangle$ and $|\psi^a\rangle = J^a_{-1}|\phi\rangle$. We can write

$$\psi^a(w_i) = \frac{1}{2\pi i}\oint_{w_i}\frac{J^a(z)\phi^a(w_i)}{z-w_i}dz \tag{4.94}$$

[2]It is important to note that our expression for the KZ equations will depend on the choice of basis. However, the correlators obtained from the KZ equations are independent of this choice.

[3]Our fields are not necessarily annihilated by E_0 and so they do not have to be $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ primaries.

and place the field $\psi(w_i)$ in an appropriate correlator to obtain

$$\begin{aligned}
&\rho_i(J^a_{-1}(J_a)_0)\,\langle\phi_1(w_1)\cdots\phi_n(w_n)\rangle \\
&\qquad = \langle\phi_1(w_1)\cdots\phi_{i-1}(w_{i-1})\psi^a(w_i)\phi_{i+1}(w_{i+1})\cdots\phi_n(w_n)\rangle \qquad (4.95)\\
&\qquad = \frac{1}{2\pi i}\oint_{w_i}\frac{1}{z-w_i}\,\langle\phi_1(w_1)\cdots J^a(z)\phi^a(w_i)\phi_{i+1}(w_{i+1})\cdots\phi_n(w_n)\rangle\,dz \qquad (4.96)
\end{aligned}$$

We can rewrite this as a sum of contour integrals around the w_j for $j\neq i$ and the local coordinate at infinity. Changing to the local coordinate at infinity, we see that the integrand is holomorphic and so the contour integral around infinity vanishes. So, we obtain

$$\begin{aligned}
&\rho_i(J^a_{-1}(J_a)_0)\,\langle\phi_1(w_1)\cdots\phi_n(w_n)\rangle \\
&\qquad = \frac{-1}{2\pi i}\sum_{j\neq i}\oint_{w_j}\frac{1}{z-w_i}\,\langle\phi_1(w_1)\cdots J^a(z)\phi^a(w_i)\phi_{i+1}(w_{i+1})\cdots\phi_n(w_n)\rangle\,dz. \qquad (4.97)
\end{aligned}$$

We now consider the case where J^a is E or F. In these cases, we pick up a factor of -1 every time we commute $J^a(z)$ through the fields ϕ_i. For an explanation of this, see [28]. At the end of Appendix F, we verify this for the case of $k=1$. This implies that we also pick up a factor of -1 when we commute $J^a(z)$ through the field ϕ^a since we have

$$\phi^a(w_i) = \,:J_a(w_i)\phi_i(w_i):\, = \frac{1}{2\pi i}\oint_{w_i}\frac{J_a(z)\phi_i(w_i)}{z-w_i}dz \qquad (4.98)$$

and $J^a(z)$ commutes through $J_a(w)$. Using these results, we have

$$\begin{aligned}
&\rho_i(J^a_{-1}(J_a)_0)\,\langle\phi_1(w_1)\cdots\phi_n(w_n)\rangle \\
&= \sum_{j\neq i}\oint_{w_j}\frac{(-1)^{i+j+1}}{2\pi i(z-w_i)}\,\langle\phi_1(w_1)\cdots\phi^a(w_i)\cdots J^a(z)\phi_j(w_j)\phi_{i+1}(w_{i+1})\cdots\phi_n(w_n)\rangle\,dz \qquad (4.99)\\
&= \frac{-1}{2\pi i}\sum_{j\neq i}(-1)^{i+j}\oint_{w_j}\frac{\rho_i((J_a)_0)}{z-w_i}\,\langle\phi_1(w_1)\cdots J^a(z)\phi_j(w_j)\phi_{i+1}(w_{i+1})\cdots\phi_n(w_n)\rangle\,dz. \qquad (4.100)
\end{aligned}$$

The state-field correspondence gives the OPE

$$J^a(z)\phi_j(w_j)\sim\frac{J^a_0\phi_j(w_j)}{z-w_j}. \qquad (4.101)$$

If we insert this into our above equation, we obtain

$$\rho_i(J^a_{-1}(J_a)_0)\,\langle\phi_1(w_1)\cdots\phi_n(w_n)\rangle$$

$$= \frac{-1}{2\pi i} \sum_{j \neq i} (-1)^{i+j} \oint_{w_j} \frac{\rho_i((J_a)_0) \otimes \rho_j(J_0^a)}{(z-w_i)(z-w_j)} \langle \phi_1(w_1) \cdots \phi_n(w_n) \rangle \, dz \tag{4.102}$$

$$= \frac{-1}{2\pi i} \sum_{j \neq i} (-1)^{i+j} \frac{\rho_i((J_a)_0) \otimes \rho_j(J_0^a)}{w_j - w_i} \langle \phi_1(w_1) \cdots \phi_n(w_n) \rangle . \tag{4.103}$$

For the case $J^a = H$, we can commute $J^a(z)$ through the primary fields without picking up any factors and so we have

$$\rho_i(J_{-1}^a (J_a)_0) \langle \phi_1(w_1) \cdots \phi_n(w_n) \rangle = \frac{-1}{2\pi i} \sum_{j \neq i} \frac{\rho_i((J_a)_0) \otimes \rho_j(J_0^a)}{w_j - w_i} \langle \phi_1(w_1) \cdots \phi_n(w_n) \rangle . \tag{4.104}$$

If we substitute the above equations into Equation (4.93) and note that the dual Coxeter number of $\mathfrak{sl}_2(\mathbb{C})$ is 2, we arrive at the KZ equations for our choice of basis

$$0 = \left\{ \partial_{w_i} + \frac{1}{k+2} \sum_{j \neq i} \left(\frac{\rho_i(H_0) \otimes \rho_j(H_0)}{w_j - w_i} \right.\right.$$
$$\left.\left. + (-1)^{i+j} \frac{\rho_i(E_0) \otimes \rho_j(F_0)}{w_j - w_i} + (-1)^{i+j} \frac{\rho_i(F_0) \otimes \rho_j(E_0)}{w_j - w_i} \right) \right\} \langle \phi_1(w_1) \cdots \phi_n(w_n) \rangle . \tag{4.105}$$

4.4 Computing 4-Point Functions with KZ

We are now interested in the level 1 $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ WZW model. As an example, we will compute the correlation function $\langle \phi(z_1) \psi(z_2) \phi(z_3) \psi(z_4) \rangle$ where ϕ and ψ are chosen as follows. We use the basis given by Equation (4.177). We then consider the irreducible integrable highest weight module of $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ given in Example 2.5.2. The highest weight λ is defined by

$$\lambda = 0 \cdot \omega_0 + \omega_1 - \frac{1}{4}\delta. \tag{4.106}$$

We denote the highest weight vector by $|\phi\rangle$. We then define $|\psi\rangle = F_0 |\phi\rangle$. Our module has been obtained by setting to zero the submodule generated by the singular vectors

$$\left\{ F_0^2 v_\lambda, E_{-1} v_\lambda \right\} \tag{4.107}$$

in the Verma module. We are particularly interested in the relation $F_0^2 |\phi\rangle = 0$ in the irreducible quotient, which implies $F_0 |\psi\rangle = 0$. Now, in our basis $\omega_1(H_0) = \frac{1}{\sqrt{2}}$ and so we have

$$H_0 |\phi\rangle = \frac{1}{\sqrt{2}} |\phi\rangle \tag{4.108}$$

$$H_0 |\psi\rangle = -\frac{1}{\sqrt{2}} |\psi\rangle . \tag{4.109}$$

Furthermore,

$$E_0 |\psi\rangle = |\phi\rangle . \tag{4.110}$$

Recall Equation (4.79), which is

$$L_0 |\phi\rangle = \gamma \sum_i \lambda (H_0^i)^2 |\phi\rangle + \gamma \sum_{\alpha \in \Phi_+} [E_0^\alpha, E_0^{-\alpha}] |\phi\rangle . \tag{4.111}$$

Using

$$\gamma = \frac{1}{2(k + h^\vee)} = \frac{1}{2(1+2)} = \frac{1}{6}, \tag{4.112}$$

we therefore obtain

$$L_0 |\phi\rangle = \frac{3\gamma}{2} |\phi\rangle = \frac{1}{4} |\phi\rangle . \tag{4.113}$$

Recall that

$$[L_n, J_m^a] = -m J_{n+m}^a . \tag{4.114}$$

So,

$$L_0 |\psi\rangle = F_0 L_0 |\phi\rangle = \frac{1}{4} |\psi\rangle . \tag{4.115}$$

Notice that our choice of λ implies that the conformal weight of $|\phi\rangle$ is consistent with the identification $d = -L_0$.

If we recall Equation (3.63), we know that the four point function above takes the form

$$\langle \phi(z_1)\psi(z_2)\phi(z_3)\psi(z_4)\rangle = f(\eta) \prod_{1 \le i < j \le 4} z_{ij}^{h/3 - h_i - h_j} . \tag{4.116}$$

Here, $z_{ij} = z_i - z_j$ and f is an unknown function of the cross ratio η, which we define as[4]

$$\eta = \frac{z_{14} z_{23}}{z_{12} z_{34}} . \tag{4.117}$$

Also, $h_i = \frac{1}{4}$ is the conformal weight of the ith field in the correlation function of interest. We also define

$$h := \sum_{i=1}^{4} h_i = 1. \tag{4.118}$$

In our case, we have $h_i = h_j$ for all i, j. Now, we can write

$$\langle \phi(z_1)\psi(z_2)\phi(z_3)\psi(z_4)\rangle = f_1(\eta) \prod_{1 \le i < j \le 4} z_{ij}^{h/3 - h_i - h_j} \tag{4.119}$$

$$\langle \psi(z_1)\phi(z_2)\phi(z_3)\psi(z_4)\rangle = f_2(\eta) \prod_{1 \le i < j \le 4} z_{ij}^{h/3 - h_i - h_j} \tag{4.120}$$

[4]Note that our cross ratio is $\frac{\eta}{\eta - 1}$ where η is the cross ratio as defined in [8].

$$\langle\psi(z_1)\psi(z_2)\phi(z_3)\phi(z_4)\rangle = f_3(\eta)\prod_{1\le i<j\le 4} z_{ij}^{h/3-h_i-h_j} \tag{4.121}$$

$$\langle\phi(z_1)\phi(z_2)\psi(z_3)\psi(z_4)\rangle = f_4(\eta)\prod_{1\le i<j\le 4} z_{ij}^{h/3-h_i-h_j} \tag{4.122}$$

$$\langle\phi(z_1)\psi(z_2)\psi(z_3)\phi(z_4)\rangle = f_5(\eta)\prod_{1\le i<j\le 4} z_{ij}^{h/3-h_i-h_j} \tag{4.123}$$

$$\langle\psi(z_1)\phi(z_2)\psi(z_3)\phi(z_4)\rangle = f_6(\eta)\prod_{1\le i<j\le 4} z_{ij}^{h/3-h_i-h_j}. \tag{4.124}$$

Recall the $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ KZ equations from Section 4.3. At level 1, the $i = 1$ equation is

$$\begin{aligned}0 = \Bigg\{ \partial_{z_1} + \frac{1}{3}\sum_{j\neq 1}\Bigg(&\frac{\rho_1(H_0)\otimes\rho_j(H_0)}{z_j - z_1} + (-1)^{1+j}\frac{\rho_1(E_0)\otimes\rho_j(F_0)}{z_j - z_1} \\ &+ (-1)^{1+j}\frac{\rho_1(F_0)\otimes\rho_j(E_0)}{z_j - z_1}\Bigg)\Bigg\}\langle\phi_1(z_1)\cdots\phi_n(z_n)\rangle. \end{aligned} \tag{4.125}$$

We begin by substituting $\langle\phi(z_1)\psi(z_2)\phi(z_3)\psi(z_4)\rangle$ into this equation. Note first that the action of $\rho_1(E_0)\otimes\rho_j(F_0)$ on our correlator is 0 for all j. The other actions we need to compute are

$$\rho_1(F_0)\otimes\rho_2(E_0)\langle\phi(z_1)\psi(z_2)\phi(z_3)\psi(z_4)\rangle = \langle\psi(z_1)\phi(z_2)\phi(z_3)\psi(z_4)\rangle \tag{4.126}$$

$$\rho_1(F_0)\otimes\rho_3(E_0)\langle\phi(z_1)\psi(z_2)\phi(z_3)\psi(z_4)\rangle = 0 \tag{4.127}$$

$$\rho_1(F_0)\otimes\rho_4(E_0)\langle\phi(z_1)\psi(z_2)\phi(z_3)\psi(z_4)\rangle = \langle\psi(z_1)\psi(z_2)\phi(z_3)\phi(z_4)\rangle \tag{4.128}$$

$$\rho_1(H_0)\otimes\rho_2(H_0)\langle\phi(z_1)\psi(z_2)\phi(z_3)\psi(z_4)\rangle = -\frac{1}{2}\langle\phi(z_1)\psi(z_2)\phi(z_3)\psi(z_4)\rangle \tag{4.129}$$

$$\rho_1(H_0)\otimes\rho_3(H_0)\langle\phi(z_1)\psi(z_2)\phi(z_3)\psi(z_4)\rangle = \frac{1}{2}\langle\phi(z_1)\psi(z_2)\phi(z_3)\psi(z_4)\rangle \tag{4.130}$$

$$\rho_1(H_0)\otimes\rho_4(H_0)\langle\phi(z_1)\psi(z_2)\phi(z_3)\psi(z_4)\rangle = -\frac{1}{2}\langle\phi(z_1)\psi(z_2)\phi(z_3)\psi(z_4)\rangle. \tag{4.131}$$

We can also rewrite ∂_{z_1} as

$$\partial_{z_1} = \frac{\partial\eta}{\partial z_1}\partial_\eta = \left(\frac{\eta}{z_{14}} - \frac{\eta}{z_{12}}\right)\partial_\eta. \tag{4.132}$$

We now have

$$\begin{aligned}0 = &\left(\frac{\eta}{z_{14}} - \frac{\eta}{z_{12}}\right)\partial_\eta\left(f_1(\eta)\prod_{1\le i<j\le 4} z_{ij}^{-1/6}\right) - \frac{1}{6}\frac{\langle\phi(z_1)\psi(z_2)\phi(z_3)\psi(z_4)\rangle}{z_2 - z_1} \\ &+ \frac{1}{6}\frac{\langle\phi(z_1)\psi(z_2)\phi(z_3)\psi(z_4)\rangle}{z_3 - z_1} - \frac{1}{6}\frac{\langle\phi(z_1)\psi(z_2)\phi(z_3)\psi(z_4)\rangle}{z_4 - z_1} - \frac{1}{3}\frac{\langle\psi(z_1)\phi(z_2)\phi(z_3)\psi(z_4)\rangle}{z_2 - z_1} \\ &- \frac{1}{3}\frac{\langle\psi(z_1)\psi(z_2)\phi(z_3)\phi(z_4)\rangle}{z_4 - z_1}. \end{aligned} \tag{4.133}$$

Applying the product rule to the first term gives

$$\left(\frac{\eta}{z_{14}}-\frac{\eta}{z_{12}}\right)f_1'(\eta)\prod_{1\le i<j\le 4}z_{ij}^{-1/6}+\left(\frac{\eta}{z_{14}}-\frac{\eta}{z_{12}}\right)\frac{f(\eta)}{z_{12}^{-1/6}z_{13}^{-1/6}z_{14}^{-1/6}}\prod_{1\le i<j\le 4}z_{ij}^{-1/6}$$
$$\cdot\left[-\frac{1}{6}z_{12}^{-7/6}z_{13}^{-1/6}z_{14}^{-1/6}-\frac{1}{6}z_{12}^{-1/6}z_{13}^{-7/6}z_{14}^{-1/6}-\frac{1}{6}z_{12}^{-1/6}z_{13}^{-1/6}z_{14}^{-7/6}\right] \tag{4.134}$$
$$=\left(\frac{\eta}{z_{14}}-\frac{\eta}{z_{12}}\right)\left[f_1'(\eta)-\frac{1}{6}f_1(\eta)\left(\frac{1}{z_{12}}+\frac{1}{z_{13}}+\frac{1}{z_{14}}\right)\right]\prod_{1\le i<j\le 4}z_{ij}^{-1/6}. \tag{4.135}$$

Substituting this into Equation (4.133) and multiplying the entire equation by $\prod_{1\le i<j\le 4}z_{ij}^{1/6}$ gives

$$0=\left(\frac{\eta}{z_{14}}-\frac{\eta}{z_{12}}\right)\left[f_1'(\eta)-\frac{1}{6}f_1(\eta)\left(\frac{1}{z_{12}}+\frac{1}{z_{13}}+\frac{1}{z_{14}}\right)\right]-\frac{1}{6}\frac{f_1(\eta)}{z_2-z_1}+\frac{1}{6}\frac{f_1(\eta)}{z_3-z_1}$$
$$-\frac{1}{6}\frac{f_1(\eta)}{z_4-z_1}-\frac{1}{3}\frac{f_2(\eta)}{z_2-z_1}-\frac{1}{3}\frac{f_3(\eta)}{z_4-z_1}. \tag{4.136}$$

To simplify the above expression, we choose a convenient collection of coordinates for z_i. In particular, we choose

$$z_1=0 \qquad z_2=1 \qquad z_3=z+1 \qquad z_4=\infty. \tag{4.137}$$

This choice of coordinates gives $\eta=z$. Note that we can transform these coordinates to any other choice of coordinates using a Möbius transformation. Since our correlation function is invariant under conformal transformations, we can compute it for any choice of coordinates provided that we can compute it for the previously defined coordinates. Substituting these coordinates into Equation (4.136) gives

$$0=zf_1'(z)+\frac{f_1(z)}{6}+\frac{f_1(z)}{6(z+1)}-\frac{f_1(z)}{6}+\frac{f_1(z)}{6(z+1)}-\frac{f_2(z)}{3} \tag{4.138}$$
$$=zf_1'(z)+\frac{f_1(z)}{3(z+1)}-\frac{f_2(z)}{3}. \tag{4.139}$$

To obtain a second differential equation, we will substitute $\langle\psi(z_1)\phi(z_2)\phi(z_3)\psi(z_4)\rangle$ into the $i=1$ KZ equation. We have $\rho_1(F_0)\otimes\rho_j(E_0)\langle\psi(z_1)\phi(z_2)\phi(z_3)\psi(z_4)\rangle=0$ for all j and

$$\rho_1(E_0)\otimes\rho_2(F_0)\langle\psi(z_1)\phi(z_2)\phi(z_3)\psi(z_4)\rangle=\langle\phi(z_1)\psi(z_2)\phi(z_3)\psi(z_4)\rangle \tag{4.140}$$
$$\rho_1(E_0)\otimes\rho_3(F_0)\langle\psi(z_1)\phi(z_2)\phi(z_3)\psi(z_4)\rangle=\langle\phi(z_1)\phi(z_2)\psi(z_3)\psi(z_4)\rangle \tag{4.141}$$
$$\rho_1(E_0)\otimes\rho_4(F_0)\langle\psi(z_1)\phi(z_2)\phi(z_3)\psi(z_4)\rangle=0 \tag{4.142}$$
$$\rho_1(H_0)\otimes\rho_2(H_0)\langle\psi(z_1)\phi(z_2)\phi(z_3)\psi(z_4)\rangle=-\frac{1}{2}\langle\psi(z_1)\phi(z_2)\phi(z_3)\psi(z_4)\rangle \tag{4.143}$$
$$\rho_1(H_0)\otimes\rho_3(H_0)\langle\psi(z_1)\phi(z_2)\phi(z_3)\psi(z_4)\rangle=-\frac{1}{2}\langle\psi(z_1)\phi(z_2)\phi(z_3)\psi(z_4)\rangle \tag{4.144}$$

$$\rho_1(H_0) \otimes \rho_4(H_0) \langle \psi(z_1)\phi(z_2)\phi(z_3)\psi(z_4)\rangle = \frac{1}{2} \langle \psi(z_1)\phi(z_2)\phi(z_3)\psi(z_4)\rangle . \tag{4.145}$$

We then obtain

$$\begin{aligned}0 = &\left(\frac{\eta}{z_{14}} - \frac{\eta}{z_{12}}\right) \partial_\eta \left(f_2(\eta) \prod_{1\le i<j\le 4} z_{ij}^{-1/6}\right) - \frac{1}{6}\frac{\langle \psi(z_1)\phi(z_2)\phi(z_3)\psi(z_4)\rangle}{z_2 - z_1} \\ &- \frac{1}{6}\frac{\langle \psi(z_1)\phi(z_2)\phi(z_3)\psi(z_4)\rangle}{z_3 - z_1} + \frac{1}{6}\frac{\langle \psi(z_1)\phi(z_2)\phi(z_3)\psi(z_4)\rangle}{z_4 - z_1} - \frac{1}{3}\frac{\langle \phi(z_1)\psi(z_2)\phi(z_3)\psi(z_4)\rangle}{z_2 - z_1} \\ &+ \frac{1}{3}\frac{\langle \phi(z_1)\phi(z_2)\psi(z_3)\psi(z_4)\rangle}{z_3 - z_1}.\end{aligned} \tag{4.146}$$

Repeating the same process as before, this simplifies to

$$\begin{aligned}0 = &\left(\frac{\eta}{z_{14}} - \frac{\eta}{z_{12}}\right) \left[f_2'(\eta) - \frac{1}{6} f_2(\eta) \left(\frac{1}{z_{12}} + \frac{1}{z_{13}} + \frac{1}{z_{14}}\right)\right] - \frac{1}{6}\frac{f_2(\eta)}{z_2 - z_1} - \frac{1}{6}\frac{f_2(\eta)}{z_3 - z_1} \\ &+ \frac{1}{6}\frac{f_2(\eta)}{z_4 - z_1} - \frac{1}{3}\frac{f_1(\eta)}{z_2 - z_1} + \frac{1}{3}\frac{f_4(\eta)}{z_3 - z_1}.\end{aligned} \tag{4.147}$$

Changing coordinates results in

$$0 = z f_2'(z) + \frac{f_2(z)}{6} + \frac{f_2(z)}{6(z+1)} - \frac{f_2(z)}{6} - \frac{f_2(z)}{6(z+1)} - \frac{f_1(z)}{3} + \frac{f_4(z)}{3(z+1)} \tag{4.148}$$

$$= z f_2'(z) - \frac{f_1(z)}{3} + \frac{f_4(z)}{3(z+1)}. \tag{4.149}$$

To simplify our system of differential equations given by Equation (4.139) and Equation (4.149), we will use an identity called an affine Ward identity. This can easily be derived by noting that

$$0 = \langle f_0 \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\psi_4(z_4)\rangle \tag{4.150}$$

$$\begin{aligned}= &\langle \{f_0, \phi_1(z_1)\}\phi_2(z_2)\phi_3(z_3)\psi_4(z_4)\rangle - \langle \phi_1(z_1)\{f_0, \phi_2(z_2)\}\phi_3(z_3)\psi_4(z_4)\rangle \\ &+ \langle \phi_1(z_1)\phi_2(z_2)\{f_0, \phi_3(z_3)\}\psi_4(z_4)\rangle - \langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\{f_0, \psi_4(z_4)\}\rangle .\end{aligned} \tag{4.151}$$

Here, $\{\cdot,\cdot\}$ denotes the anticommutator. Now, writing

$$F_0 = \oint_0 F(z)dz \tag{4.152}$$

and manipulating contours gives us

$$\{F_0, \phi(z)\} = \frac{1}{2\pi i}\oint_0 F(z)\phi(w)dz + \frac{1}{2\pi i}\oint_0 \phi(w)F(z)dz \tag{4.153}$$

$$= \frac{1}{2\pi i}\left[\underset{|z|>|w|}{\oint_0} - \underset{|z|<|w|}{\oint_0} \right] \mathcal{R}(F(z)\phi(w)) \tag{4.154}$$

$$= \frac{1}{2\pi i} \oint_w F(z)\phi(w) dz \tag{4.155}$$

where $\phi(z)$ is any of our chosen fields. Here, we have used the result

$$\mathcal{R}(F(z)\phi(w)) = -\mathcal{R}(\phi(w)F(z)), \tag{4.156}$$

which we used in our derivation of the KZ equations. For our choice of fields, the state field correspondence gives

$$F(z)\phi_i(w_i) \sim \frac{F_0\phi_i(w_i)}{z - w_i}. \tag{4.157}$$

We conclude that $\{F_0, \psi(z_i)\} = 0$ and $\{F_0, \phi(z_i)\} = \psi(z_i)$. Substituting this into Equation (4.151), we arrive at

$$\begin{aligned} 0 = \langle \psi_1(z_1)\phi_2(z_2)\phi_3(z_3)\psi_4(z_4)\rangle - \langle \phi_1(z_1)\psi_2(z_2)\phi_3(z_3)\psi_4(z_4)\rangle \\ + \langle \phi_1(z_1)\phi_2(z_2)\psi_3(z_3)\psi_4(z_4)\rangle . \end{aligned} \tag{4.158}$$

This implies

$$0 = f_2(z) - f_1(z) + f_4(z) \tag{4.159}$$

and so we obtain the coupled system of two differential equations

$$0 = zf_1'(z) + \frac{f_1(z)}{3(z+1)} - \frac{f_1(z)}{3} + \frac{f_4(z)}{3} \tag{4.160}$$

$$0 = z(f_1'(z) - f_4'(z)) - \frac{f_1(z)}{3} + \frac{f_4(z)}{3(z+1)}. \tag{4.161}$$

The first of these equations allows us to write

$$f_4(z) = \frac{z}{z+1} f_1(z) - 3zf_1'(z). \tag{4.162}$$

Substituting this into the second equation above and simplifying gives

$$0 = 9z^2(z+1)f_1''(z) + 9z(z+1)f_1'(z) - \left(\frac{z^2+4z+1}{z+1}\right) f_1(z). \tag{4.163}$$

Solving this in Mathematica gives

$$f_1(z) = \frac{c_1(z+1)^{2/3}}{z^{1/3}} - \frac{3}{2}c_2 z^{1/3}(z+1)^{1/3}\left((z+1)^{1/3}\, {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{5}{3}; -z\right) - 2\right) \tag{4.164}$$

where c_1 and c_2 are unknown constants. Our expression for $f_4(z)$ in terms of $f_1(z)$ then gives

$$f_4(z) = \frac{2c_1 - 3c_2 z^{2/3}\, {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{5}{3}; -z\right)}{2z^{1/3}(z+1)^{1/3}}. \tag{4.165}$$

Our affine Ward identity above gives

$$f_2(z) = \frac{1}{2}\left(\frac{z}{z+1}\right)^{1/3}\left(-3c_2 z\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{5}{3};-z\right) + 2c_1 z^{1/3} + 6c_2(z+1)^{2/3}\right). \tag{4.166}$$

We can obtain the remaining 3 correlators by using more of the affine Ward identities. Repeating our derivation for the affine Ward identities using the correlators $\langle f_0\psi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)\rangle$, $\langle f_0\phi(z_1)\psi(z_2)\phi(z_3)\phi(z_4)\rangle$, and $\langle f_0\phi(z_1)\phi(z_2)\psi(z_3)\phi(z_4)\rangle$ respectively gives

$$0 = -f_3 + f_6 - f_2 \tag{4.167}$$
$$0 = f_3 + f_5 - f_1 \tag{4.168}$$
$$0 = f_6 - f_5 - f_4. \tag{4.169}$$

From this, we can deduce

$$f_3(z) = \frac{1}{2}(f_1(z) + f_4(z) - f_2(z)) \tag{4.170}$$
$$= \frac{1}{4z^{1/3}(z+1)^{1/3}}\Bigg[-6c_2 z^{2/3}\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{5}{3};-z\right) - 6c_2\left(-(z+1)^{2/3}z^{1/3} + \left(\frac{z}{z+1}\right)^{1/3} z + \left(\frac{z}{z+1}\right)^{1/3}\right) z^{1/3} + 4c_1\Bigg]. \tag{4.171}$$

$$f_5(z) = f_1(z) - f_3(z) \tag{4.172}$$
$$= \frac{1}{2(z+1)^{1/3}}\Bigg[-3c_2 z^{4/3}\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{5}{3};-z\right) + 2c_1 z^{2/3} + 3c_2(z+1)^{2/3}z^{1/3} + 3c_2\left(\frac{z}{z+1}\right)^{1/3} z + 3c_2\left(\frac{z}{z+1}\right)^{1/3}\Bigg]. \tag{4.173}$$

$$f_6 = f_4 + f_5 \tag{4.174}$$
$$= \frac{1}{2z^{1/3}(z+1)^{1/3}}\Bigg[-3c_2(z+1)z^{2/3}\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{5}{3};-z\right) + 3c_2\left((z+1)^{2/3}z^{1/3} + \left(\frac{z}{z+1}\right)^{1/3} z + \left(\frac{z}{z+1}\right)^{1/3}\right) z^{1/3} + 2c_1(z+1)\Bigg]. \tag{4.175}$$

Recall that the correlators we are solving for are given by multiplying the functions f_i by $\prod_{1\le i<j\le 4} z_{ij}^{-1/6}$.

4.5 Free Field Realisation

In this section, we give an example of a free field realisation for the $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ level 1 WZW model, which is provided in [8, p. 654]. A free field realisation allows us to realise fields in this WZW model as fields of the free boson. We will use a free field realisation in Section 5.5 to compute the four-point functions from Section 4.4. We avoid introducing the free boson and vertex operators in this section and instead refer the unfamiliar reader to Appendix F.

Consider the fields

$$E(z) = \sum_{n \in \mathbb{Z}} E_n z^{-n-1} \qquad F(z) = \sum_{n \in \mathbb{Z}} F_n z^{-n-1} \qquad H(z) = \sum_{n \in \mathbb{Z}} H_n z^{-n-1}, \tag{4.176}$$

where we take $\{E, F, H\}$ to be the basis of $\mathfrak{sl}_2(\mathbb{C})$ defined in Example 1.2.1. This basis is given by

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}. \tag{4.177}$$

This choice gives us the commutation relations

$$[E, F] = \sqrt{2} H \qquad [H, E] = \sqrt{2} E \qquad [H, F] = -\sqrt{2} F. \tag{4.178}$$

We can then compute the relevant OPEs using Equation (3.30). The results are

$$H(z)H(w) \sim \frac{1}{(z-w)^2} \tag{4.179}$$

$$H(z)E(w) \sim \frac{\sqrt{2}E(w)}{(z-w)} \tag{4.180}$$

$$H(z)F(w) \sim \frac{-\sqrt{2}F(w)}{(z-w)} \tag{4.181}$$

$$E(z)F(w) \sim \frac{\sqrt{2}H(w)}{(z-w)} + \frac{1}{(z-w)^2} \tag{4.182}$$

$$E(z)E(w) \sim 0 \tag{4.183}$$

$$F(z)F(w) \sim 0. \tag{4.184}$$

Consider the maps

$$\partial\phi \mapsto H \qquad V_{\sqrt{2}} \mapsto E \qquad V_{-\sqrt{2}} \mapsto F. \tag{4.185}$$

We now wish to show that OPEs are preserved under the maps defined above. That is, we would like to show

$$\partial\phi(z) V_{\sqrt{2}}(w) \sim \frac{\sqrt{2} V_{\sqrt{2}}(w)}{z-w} \tag{4.186}$$

$$\partial\phi(z)V_{-\sqrt{2}}(w) \sim \frac{-\sqrt{2}V_{-\sqrt{2}}(w)}{z-w} \tag{4.187}$$

$$V_{\sqrt{2}}(z)V_{-\sqrt{2}}(w) \sim \frac{\sqrt{2}\partial\phi(w)}{z-w} + \frac{1}{(z-w)^2}. \tag{4.188}$$

For the first two OPEs, we use the state-field correspondence and the result that the field $V_p(z)$ corresponds to the state $|p\rangle$. Using

$$\partial\phi(z)V_p(w) = \sum_n \psi_n(w)(z-w)^{-n-1} \tag{4.189}$$

gives

$$|\psi_n\rangle = a_n\,|p\rangle \tag{4.190}$$

for $n \in \mathbb{Z}$. For $n > 0$, we know this is zero. Also, $|\psi_0\rangle = p\,|p\rangle$. This gives the first two OPEs. For the final OPE, we simply note that

$$V_p(z)V_q(w) = (z-w)^{pq}\Big\{V_{p+q}(w) + p : \partial\phi(w)V_{p+q}(w) : (z-w)$$
$$+\frac{1}{2}[p^2 : \partial\phi(w)\partial\phi(w)V_{p+q}(w) : +p : \partial^2\phi(w)V_{p+q}(w) :](z-w)^2 + \cdots\Big\}. \tag{4.191}$$

This follows from Equations (F.36)-(F.38). Substituting $p = \sqrt{2} = -q$ and using the fact that $V_0(w)$ is the identity field gives the final OPE.

Note that the OPE $V_p(z)V_q(w)$ allows us to also show that the energy-momentum tensor is preserved under our free field realisation. The energy-momentum tensor in our basis for $\mathfrak{sl}_2(\mathbb{C})$ is

$$T(z) = \frac{1}{6}\left(: H(z)H(z) : + : E(z)F(z) : + : F(z)E(z) :\right). \tag{4.192}$$

We then have

$$T(z) \mapsto \frac{1}{6}\big(: \partial\phi(z)\partial\phi(z) : + : V_{\sqrt{2}}(z)V_{\sqrt{-2}}(z) : + : V_{-\sqrt{2}}(z)V_{\sqrt{2}}(z) :\big) \tag{4.193}$$
$$= \frac{1}{2} : \partial\phi(z)\partial\phi(z) : . \tag{4.194}$$

This gives the energy-momentum tensor of the free boson. In Chapter 5, we will formalise this free field realisation. Formally, our free field realisation defines a homomorphism of vertex operator algebras. This will allow us to compute the 4-point functions of Section 4.4 using vertex operators.

Chapter 5

Vertex Operator Algebras

In this chapter, we will discuss vertex operator algebras. These mathematical objects give a formalisation of the chiral sector of a CFT. In particular, we will give constructions that are analogous to the CFTs discussed previously, namely the free boson and WZW models. One of our main goals in this chapter will be developing the mathematical machinery required to give an alternative calculation of the four-point functions we considered earlier. The main definitions and theorems presented in this chapter were obtained from [12]. The texts [27] and [20] were also found to be useful.

5.1 Introduction

We saw earlier that affine Kac-Moody algebras are used to construct the fields of the WZW models. In this chapter, we will see them in the context of vertex operator algebras. However, we will modify the definition slightly. This modified definition of affine Kac-Moody algebra in this section will not be important for our purposes. In the geometric realisation of vertex algebras, it is important as it allows one to view elements of the loop algebra as Lie algebra-valued meromorphic functions. We introduce the modified definition to maintain consistency with [12].

Definition 5.1.1. In this chapter, we commonly denote by $\mathbb{C}[[z]]$ the space of formal power series in the variable z. We will denote by $\mathbb{C}((z))$ the space of formal Laurent series in the variable z. We call a formal power series

$$\sum_{n\in\mathbb{Z}} a_n z^n \tag{5.1}$$

a formal Laurent series if there exists some $N \in \mathbb{Z}$ such that $a_n = 0$ for all $n \leq N$.[1]

Definition 5.1.2. Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra. The loop algebra of $\mathfrak{g}$ is defined as

$$\mathfrak{g} \otimes \mathbb{C}((t)). \tag{5.2}$$

[1] This condition allows us to multiply formal Laurent series.

The bracket on $\mathfrak{g} \otimes \mathbb{C}((t))$ is given by Equation (2.11), extended by linearity to formal power series in t^n.

Definition 5.1.3. Given a complex finite-dimensional simple Lie algebra $\mathfrak{g}$ with Killing form κ, its corresponding affine Kac-Moody algebra $\hat{\mathfrak{g}}$ is defined by the central extension

$$0 \to \mathbb{C}K \to \hat{\mathfrak{g}} \to \mathfrak{g} \otimes \mathbb{C}((t)) \to 0, \tag{5.3}$$

where the bracket on $\hat{\mathfrak{g}}$ is given by Equation (2.13).

Note that in this definition, we did not add the element d given in our previous definition. This is not necessary for us since the Sugawara construction allows us to realise L_0 as an element satisfying the same relations as d.

Definition 5.1.4. Let $\mathfrak{g}$ be a Lie algebra with a triangular decomposition $\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$. Let V be a representation of $\mathfrak{g}_0$. Then V is a representation of $\mathfrak{g}_0 \oplus \mathfrak{g}_+$ where $\mathfrak{g}_+$ acts trivially. Define

$$\operatorname{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_+}^{\mathfrak{g}} V = U(\mathfrak{g}) \underset{U(\mathfrak{g}_0 \oplus \mathfrak{g}_+)}{\otimes} V. \tag{5.4}$$

This is a representation of $\mathfrak{g}$ called the induced representation. Given $X \in \mathfrak{g}$, $Y \in U(\mathfrak{g})$, $v \in V$, the action is defined by

$$X \cdot (Y \otimes v) = (X \cdot Y) \otimes v. \tag{5.5}$$

We will now define the main object of interest in this chapter.

Definition 5.1.5. A vertex algebra consists of

1. A $\mathbb{Z}$-graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$ satisfying the following[2]:
   (a) $\dim V_n < \infty$.
   (b) There exists $N \in \mathbb{N}$ such that $V_n = 0$ for all $n < N$.
2. A linear map $Y(\cdot, z) : V \to (\operatorname{End} V)[[z]]$ such that for $A \in V_m$, the operators A_n of the field
   $$Y(A, z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-1} \tag{5.6}$$
   have degree $m - n - 1$. An operator $f \in \operatorname{End} V$ has degree α if $f(V_n) \subseteq V_{n+\alpha}$ for all n. Alternatively, we will say that the field $Y(A, z)$ has degree m.
3. A vector $|0\rangle \in V_0$.
4. A linear map $T : V \to V$ of degree 1.

We require that the objects defined above satisfy the following axioms:

[2] This space only aims to formalise certain CFTs. There are examples of CFTs where these conditions do not hold.

1. For all $A \in V$, $Y(A, z) |0\rangle$ is well-defined at $z = 0$ and we have
$$Y(A, z) |0\rangle |_{z=0} = A. \tag{5.7}$$
2. For all $A \in V$,
$$\sum_{n \in \mathbb{Z}} [T, A_n] z^{-n-1} = \partial_z Y(A, z). \tag{5.8}$$
3. $T |0\rangle = 0$.
4. Given two fields $Y(A, z)$ and $Y(B, w)$, there exists $n \in \mathbb{N}$ such that
$$(z - w)^n [Y(A, z), Y(B, w)] = 0 \tag{5.9}$$
as an element of $\operatorname{End} V[[z, z^{-1}, w, w^{-1}]]$. If $Y(A, z)$ and $Y(B, w)$ satisfy this equation, we say they are mutually local.

By the map $Y(\cdot, z) : V \to (\operatorname{End} V)[[z]]$ being linear, we mean that
$$Y\left(\sum_{n \in \mathbb{Z}} a_n v z^{-n-1}, z\right) = \sum_{n \in \mathbb{Z}} z^{-n-1} Y(a_n v, z). \tag{5.10}$$
We can think of this map as the state-field correspondence, which we discussed earlier in the context of conformal field theory. We will now give motivation for the rest of the axioms. Recall that in the Wess-Zumino-Witten models, states are given by elements of the form
$$J^{a_1}_{i_1} J^{a_2}_{i_2} \cdots J^{a_n}_{i_n} |p\rangle \tag{5.11}$$
with $|p\rangle$ a highest weight vector with weight p and J^{a_i} generators of the Lie algebra of interest. We can define a $\mathbb{Z}$-grading on the space of states V with the function
$$f : V \to \mathbb{Z} \tag{5.12}$$
$$J^{a_1}_{i_1} J^{a_2}_{i_2} \cdots J^{a_n}_{i_n} |p\rangle \mapsto -\sum_{k=1}^{n} i_k. \tag{5.13}$$
Note that this function is chosen so that $V_n = 0$ for sufficiently small n. Furthermore, the function ensures that the dimension of each V_n is finite.

Our assumption that there exists a vector $|0\rangle \in V_0$ is the analogue of assuming the existence of our vacuum state in the context of conformal field theory. We will also see shortly that the linear map $T : V \to V$ generalises the mode L_{-1} of the energy-momentum tensor of a CFT. The first axiom of the objects of a vertex algebra is the state-field correspondence. It is easy to show that the second axiom is equivalent to $[T, A_n] = -nA_{n-1}$. This generalises the relation $[L_n, J^a_m] = -mJ^a_{n+m}$ of the WZW models. We can see this by looking at the definition of a vertex operator algebra, where T corresponds to L_{-1} and L_n are operators corresponding to a field which is an analogue of the energy-momentum tensor. Furthermore, we can recognise the third axiom as a generalisation of a defining property of the vacuum in CFT: namely, that $L_{-1} |0\rangle = 0$. Finally, the locality axiom ensures that OPEs have finitely many singular terms.

Definition 5.1.6. A vertex operator algebra of central charge c is a vertex algebra V_2 with a vector $\omega \in V$ called a conformal vector. This vector satisfies the following:

1. The commutation relations of the operators L_n corresponding to the field

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \tag{5.14}$$

agree with the Lie bracket on the Virasoro algebra with central charge c. We call $Y(\omega, z)$ the energy-momentum tensor of the vertex operator algebra.

2. $L_{-1} = T$.

3. $L_0|_{V_n} = n\ id_{V_n}$.

Definition 5.1.7. We define the normally ordered product of the two fields

$$A(z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-1}, \qquad B(w) = \sum_{n \in \mathbb{Z}} B_n w^{-n-1} \tag{5.15}$$

as

$$: A(z)B(w) := \sum_{n \in \mathbb{Z}} \left(\sum_{m \leq -1} A_m B_n z^{-m-1} + \sum_{m > -1} B_n A_m z^{-m-1} \right) w^{-n-1}. \tag{5.16}$$

Note the similarity to Equation (3.49) in the context of CFT. In the case of $A(z)$ and $B(z)$, we have

$$: A(z)B(z) := \sum_{n \in \mathbb{Z}} \left(\sum_{m \leq -1} A_m B_{n-m} + \sum_{m > -1} B_{n-m} A_m \right) z^{-n-2}. \tag{5.17}$$

This agrees with the expression we would have in CFT if $A(z)$ and $B(z)$ had conformal weight 1. Before we give an example of a vertex operator algebra, we prove an important theorem that can help us verify that a particular structure defines a vertex operator algebra. To do this, we begin with a lemma.

Lemma 5.1.1. *Let $A(z)$ and $B(z)$ be fields of degree h_A and h_B respectively. Then $: A(z)B(z) :$ is a field of degree dimension $h_A + h_B$.*

Proof. Assume $A(z)$ and $B(z)$ are fields of degree h_A and h_B respectively. Writing

$$: A(z)B(z) := \sum_{n \in \mathbb{Z}} \left(\sum_{m \leq -1} A_m B_{n-m-1} + \sum_{m > -1} B_{n-m-1} A_m \right) z^{-n-1} \tag{5.18}$$

we see that it suffices to show the operator

$$\sum_{m \leq -1} A_m B_{n-m-1} + \sum_{m > -1} B_{n-m-1} A_m \tag{5.19}$$

is homogeneous of degree $h_A + h_B - n - 1$. Let $v \in V_l$. Then

$$A_m V_l \in V_{h_A - m - 1 + l} \tag{5.20}$$

and so

$$B_{n-m-1} A_m v_l \in V_{h_A + h_B - n - 1 + l}. \tag{5.21}$$

Similarly,

$$A_m B_{n-m-1} v_l \in V_{h_A + h_B - n - 1 + l}. \tag{5.22}$$

Therefore, the operator above is homogeneous of degree $h_A + h_B - n - 1$ as required. □

Definition 5.1.8. The formal delta function is

$$\delta(z-w) = \sum_{m \in \mathbb{Z}} z^m w^{-m-1}. \tag{5.23}$$

Note that we can write

$$\delta(z-w) = \frac{1}{z} \sum_{n \geq 0} \left(\frac{w}{z}\right)^n + \frac{1}{z} \sum_{n > 0} \left(\frac{z}{w}\right)^n. \tag{5.24}$$

We call the first term $\delta(z-w)_-$ and the second term $\delta(z-w)_+$.

Lemma 5.1.2. *We have [12, p. 12]*

$$(z-w)^{n+1} \partial_w^n \delta(z-w) = 0. \tag{5.25}$$

Lemma 5.1.3. *We have the equality*

$$: A(w)B(w) := \underset{z=0}{Res} \left[\delta(z-w)_- A(z)B(w) + \delta(z-w)_+ B(w)A(z)\right]. \tag{5.26}$$

Proof. We have

$$\delta(z-w)_- A(z)B(w) = \sum_{n \geq 0} \sum_{m,l \in \mathbb{Z}} A_m B_l w^{n-l-1} z^{-m-n-2} \tag{5.27}$$

$$\delta(z-w)_+ B(w)A(z) = \sum_{n > 0} \sum_{m,l \in \mathbb{Z}} B_m A_l z^{n-l-2} w^{-n-m-1}. \tag{5.28}$$

So,

$$\underset{z=0}{\text{Res}} \left[\delta(z-w)_- A(z)B(w)\right] = \sum_{n \geq 0} \sum_{m,l \in \mathbb{Z}} A_m B_l w^{n-l-1} \delta_{m+n,-1} \tag{5.29}$$

$$= \sum_{l \in \mathbb{Z}} \sum_{m \leq -1} A_m B_l w^{-l-m-2}. \tag{5.30}$$

$$\underset{z=0}{\text{Res}} \left[\delta(z-w)_+ B(w)A(z)\right] = \sum_{n > 0} \sum_{m,l \in \mathbb{Z}} B_m A_l w^{-n-m-1} \delta_{n-l,1} \tag{5.31}$$

$$= \sum_{l \in \mathbb{Z}} \sum_{m > -1} B_l A_m w^{-l-m-2}. \tag{5.32}$$

Adding these two residues gives the result. □

Lemma 5.1.4. *Dong's Lemma.* *If $A(z), B(z)$, and $C(z)$ are mutually local then $: A(z)B(z) :$ and $C(z)$ are mutually local.*

For a proof of this lemma, see [12, p. 35].

5.2 Examples

Example 5.2.1. Fock Representation [12, p. 28]. The vacuum Fock representation is the analogue of the vacuum module in the chiral sector of the free boson in CFT. To construct this, we begin by defining the Weyl algebra $\widetilde{\mathcal{H}}$. The Heisenberg Lie algebra $\mathcal{H}$ is defined by the central extension

$$0 \to \mathbb{C}K \to \mathcal{H} \to \mathbb{C}((t)) \to 0 \tag{5.33}$$

with cocycle $c(f, g) = -\operatorname*{Res}_{t=0} f dg$. From this, it is easy to deduce the brackets

$$[t^n, t^m] = n\delta_{n,-m}K \qquad [t^n, c] = 0. \tag{5.34}$$

Note that we have a basis for $\mathcal{H}$ given by $a_n := t^n$ for $n \in \mathbb{Z}$ and K. We now define

$$\widetilde{\mathcal{H}} = U(\mathcal{H})/(K - 1) \tag{5.35}$$

where 1 is the unit in $U(\mathcal{H})$.

Let $\widetilde{\mathcal{H}}_+$ be the subalgebra of $\widetilde{\mathcal{H}}$ generated by a_n for $n \in \mathbb{Z}_{\geq 0}$. We define the Fock representation to be

$$\pi = \operatorname{Ind}_{\widetilde{\mathcal{H}}_+}^{\widetilde{\mathcal{H}}} \mathbb{C} \tag{5.36}$$

where $\mathbb{C}$ is the one-dimensional trivial representation of $\widetilde{\mathcal{H}}_+$. We can now endow π with a vertex algebra structure. First, let $\widetilde{\mathcal{H}}_-$ be the subalgebra of $\widetilde{\mathcal{H}}$ generated by a_n for $n < 0$. The Poincaré-Birkhoff-Witt theorem then gives us the vector space isomorphism $\widetilde{\mathcal{H}} \cong \widetilde{\mathcal{H}}_- \otimes \widetilde{\mathcal{H}}_+$ and so

$$\pi \cong (\widetilde{\mathcal{H}}_- \otimes \widetilde{\mathcal{H}}_+) \otimes_{\widetilde{\mathcal{H}}_+} \mathbb{C} \cong \widetilde{\mathcal{H}}_- \tag{5.37}$$

as vector spaces. Here, we have also used the associativity of the tensor product and the result that

$$R \otimes_R M \cong M \tag{5.38}$$

for a left R-module M. We can now choose our vacuum vector $|0\rangle$ to be the unit 1. By the isomorphism above, any element of π can be written in the form $a_{i_1} a_{i_2} \cdots a_{i_k} |0\rangle$ with $i_j \leq i_{j+1}$ and $i_j < 0$. We can then introduce a grading by $\deg(a_{i_1} a_{i_2} \cdots a_{i_k} |0\rangle) = -\sum_j i_j$. For the Fock representation, we only have the one generating field

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}. \tag{5.39}$$

From these facts, we see that π satisfies the first three requirements of Theorem G.0.1. We now define a translation operator T by $T(1) = 0$ and $[T, a_n] = -na_{n-1}$. Given some element $a_{i_1} a_{i_2} \cdots a_{i_k} \cdot 1$, the action of T is given by "commuting T through" the modes a_n. For example, given $a_{-1}a_{-2} \cdot 1$, we have

$$\begin{aligned} T(a_{-1}a_{-2} \cdot 1) &= [T, a_{-1}]a_{-2} \cdot 1 + a_{-1}T(a_{-2} \cdot 1) && (5.40) \\ &= a_{-2}a_{-2} \cdot 1 + a_{-1}[T, a_{-2}] \cdot 1 + a_{-1}a_{-2}T(1) && (5.41) \end{aligned}$$

$$= a_{-2}a_{-2} \cdot 1 + 2a_{-1}a_{-3} \cdot 1. \tag{5.42}$$

We also have

$$[T, a(z)] = \sum_{n\in\mathbb{Z}} [T, a_n] z^{-n-1} = \sum_{n\in\mathbb{Z}} -na_{n-1} z^{-n-1} = \sum_{n\in\mathbb{Z}} -(n+1)a_n z^{-n-2} = \partial_z a(z). \tag{5.43}$$

Finally, note that T has degree 1 since every time we commute T through some element $a_{i_1}a_{i_2}\cdots a_{i_k}|0\rangle$, we obtain an element of degree $1 + \deg(a_{i_1}a_{i_2}\cdots a_{i_k}|0\rangle)$. By the reconstruction theorem, we conclude that π has the structure of a vertex algebra where we define

$$Y(a_{i_1}a_{i_2}\cdots a_{i_k}, z) = \frac{:\partial_z^{-i_1-1}a(z)\cdots\partial_z^{-i_k-1}a(z):}{(-i_1-1)!\cdots(-i_k-1)!}. \tag{5.44}$$

In fact, π is a vertex operator algebra with energy-momentum tensor given by

$$\frac{1}{2} : a(z)a(z) : . \tag{5.45}$$

Example 5.2.2. The Vacuum Representation [12, p. 39]. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. The vacuum representation $V_k(\mathfrak{g})$ is a vertex algebra that can be used to construct the chiral sector of the $\widehat{\mathfrak{g}}$ WZW model at level k. We define

$$V_k(\mathfrak{g}) = \operatorname{Ind}_{\mathfrak{g}[[t]]\oplus\mathbb{C}K}^{\hat{\mathfrak{g}}}\mathbb{C}. \tag{5.46}$$

Here, $\mathfrak{g}[[t]] = \mathfrak{g}\otimes\mathbb{C}[[t]]$ is a subalgebra of the loop algebra of $\mathfrak{g}$. The element K is the central element. We view $\mathbb{C}$ as a one-dimensional representation of $\mathfrak{g}[[t]]\oplus\mathbb{C}K$ with action defined by

$$(X \oplus \alpha K)\cdot\beta = \alpha k \beta. \tag{5.47}$$

Note that we have the vector space isomorphisms

$$V_k(\mathfrak{g}) \cong \big(U(\mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}])\otimes U(\mathfrak{g}[[t]]\oplus\mathbb{C}K)\big) \underset{U(\mathfrak{g}[[t]]\oplus\mathbb{C}K)}{\otimes} \mathbb{C} \tag{5.48}$$

$$\cong U(\mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}]). \tag{5.49}$$

The first isomorphism above follows from the PBW theorem. The second isomorphism follows from the associativity of the tensor product and Equation (5.38). Let $\{J^a\}$ be an ordered basis for $\mathfrak{g}$. Then we have an ordered basis for $\hat{\mathfrak{g}}$ given by $\{K\}\cup\{J_n^a\}_{n\in\mathbb{Z}}$ with the ordering defined by

$$J_n^a \le J_m^b \text{ for } n \le m \tag{5.50}$$

$$J_m^a \le J_m^b \text{ for } a \le b \tag{5.51}$$

$$K \le J_n^a \text{ for all } n, a. \tag{5.52}$$

This also gives the ordered basis $\{J_n^a\}_{n\in\mathbb{Z}_{<0}}$ of $\mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}]$. By the PBW theorem, $U(\mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}])$ has a basis (as a vector space) given by the unit 1 and the elements

$$J_{n_1}^{a_1}J_{n_2}^{a_2}\cdots J_{n_k}^{a_k} \tag{5.53}$$

where $J^{a_i}_{n_i} \leq J^{a_{i+1}}_{n_{i+1}}$ and $n_i < 0$. Our vector space isomorphism then gives us a basis for $V_k(\mathfrak{g})$.

Again, we can use the reconstruction theorem to show that $V_k(\mathfrak{g})$ has the structure of a vertex algebra. Let our vacuum vector be the unit 1 and note that our vector space is generated by elements of the form $J^{a_1}_{n_1} J^{a_2}_{n_2} \cdots J^{a_k}_{n_k} \cdot 1$. Define the translation operator T by $T \cdot 1 = 0$ and $[T, J^a_n] = -nJ^a_{n-1}$. By the same argument as in the Fock representation example, this is a linear operator of degree 1. Furthermore, the first three conditions of the reconstruction theorem are satisfied. Our fields are

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}. \tag{5.54}$$

We can use these fields to define our vertex operators

$$Y(J^{a_1}_{i_1} J^{a_2}_{i_2} \cdots J^{a_k}_{i_k} \cdot 1, z) = \frac{: \partial_z^{-i_1-1} J^{a_1}(z) \cdots \partial_z^{-i_k-1} J^{a_k}(z) :}{(-i_1-1)! \cdots (-i_k-1)!}. \tag{5.55}$$

It remains to verify that our fields are mutually local. We compute

$$\begin{aligned}
[J^a(z), J^b(w)] &= \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} [J^a_n, J^b_m] z^{-n-1} w^{-m-1} &(5.56)\\
&= \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} [J^a, J^b]_{n+m} z^{-n-1} w^{-m-1} + \sum_{n\in\mathbb{Z}} n\kappa(J^a, J^b) K z^{-n-1} w^{n-1} &(5.57)\\
&= \sum_{k\in\mathbb{Z}} [J^a, J^b]_k \left(\sum_{n\in\mathbb{Z}} z^{-n-1} w^n \right) w^{k-1} + \sum_{n\in\mathbb{Z}} n\kappa(J^a, J^b) K z^{-n-1} w^{n-1} &(5.58)\\
&= [J^a, J^b](w)\delta(z-w) + \kappa(J^a, J^b) K \partial_w \delta(z-w). &(5.59)
\end{aligned}$$

Our result follows by Lemma 5.1.2.

Note that the Sugawara construction of Section 4.1 amounts to the existence of a conformal vector

$$\omega = \frac{1}{2(k+h^\vee)} \sum_{i=1}^{\dim\mathfrak{g}} J_{a,-1} J^a_{-1} 1 \tag{5.60}$$

of central charge

$$\frac{k \dim\mathfrak{g}}{k+h^\vee}. \tag{5.61}$$

As a consequence, the vacuum representation has the structure of a vertex operator algebra.

Example 5.2.3. [12, p. 71] Consider $V_k(\mathfrak{sl}_2(\mathbb{C}))$ as an $\widehat{\mathfrak{sl}_2}(\mathbb{C})$-module. Then if we quotient by the submodule generated by $E^{k+1}_{-1} \cdot 1$, we obtain an irreducible $\widehat{\mathfrak{sl}_2}(\mathbb{C})$-module. We denote the irreducible quotient by $L_k(\mathfrak{sl}_2(\mathbb{C}))$. Notice that the image of our basis under the quotient map gives us a spanning set for the module. However, this is no longer a basis and so we cannot apply the reconstruction theorem to show that our quotient is still a vertex algebra. This is fixed by proving a generalisation of the reconstruction theorem where we remove the requirement that our spanning set be a basis. The generalisation is called the strong reconstruction theorem and is proven in [12, p. 71]. We avoid proving the theorem in this thesis as it first requires establishing more results.

5.3 Vertex Algebra Modules

In this section, we will use the following theorem.

Theorem 5.3.1. *[12, p. 49] For any vertex algebra $(V, |0\rangle, T, Y)$ and for all $A, B, C \in V$, the expansions $Y(A,z)Y(B,w)C$, $Y(B,w)Y(A,z)C$, and $Y(Y(A,z-w),w)C$ are all equal in $W[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$.*

Definition 5.3.1. Let (V, Y) and (V', Y') be vertex operator algebras with conformal vectors ω and ω' respectively. Let $\{Y(A_i, z)\}_{i\in S}$ be the collection of fields of (V, Y) with $A_i \in V$ for all $i \in S$. Let $\{Y'(B_j, z)\}_{j\in T}$ be the collection of fields in (V', Y') with $B_j \in V'$ for all $j \in T$. A homomorphism between (V, Y) and (V', Y') is a map

$$\phi : \{1, Y(A_i, z)\}_{i\in S} \to \{1', Y'(B_j, z)\}_{j\in T} \tag{5.62}$$

satsifying

$$\phi(Y(A_i, z)Y(A_j, w)) = \phi(Y(A_i, z)\phi(Y(A_j, w)) \text{ for all } i, j \in S \tag{5.63}$$
$$\phi(Y(\omega, z)) = Y'(\omega', z) \tag{5.64}$$
$$\phi(1) = 1' \tag{5.65}$$

where 1 and $1'$ are the identity fields of (V, Y) and (V', Y') respectively.

Recall that our state space in the $\hat{\mathfrak{g}}$ Wess-Zumino-Witten model at level k was of the form

$$\bigoplus_{p\in X_{\hat{\mathfrak{g}}_k}} (V_p \otimes V_p) \tag{5.66}$$

where the V_p are integrable highest weight representations of $\hat{\mathfrak{g}}$ at level k and $X_{\hat{\mathfrak{g}}_k}$ are the dominant integral weights of $\hat{\mathfrak{g}}$ at level k. The aim of vertex algebra modules is to formalise these representations, and the corresponding representations for arbitrary chiral CFTs.

Definition 5.3.2. A module over a vertex algebra $(V, |0\rangle, T, Y)$ is a vector space W equipped with a map $Y_W : V \to (\operatorname{End} W)[[z, z^{-1}]]$ satisfying[3]:

1. $Y_W(|0\rangle, z) = id_W$.

[3]Here, id_W denotes the identity field. That is, $id_W = \sum_{n\in\mathbb{Z}} A_n z^{-n-1}$ where A_n acts as zero on W for $n \neq -1$ and A_{-1} acts as the identity on W.

2. For all $A, B \in V$, $C \in W$, the expansions

$$Y_W(A,z)Y_W(B,w)C, \quad Y_W(B,w)Y_W(A,z)C, \quad Y_W(Y(A,z-w),w)C \tag{5.67}$$

are all equal in $W[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$.

Recall that we introduced the Weyl algebra, which is defined by Equation 5.35. Before considering our first example of a module over a vertex algebra, we define a module π_λ over the Weyl algebra, which we will call a Fock module. Formally, this is the Verma module generated by some vector $|\lambda\rangle$ satisfying[4]

$$a_n |\lambda\rangle = 0 \text{ for } n > 0, \qquad a_0 |\lambda\rangle = \lambda |\lambda\rangle . \tag{5.68}$$

Example 5.3.1. Fock Lattice [12, p. 81]. Let $N \in \mathbb{N}$ be even. Define

$$V_{\sqrt{N}\mathbb{Z}} = \bigoplus_{n\in\mathbb{Z}} \pi_{n\sqrt{N}}. \tag{5.69}$$

We can view this as both a π-module and as a vertex operator algebra on its own. Our grading on π can be used to define a grading on π_λ. The diagram below shows the vector space structure of $V_{\sqrt{N}\mathbb{Z}}$.

$$\cdots \quad \begin{array}{c} \vdots \\ \hdashline a_{-2}\left|-\sqrt{N}\right\rangle \ \ (a_{-1})^2\left|-\sqrt{N}\right\rangle \\ \hdashline a_{-1}\left|-\sqrt{N}\right\rangle \\ \hdashline \left|-\sqrt{N}\right\rangle \end{array} \bigoplus \begin{array}{c} \vdots \\ \hdashline a_{-2}\left|0\right\rangle \ \ (a_{-1})^2\left|0\right\rangle \\ \hdashline a_{-1}\left|0\right\rangle \\ \hdashline \left|0\right\rangle \end{array} \bigoplus \begin{array}{c} \vdots \\ \hdashline a_{-2}\left|\sqrt{N}\right\rangle \ \ (a_{-1})^2\left|\sqrt{N}\right\rangle \\ \hdashline a_{-1}\left|\sqrt{N}\right\rangle \\ \hdashline \left|\sqrt{N}\right\rangle \end{array} \quad \cdots$$

To define the π-module structure, note that the vertex operator Y of the vacuum Fock representation is a map $Y : \pi \to (\operatorname{End} V_{\sqrt{N}\mathbb{Z}})[[z, z^{-1}]]$ satisfying $Y(|0\rangle) = id_{V_{\sqrt{N}\mathbb{Z}}}$. For example, we have

$$Y(a_{-1}|0\rangle, z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}. \tag{5.70}$$

Each mode a_n is an element of the Weyl algebra and so acts on all of the Fock modules $\pi_{n\sqrt{N}}$. We can thus view a_n as an element of $\operatorname{End} V_{\sqrt{N}\mathbb{Z}}$. Our map defines the structure of a vertex algebra module by Theorem 5.3.1.

Example 5.3.2. Any vertex algebra is a module over itself. If $(V, |0\rangle, T, Y)$ is a vertex algebra then the map defining the vertex algebra module is Y itself. In this case, the first condition of Definition 5.3.2 is clearly satisfied and the second condition is true by Theorem 5.3.1.

[4] This module is essentially the Verma module of the Heisenberg algebra with highest weight λ where the eigenvalue of the central element is fixed to be 1.

5.4 Level 1 $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ Free Field Realisation

In this section, we will give a correspondence between primary fields of the level 1 $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ WZW model and free boson vertex operators.

Theorem 5.4.1. *We have an isomorphism of vertex operator algebras*

$$V_{\sqrt{2}\mathbb{Z}} \cong L_1(\mathfrak{sl}_2(\mathbb{C})). \tag{5.71}$$

Proof. By definition of the vertex operators of $L_1(\mathfrak{sl}_2(\mathbb{C}))$, we know that the fields

$$E(z) \qquad H(z) \qquad F(z) \tag{5.72}$$

generate every field of $L_1(\mathfrak{sl}_2(\mathbb{C}))$ in the sense that any field of $L_1(\mathfrak{sl}_2(\mathbb{C}))$ can be obtained by taking linear combinations, derivatives, and normally ordered products of the fields $E(z)$, $H(z)$, and $F(z)$. We now define a map

$$E(z), H(z), F(z) \mapsto V_{\sqrt{2}}(z), a(z), V_{-\sqrt{2}}(z). \tag{5.73}$$

By Section 4.5, we know that the corresponding OPEs are preserved and also that the energy-momentum tensor is preserved. Now, for our map to defined a homomorphism, we need the relations of the fields in $L_1(\mathfrak{sl}_2(\mathbb{C}))$ to be preserved. The only such relation that we need to check is

$$: E(z)E(z) := 0, \tag{5.74}$$

which is obtained from us setting to zero the submodule generated by $E_{-1}^2 \cdot 1$ in $V_1(\mathfrak{sl}_2(\mathbb{C}))$ to obtain the irreducible quotient.[5] But the OPE given by Equation (4.191) immediately tells us that $: V_{\sqrt{2}}(z)V_{\sqrt{2}}(z) := 0$.

Finally, note that any field in $V_{\sqrt{2}\mathbb{Z}}$ is obtained by taking derivatives and normally ordered products of the fields $V_{\sqrt{2}}(z)$, $a(z)$, and $V_{-\sqrt{2}}(z)$. This implies our homomorphism is surjective. Since the vertex operator algebras $V_{\sqrt{2}\mathbb{Z}}$ and $L_1(\mathfrak{sl}_2(\mathbb{C}))$ are both irreducible, our homomorphism is an isomorphism. $\square$

We can now obtain our desired free field realisation. Recall the $\widehat{\mathfrak{sl}_2(\mathbb{C})}$-module we considered in Section 4.4. We can view this as a module $\mathcal{L}$ over $L_1(\mathfrak{sl}_2(\mathbb{C}))$. Through the isomorphism $V_{\sqrt{2}\mathbb{Z}} \cong L_1(\mathfrak{sl}_2(\mathbb{C}))$, we obtain a module over $V_{\sqrt{2}\mathbb{Z}}$. We can identify this with the $V_{\sqrt{2}\mathbb{Z}}$-module

$$\bigoplus_{n\in\mathbb{Z}} \pi_{(n+\frac{1}{2})\sqrt{2}}. \tag{5.75}$$

This allows us to identify the states

$$|\phi\rangle \leftrightarrow \left|\frac{1}{\sqrt{2}}\right\rangle \tag{5.76}$$

$$|\psi\rangle \leftrightarrow \left|\frac{-1}{\sqrt{2}}\right\rangle. \tag{5.77}$$

In the next section, we will use this identification to compute the 4-point correlators of Section 4.4 using the vertex operators $V_{\frac{1}{\sqrt{2}}}(z)$ and $V_{\frac{-1}{\sqrt{2}}}(z)$.

[5] There are no other relations to check since the state corresponding to $: E(z)E(z) :$ generates the maximal proper ideal in the vertex algebra $V_1(\mathfrak{sl}_2(\mathbb{C}))$.

5.5 Computing 4-point Functions with Vertex Operators

We wish to compute

$$F_{p_1,p_2,p_3,p_4} := \langle 0| V_{p_1}(z_1)V_{p_2}(z_2)V_{p_3}(z_3)V_{p_4}(z_4)|0\rangle \tag{5.78}$$

where $p_1+p_2+p_3+p_4=0$. Recall from Appendix F that

$$V_p(z) = \left[e^{p\phi_0}e^{pa_0 \log z}\right]\prod_{n=1}^{\infty}\left[e^{pa_{-n}z^n/n}e^{-pa_n z^{-n}/n}\right] \tag{5.79}$$

where each factor in brackets commutes with every other factor. Recall also Equation (F.20), which is

$$e^{\alpha a_0}e^{\beta\phi_0} = e^{\alpha\beta}e^{\beta\phi_0}e^{\alpha a_0}. \tag{5.80}$$

Another result we will use is

$$e^{\alpha a_n}e^{\beta a_{-n}} = e^{\alpha\beta n}e^{\beta a_{-n}}e^{\alpha a_n}. \tag{5.81}$$

This result can be found in [32, p. 53] and can be derived by using commutator identities and the bracket $[a_n, a_m]$. We will now define

$$\Omega_i = \prod_{n=1}^{\infty}\left[e^{p_i a_{-n}z_i^n/n}e^{-p_i a_n z_i^{-n}/n}\right]. \tag{5.82}$$

We begin by using Equation (5.80) to write

$$\begin{aligned}F_{p_1,p_2,p_3,p_4} &= \langle 0|\,\Omega_1\Omega_2\Omega_3\Omega_4 e^{p_1\phi_0}e^{p_1a_0\log z_1}e^{p_2\phi_0}e^{p_2a_0\log z_2}e^{p_3\phi_0}e^{p_3a_0\log z_3}e^{p_4\phi_0}e^{p_4a_0\log z_4}|0\rangle & (5.83)\\ &= \langle 0|\,\Omega_1\Omega_2\Omega_3\Omega_4 e^{p_1\phi_0}e^{p_2\phi_0}e^{p_3\phi_0}e^{p_4\phi_0}e^{p_1a_0\log z_1}e^{p_2a_0\log z_2}e^{p_3a_0\log z_3}e^{p_4a_0\log z_4}|0\rangle \times\\ &\quad z_1^{p_1p_2}z_1^{p_1p_3}z_1^{p_1p_4}z_2^{p_2p_3}z_2^{p_2p_4}z_3^{p_3p_4}. & (5.84)\end{aligned}$$

Using $e^{p\phi_0}|q\rangle = |p+q\rangle$, $p_1+p_2+p_3+p_4=0$, and $e^{p_ia_0\log z_i}|0\rangle = |0\rangle$, this simplifies to

$$F_{p_1,p_2,p_3,p_4} = \langle 0|\,\Omega_1\Omega_2\Omega_3\Omega_4|0\rangle\, z_1^{p_1p_2}z_1^{p_1p_3}z_1^{p_1p_4}z_2^{p_2p_3}z_2^{p_2p_4}z_3^{p_3p_4}. \tag{5.85}$$

To compute $\langle 0|\,\Omega_1\Omega_2\Omega_3\Omega_4|0\rangle$, we can commute the exponentials of the annihilation operators (a_n for $n>0$) through to the right. For example, using Equation (5.81) we have

$$\begin{aligned}&\Omega_3\Omega_4\\ &= \prod_{n=1}^{\infty}\left[e^{p_3a_{-n}z_3^n/n}e^{-p_3a_nz_3^{-n}/n}\right]\prod_{n=1}^{\infty}\left[e^{p_4a_{-n}z_4^n/n}e^{-p_4a_nz_4^{-n}/n}\right] & (5.86)\\ &= \prod_{n=1}^{\infty}\left(e^{p_3a_{-n}z_3^n/n}e^{p_4a_{-n}z_4^n/n}\right)\prod_{n=1}^{\infty}\left(e^{-p_3a_nz_3^{-n}/n}e^{-p_4a_nz_4^{-n}/n}\right)\prod_{n=1}^{\infty}\exp\left(-\frac{p_3p_4}{n}\left(\frac{z_4}{z_3}\right)^n\right) & (5.87)\end{aligned}$$

$$= \prod_{n=1}^{\infty} \left(e^{p_3 a_{-n} z_3^n/n} e^{p_4 a_{-n} z_4^n/n}\right) \prod_{n=1}^{\infty} \left(e^{-p_3 a_n z_3^{-n}/n} e^{-p_4 a_n z_4^{-n}/n}\right) \exp\left(-\sum_{n=1}^{\infty} \frac{p_3 p_4}{n} \left(\frac{z_4}{z_3}\right)^n\right) \tag{5.88}$$

$$= \prod_{n=1}^{\infty} \left(e^{p_3 a_{-n} z_3^n/n} e^{p_4 a_{-n} z_4^n/n}\right) \prod_{n=1}^{\infty} \left(e^{-p_3 a_n z_3^{-n}/n} e^{-p_4 a_n z_4^{-n}/n}\right) \left(1 - \frac{z_4}{z_3}\right)^{p_3 p_4}. \tag{5.89}$$

Note that this implies

$$\Omega_3 \Omega_4 |0\rangle = \prod_{n=1}^{\infty} \left(e^{p_3 a_{-n} z_3^n/n} e^{p_4 a_{-n} z_4^n/n}\right) \left(1 - \frac{z_4}{z_3}\right)^{p_3 p_4} |0\rangle. \tag{5.90}$$

We can repeat this process to obtain

$$\langle 0| \Omega_1 \Omega_2 \Omega_3 \Omega_4 |0\rangle = \left\langle 0 \left| \prod_{n=1}^{\infty} \left(e^{p_1 a_{-n} z_1^n/n} e^{p_2 a_{-n} z_2^n/n} e^{p_3 a_{-n} z_3^n/n} e^{p_4 a_{-n} z_4^n/n}\right) \right| 0 \right\rangle \times$$
$$\left(1 - \frac{z_4}{z_3}\right)^{p_3 p_4} \left(1 - \frac{z_4}{z_2}\right)^{p_2 p_4} \left(1 - \frac{z_4}{z_1}\right)^{p_1 p_4} \left(1 - \frac{z_3}{z_2}\right)^{p_2 p_3} \left(1 - \frac{z_3}{z_1}\right)^{p_1 p_3} \left(1 - \frac{z_2}{z_1}\right)^{p_1 p_2}. \tag{5.91}$$

But the correlator above is just $\langle 0|0\rangle = 1$ since we can take the adjoint of the product and act on $\langle 0|$ to get back $\langle 0|$. We therefore obtain

$$F_{p_1,p_2,p_3,p_4} = \left(1 - \frac{z_4}{z_3}\right)^{p_3 p_4} \left(1 - \frac{z_4}{z_2}\right)^{p_2 p_4} \left(1 - \frac{z_4}{z_1}\right)^{p_1 p_4} \left(1 - \frac{z_3}{z_2}\right)^{p_2 p_3} \left(1 - \frac{z_3}{z_1}\right)^{p_1 p_3} \times$$
$$\left(1 - \frac{z_2}{z_1}\right)^{p_1 p_2} z_1^{p_1 p_2} z_1^{p_1 p_3} z_1^{p_1 p_4} z_2^{p_2 p_3} z_2^{p_2 p_4} z_3^{p_3 p_4} \tag{5.92}$$
$$= \prod_{1 \le i < j \le 4} z_{ij}^{p_i p_j}. \tag{5.93}$$

We will compare the above result to the calculations performed in Section 4.4. Recall the we solved for the correlator

$$\langle \phi_1(z_1) \psi_2(z_2) \phi_3(z_3) \psi_4(z_4) \rangle = f_1(\eta) \prod_{1 \le i < j \le 4} z_{ij}^{-1/6} \tag{5.94}$$

where

$$\eta = \frac{z_{14} z_{23}}{z_{12} z_{34}} \tag{5.95}$$

is the cross ratio and we found

$$f_1(\eta) = \frac{c_1(\eta+1)^{2/3}}{\eta^{1/3}} - \frac{3}{2} c_2 \eta^{1/3} (\eta+1)^{1/3} \left((\eta+1)^{1/3} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{5}{3}; -\eta\right) - 2\right) \tag{5.96}$$

by solving the KZ equations. Here, c_1 and c_2 are constants. Our free field realisation of Section 5.4 is given by

$$\phi \mapsto V_{\frac{1}{\sqrt{2}}} \tag{5.97}$$

$$\psi \mapsto V_{-\frac{1}{\sqrt{2}}}. \tag{5.98}$$

By Equation (5.93), we have

$$F_{\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}} = \frac{z_{13}^{1/2} z_{24}^{1/2}}{z_{12}^{1/2} z_{34}^{1/2} z_{14}^{1/2} z_{23}^{1/2}}. \tag{5.99}$$

This implies

$$F_{\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}} \propto \frac{(z_{13}z_{24})^{2/3}}{(z_{12}z_{34})^{2/3}} \frac{(z_{12}z_{34})^{1/3}}{(z_{14}z_{23})^{1/3}} \prod_{1\leq i<j\leq n} z_{ij}^{-1/6} \tag{5.100}$$

$$\propto \frac{((z_{13}z_{24})/(z_{12}z_{34}))^{2/3}}{((z_{14}z_{23})/(z_{12}z_{34}))^{1/3}} \prod_{1\leq i<j\leq n} z_{ij}^{-1/6}. \tag{5.101}$$

Using the identity $z_{14}z_{23} + z_{12}z_{34} = z_{13}z_{24}$, we obtain

$$F_{\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}} \propto \frac{((z_{14}z_{23} + z_{12}z_{34})/(z_{12}z_{34}))^{2/3}}{((z_{14}z_{23})/(z_{12}z_{34}))^{1/3}} \prod_{1\leq i<j\leq n} z_{ij}^{-1/6} \tag{5.102}$$

$$= \frac{(\eta+1)^{2/3}}{\eta^{1/3}} \prod_{1\leq i<j\leq n} z_{ij}^{-1/6}. \tag{5.103}$$

Finally, we see that this result agrees with Equation (5.96) provided that $c_2 = 0$. If we set $c_2 = 0$, the other correlators we computed in Section 4.4 become

$$f_2(\eta) = c_1\eta^{1/3}\left(\frac{\eta}{\eta+1}\right)^{1/3} \tag{5.104}$$

$$f_3(\eta) = \frac{c_1}{\eta^{1/3}(\eta+1)^{1/3}} \tag{5.105}$$

$$f_4(\eta) = \frac{c_1}{\eta^{1/3}(\eta+1)^{1/3}} \tag{5.106}$$

$$f_5(\eta) = \frac{c_1\eta^{2/3}}{(\eta+1)^{1/3}} \tag{5.107}$$

$$f_6(\eta) = \frac{c_1(\eta+1)}{\eta^{1/3}(\eta+1)^{1/3}}. \tag{5.108}$$

It is easy to check that these functions also agree with the correlators obtained using the free field realisation. Note that only three of our six correlators are distinct (and two are linearly independent) since F_{p_1,p_2,p_3,p_4} is invariant under $(p_1, p_2, p_3, p_4) \mapsto -(p_1, p_2, p_3, p_4)$. In fact, although the functions calculated are multivalued, the physical constraints fix the choice of branch cuts. For example, the choice of branch cuts must be such that

$$f_2(\eta) = f_5(\eta) \tag{5.109}$$

since the free field realisation shows us that the corresponding correlators are equal.

Chapter 6

Conformal Blocks and the KZ Equations

In this chapter, we give a geometric realisation of the Knizhnik-Zamolodchikov equations. We construct the conformal block bundle and present the KZ equations as a system of differential equations for flat sections of this bundle. It is known that any flat connection on a vector bundle gives a representation of the fundamental group of the base space called the monodromy representation. We focus on the KZ connection and show that the monodromy of the KZ equations gives a representation of the braid group on n strands. The braid group is discussed in Appendix H. This chapter concludes with an explicit example of a representation of the braid group on 4 strands, which is obtained by studying the monodromy of the 4-point correlators computed in Section 4.4. In this chapter, we follow [25]. The geometric aspects of vertex algebras is discussed in more detail in [12].

6.1 Conformal Blocks for the WZW Models

Recall that in our discussion of the WZW models, we worked entirely on $\mathbb{C}P^1$, which we identified as the one-point compactification of $\mathbb{C}$. We wish to formalise the space of solutions to the KZ equations: that is, the space of conformal blocks. An element of such a space should map states in the chiral sector of a CFT (or elements of a VOA) to elements of $\mathbb{C}$, as a correlation function would. However, it should do this in a manner that depends on the coordinates chosen in $\mathbb{C}P^1$ and that is consistent with the conformal symmetries of our model. In particular, from the defining properties of our space of conformal blocks we should be able to derive the Ward identities and the KZ equations. To encode the dependence on coordinates and consistency with conformal symmetries, we need the following definitions.

Definition 6.1.1. Let $p_1, \ldots, p_n \in \mathbb{C}P^1$ be distinct points and let $\mathcal{M}_{p_1,\ldots,p_n}$ be the space of meromorphic functions on $\mathbb{C}P^1$ with poles only at $p_1, \ldots, p_n$. Given a complex semisimple Lie algebra $\mathfrak{g}$, we define the Lie algebra

$$\mathfrak{g}(p_1, \ldots, p_n) = \mathfrak{g} \otimes \mathcal{M}_{p_1,\ldots,p_n} \tag{6.1}$$

with bracket

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg \tag{6.2}$$

for $X, Y \in \mathfrak{g}$ and $f, g \in \mathcal{M}_{p_1,\ldots,p_n}$.

Definition 6.1.2. For each point p_i, we define a local coordinate t_i around p_i as follows. If $p_i = \infty$, define $t_i(z) = \frac{1}{z}$. If $p_i \neq \infty$, define $t_i(z) = z - p_i$. This choice of coordinates is natural since it implies that the local coordinates satisfy $t_i(p_i) = 0$.

Note that for such a local coordinate t_i on $\mathbb{C}P^1$, there is a canonical inclusion

$$\mathfrak{g}(p_1, \cdots, p_n) \hookrightarrow \hat{\mathfrak{g}}_i. \tag{6.3}$$

Here,

$$\hat{\mathfrak{g}}_i = \mathfrak{g} \otimes \mathbb{C}((t_i)) \oplus \mathbb{C}K \tag{6.4}$$

is given by the loop algebra construction except that it is now coordinate dependent.

Theorem 6.1.1. *[25, pp. 43-44] Let $k \in \mathbb{N}$ be a fixed level. For $1 \leq i \leq n$ let V_{λ_i} be an integrable highest weight $\widehat{\mathfrak{g}}_i$-module of highest weight λ_i. Then $\bigotimes_{i=1}^n V_{\lambda_i}$ has the structure of a $\mathfrak{g}(p_1, \ldots, p_n)$-module with action defined by*

$$\alpha \cdot (v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} v_1 \otimes \cdots \otimes \tau_i(\alpha) v_i \otimes \cdots \otimes v_n \tag{6.5}$$

for $\alpha \in \mathfrak{g}(p_1, \ldots, p_n)$. Here, we define

$$\tau_j : \mathfrak{g}(p_1, \ldots, p_n) \to \mathfrak{g} \otimes \mathbb{C}((t_j)) \oplus \mathbb{C}K \tag{6.6}$$

$$X \otimes f \mapsto X \otimes f(t_j) \oplus 0 \tag{6.7}$$

where $f(t_j)$ is the Laurent expansion of f in some local coordinate t_j containing p_j.

Proof. Let $v_1 \otimes \cdots \otimes v_n \in \bigotimes_{i=1}^n V_{\lambda_i}$. Let $X, Y \in \mathfrak{g}$ and $f, g \in \mathcal{M}_{p_1,\ldots,p_n}$. We wish to show

$$([X,Y]\otimes fg)(v_1\otimes\cdots\otimes v_n) = (X\otimes f)(Y\otimes g)(v_1\otimes\cdots\otimes v_n) - (Y\otimes g)(X\otimes f)(v_1\otimes\cdots\otimes v_n). \tag{6.8}$$

We can write the right hand side as

$$\sum_{j=1}^{n} \sum_{i=1}^{n} v_1 \otimes \cdots \otimes (X \otimes f(t_j)) v_j \otimes \cdots \otimes (Y \otimes g(t_i)) v_i \otimes \cdots \otimes v_n \tag{6.9}$$

$$- \sum_{j=1}^{n} \sum_{i=1}^{n} v_1 \otimes \cdots \otimes (Y \otimes g(t_j)) v_j \otimes \cdots \otimes (X \otimes f(t_i)) v_i \otimes \cdots \otimes v_n \tag{6.10}$$

$$= \sum_{i=1}^{n} v_1 \otimes \cdots \otimes [X \otimes f(t_i), Y \otimes g(t_i)] v_i \otimes \cdots \otimes v_n \tag{6.11}$$

$$= \sum_{i=1}^{n} v_1 \otimes \cdots \otimes ([X, Y] \otimes f(t_i) g(t_i)) v_i \otimes \cdots \otimes v_n \tag{6.12}$$

$$-\sum_{i=1}^{n} v_1 \otimes \cdots \otimes \operatorname*{Res}_{t_i=0}\left[f(t_i)\frac{d}{dt_i}(g(t_i))\right] K v_i \otimes \cdots \otimes v_n \tag{6.13}$$

$$= ([X,Y] \otimes fg)(v_1 \otimes \cdots \otimes v_n) - \operatorname*{Res}_{t_i=0}\left[f(t_i)\frac{d}{dt_i}(g(t_i))\right] k \sum_{i=1}^{n} v_1 \otimes \cdots \otimes v_n \tag{6.14}$$

$$= ([X,Y] \otimes fg)(v_1 \otimes \cdots \otimes v_n) \tag{6.15}$$

by the residue theorem for meromorphic differentials. □

Definition 6.1.3. [25, p. 44] The space of conformal blocks $\mathcal{V}(p_1, \ldots, p_n; \lambda_1, \ldots, \lambda_n)$ is the space of $\mathfrak{g}(p_1, \ldots, p_n)$-linear maps from $\bigotimes_{i=1}^n V_{\lambda_i}$ to $\mathbb{C}$ where we view $\mathbb{C}$ as a module over $\mathfrak{g}(p_1, \ldots, p_n)$ with the action sending every element to zero. We write

$$\mathcal{V}(p_1, \ldots, p_n; \lambda_1, \ldots, \lambda_n) = \operatorname{Hom}_{\mathfrak{g}(p_1,\ldots,p_n)}\left(\bigotimes_{i=1}^{n} V_{\lambda_i}, \mathbb{C}\right). \tag{6.16}$$

Note that a linear map $\Psi : \bigotimes_{i=1}^n V_{\lambda_i} \to \mathbb{C}$ is $\mathfrak{g}(p_1, \ldots, p_n)$-linear if it satisfies

$$\sum_{i=1}^{n} \Psi(v_1 \otimes \cdots \otimes \tau_i(X) v_i \otimes \cdots \otimes v_n) = 0 \tag{6.17}$$

for all $v_1 \otimes \cdots \otimes v_n \in \bigotimes_{i=1}^n V_{\lambda_i}$ and $X \in \mathfrak{g}(p_1, \ldots, p_n)$.

6.2 Connections

This section assumes an understanding of vector bundles, which are discussed in Appendix B.

Definition 6.2.1. Let M be a smooth manifold and $E \to M$ a smooth vector bundle. A connection on E is an $\mathbb{R}$-linear map

$$\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M) \tag{6.18}$$

satisfying the Leibniz rule

$$\nabla(\sigma f) = (\nabla\sigma) f + \sigma \otimes df \tag{6.19}$$

for all $f \in C^\infty(M)$. Here, $\Gamma(E)$ is the collection of smooth sections of E and T^*M is the cotangent bundle on M. The section σf of E is defined by $(\sigma f)(p) = f(p) \cdot \sigma(p)$. The object df is called the exterior derivative of f and turns out to be a one-form on M since $f \in C^\infty(M)$. This one-form is defined by the map $p \mapsto (p, df_p)$ where $df_p : T_pM \to \mathbb{C}$ is the pushforward of f at p, which we can view as an element of the cotangent space T_pM^*.

Let σ be a smooth section of E. Let σ_i be a local frame of E as defined in Chapter 1. Recall that we can locally write[1]

$$\sigma = \sigma_i \alpha^i \tag{6.20}$$

where $\alpha^i \in C^\infty(M)$. Applying the Leibniz rule gives us

$$\nabla(\sigma) = \nabla(\sigma_i)\alpha^i + \sigma_i d\alpha^i. \tag{6.21}$$

We can then expand $\nabla(\sigma_i)$ in terms of the local frame again to obtain

$$\nabla(\sigma) = \omega_i^j \sigma_j \alpha^i + \sigma_j d\alpha^j \tag{6.22}$$
$$= (d\alpha^j + \omega_i^j \alpha^i)\sigma_j. \tag{6.23}$$

We can therefore locally write

$$\nabla = d + \omega \tag{6.24}$$

where we define d and ω such that Equation (6.23) holds.

Example 6.2.1. Consider the trivial vector bundle $\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$. We have a local (and in fact global) frame defined by

$$\sigma_1(p) = (p, (1, 0)) \tag{6.25}$$
$$\sigma_2(p) = (p, (0, 1)). \tag{6.26}$$

Let $\sigma : \mathbb{R} \to \mathbb{R} \times \mathbb{R}^2$ be an arbitrary smooth section and write

$$\sigma = f_1\sigma_1 + f_2\sigma_2 \tag{6.27}$$

for some smooth functions $f_1, f_2 \in C^\infty(\mathbb{R})$. We can write this in the matrix notation

$$\sigma = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{6.28}$$

Let $\omega = \alpha_{ij} dx$ with α_{ij} a 2×2 matrix of smooth functions on $\mathbb{R}$. We then have a connection ∇ defined by

$$\nabla = d + \omega. \tag{6.29}$$

The action on the section σ is then given by

$$\nabla(\sigma) = \begin{pmatrix} df_1 + (\alpha_{11}f_1 + \alpha_{12}f_2)dx \\ df_2 + (\alpha_{21}f_1 + \alpha_{22}f_2)dx \end{pmatrix}. \tag{6.30}$$

Definition 6.2.2. Given a connection ∇ on a smooth vector bundle $\pi : E \to M$ and a vector field $X \in \Gamma(TM)$, the covariant derivative in the direction of X is defined to be the map

$$\nabla_X : \Gamma(E) \to \Gamma(E) \tag{6.31}$$

[1] Here we use Einstein notation meaning that we sum over repeated indices.

$$\sigma \mapsto (\nabla\sigma)(X). \tag{6.32}$$

Here, we view $(\nabla\sigma)(X)$ as an element of $\Gamma(E)$ as follows. Given $p \in M$, we obtain an element $(\nabla\sigma)(p)$ of the fibre of $E \otimes T^*M$ at p, which we can write as

$$(\nabla\sigma)(p) = \sum_i \alpha_i(e_i \otimes f_i). \tag{6.33}$$

Now, $X(p)$ gives us an element v of the tangent space T_pM, and noting that $f_i \in (T_pM)^*$, we obtain a collection of scalars $f_i(v)$. We can then define

$$(\nabla_X\sigma)(p) := ((\nabla\sigma)(X))(p) = \sum_i \alpha_i f_i(v) e_i. \tag{6.34}$$

From our definition of covariant derivative, it is easy to see that we have the important properties

$$\nabla_{fX+gY}\sigma = f\nabla_X\sigma + g\nabla_Y\sigma \tag{6.35}$$

$$\nabla_X(f\sigma) = (Xf)\sigma + f\nabla_X\sigma \tag{6.36}$$

for all $X, Y \in \Gamma(TM)$, $\sigma \in \Gamma(E)$, $f, g \in C^\infty(M)$. Covariant derivatives are important as they allow us to talk about parallel transport, which (in the case of the tangent bundle) allows us to move vectors between different tangent spaces along a curve in the manifold such that the vectors remain "parallel" with respect to the connection. This will be useful in Section 6.5 where we make this notion rigorous.

6.3 The Conformal Block Bundle

Geometrically, we will view the KZ equations as a system of PDEs for sections of a vector bundle called the conformal block bundle. In this section, we will construct the conformal block bundle as a subbundle of a trivial bundle. This construction follows [25].

We begin by describing the sheaf of smooth sections that gives rise to the conformal block bundle. Recall that for a collection of points $p_1, \cdots, p_n$ on $\mathbb{C}P^1$, there is a canonical choice of local coordinates $t_1, \cdots, t_n$ given by Definition 6.1.2.

Definition 6.3.1. Consider the projection

$$\pi : (\mathbb{C}P^1)^{n+1} \to (\mathbb{C}P^1)^n. \tag{6.37}$$

We can define a coordinate on $(\mathbb{C}P^1)^{n+1}$ by $(t_1, \cdots, t_n, z_{n+1})$ for some coordinate z_{n+1} on $\mathbb{C}P^1$. For $1 \le i \le n$, we denote by D_i the hyperplane of $(\mathbb{C}P^1)^{n+1}$ given by setting $t_i = z_{n+1}$. We denote by $\mathcal{M}_{D_1,\ldots,D_n}(U)$ the space of meromorphic functions on $\pi^{-1}(U)$ with poles only on the hyperplanes $D_1, \ldots, D_n$.

Note that an arbitrary element of $\mathcal{M}_{D_1,\dots,D_n}(U)$ has a Laurent expansion on the hyperplane D_i[2]

$$f_{D_i}(z) = \sum_{n=-N}^{\infty} a_n(z_1, \dots, z_n)(z - z_i)^n \tag{6.38}$$

where $a_n(z_1, \dots, z_n)$ is holomorphic in each coordinate z_i. Fixing $(p_1, \dots, p_n) \in (\mathbb{C}P^1)^n$, we can therefore view $f_{D_i}(z)$ as the element of $\mathbb{C}((t_i))$ given by

$$f_{D_i}(t_i) = \sum_{n=-N}^{\infty} a_n(p_1, \dots, p_n)(t_i)^n \tag{6.39}$$

where we note that $t_i = z - p_i$ for $p_i \neq \infty$.[3]

Definition 6.3.2. From $\mathcal{M}_{D_1,\dots,D_n}(U)$, we can construct a Lie algebra

$$\mathfrak{g} \otimes \mathcal{M}_{D_1,\dots,D_n}(U) \tag{6.40}$$

with bracket defined by

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg \tag{6.41}$$

for $X, Y \in \mathfrak{g}$, $f, g \in \mathcal{M}_{D_1,\dots,D_n}(U)$.

Recall that we can associate an untwisted affine Lie algebra $\hat{\mathfrak{g}}_i$ to each coordinate t_i by Equation (6.4). The Laurent expansion given by Equation (6.38) defines a canonical map

$$\hat{\tau}_i : \mathfrak{g} \otimes \mathcal{M}_{D_1,\dots,D_n}(U) \to \mathfrak{g} \otimes \mathcal{O}(U) \otimes \mathbb{C}((t_i)) \tag{6.42}$$

where $\mathcal{O}(U)$ is the set of holomorphic functions on U. If we fix $(p_1, \dots, p_n) \in (\mathbb{C}P^1)^n$, then we can view $\hat{\tau}_i(f)$ as an element of $\hat{\mathfrak{g}}_i$ for $f \in \mathfrak{g} \otimes \mathcal{M}_{D_1,\dots,D_n}(U)$. For each of our local coordinates t_i of $\mathbb{C}P^1$ we associate an integrable highest weight $\hat{\mathfrak{g}}_i$-module V_{λ_i} of highest weight λ_i. Similar to Theorem 6.1.1, the map $\hat{\tau}_i$ allows us to define an action of $\mathfrak{g} \otimes \mathcal{M}_{D_1,\dots,D_n}$ on $\bigotimes_{i=1}^n V_{\lambda_i}$. Now, consider the trivial bundle

$$E := \mathrm{Conf}_n(\mathbb{C}P^1) \times \mathrm{Hom}_{\mathbb{C}}\left(\bigotimes_{i=1}^{n} V_{\lambda_i}, \mathbb{C}\right) \to \mathrm{Conf}_n(\mathbb{C}P^1). \tag{6.43}$$

Here,

$$\mathrm{Conf}_n(X) = \prod^{n} X \backslash \{(x_1, \dots, x_n) \in X^n \mid x_i = x_j \text{ for some } i \neq j\} \tag{6.44}$$

is Fadell's configuration space, defined for any topological space X.

[2]Such an expansion only exists because the poles of f on $\pi^{-1}(z_1, \dots, z_n) = \{(z_1, \dots, z_n, z) \mid z \in \mathbb{C}P^1\}$ can only occur when $z = z_i$ by definition of $\mathcal{M}_{D_1,\dots,D_n}(U)$.

[3]If $p_i = \infty$, our expression still holds true. In that case, our coordinate z_i would be a local coordinate of infinity and so we would have to replace $z - z_i$ with $1/z$ in Equation (6.38).

Definition 6.3.3. For each open set $U \subseteq \mathrm{Conf}_n(\mathbb{C}P^1)$, we denote by $\mathcal{E}_{\lambda_1,\ldots,\lambda_n}(U)$ the space of smooth sections $\sigma : U \to E$ satisfying

$$\sum_{i=1}^{n} (\sigma(p_1,\ldots,p_n))(v_1 \otimes \cdots \otimes \hat{\tau}_i(f) v_i \otimes \cdots \otimes v_n) = 0 \tag{6.45}$$

for all $f \in \mathfrak{g} \otimes \mathcal{M}_{D_1,\ldots,D_n}(U)$ and $v_1 \otimes \cdots \otimes v_n \in \bigotimes_{i=1}^{n} V_{\lambda_i}$.

In Section 6.4, we show that the trivial bundle E admits a flat connection called the KZ connection. The conformal block bundle

$$\mathcal{E}_{\lambda_1,\ldots,\lambda_n} := \coprod_{(p_1,\ldots,p_n)\in \mathrm{Conf}_n(\mathbb{C}P^1)} \mathcal{V}(p_1,\ldots,p_n;\lambda_1,\ldots,\lambda_n) \to \mathrm{Conf}_n(\mathbb{C}P^1) \tag{6.46}$$

is constructed from the flat sections of the trivial bundle with respect to the KZ connection [25, p. 56].

6.4 The KZ Equations on the Conformal Block Bundle

Note that we define the notion of a vector space-valued differential k-form in Appendix C. We use this in the following definition.

Definition 6.4.1. The curvature F^{∇} of a connection ∇ on a smooth vector bundle $\pi : E \to M$ is a differential 2-form valued in the vector space $\mathrm{End}(E)$. Given two vector fields $X, Y \in \Gamma(TM)$, define F^{∇} by

$$F^{\nabla}(X,Y) : \Gamma(E) \to \Gamma(E) \tag{6.47}$$

$$\sigma \mapsto \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X,Y]}\sigma. \tag{6.48}$$

Viewing a vector field as a derivation on the space $C^{\infty}(M)$, we define $[X, Y]$ by

$$[X,Y](f) = X(Y(f)) - Y(X(f)). \tag{6.49}$$

Theorem 6.4.1. *Recall Section 6.2 where we showed that given a local frame $\{\sigma_i\}$, we can locally write a connection as*

$$\nabla = d + \omega \tag{6.50}$$

where $\nabla\sigma_i = \omega_i^j \sigma_j$ and we can view ω is a matrix of one-forms defined by ω_i^j. The connection ∇ is flat if for any such local expression, we have

$$d\omega + \omega \wedge \omega = 0 \tag{6.51}$$

where $\omega \wedge \omega$ is the square of the matrix ω with the product of the one-form components being the wedge product.

Recall that the Sugawara construction given in Chapter 4 allows us to define an action of the Virasoro algebra on any highest weight $\widehat{\mathfrak{g}}_i$-module. We will use this to define the KZ connection. From now on, we will consider the conformal block bundle as a vector bundle over $\mathrm{Conf}_n(\mathbb{C})$ (constructed in the same manner as the vector bundle over $\mathrm{Conf}_n(\mathbb{C}P^1)$), since we can identity the pure braid group with $\pi_1(\mathrm{Conf}_n(\mathbb{C}))$.

Definition 6.4.2. Let $\psi \in \mathcal{E}_{\lambda_1,\ldots,\lambda_n}(U)$ be defined by

$$\psi(p, z_1, \ldots, z_n) = (p, \widetilde{\psi}) \tag{6.52}$$

for some $\widetilde{\psi} \in \mathrm{Hom}_{\mathbb{C}}(\bigotimes_{i=1}^n H_{\lambda_i}, \mathbb{C})$. Then we define $L_{-1}^{(i)}\psi$ by

$$(L_{-1}^{(i)}\psi)(p, z_1, \ldots, z_n) = (p, L_{-1}^{(i)}\widetilde{\psi}) \tag{6.53}$$

where

$$(L_{-1}^{(i)}\widetilde{\psi})(v_1 \otimes \cdots \otimes v_n) = \widetilde{\psi}(v_1 \otimes L_{-1}v_i \otimes \cdots \otimes v_n). \tag{6.54}$$

Consider the connection

$$\nabla : \Gamma(E) \to \Gamma(E \otimes T^*\mathrm{Conf}_n(\mathbb{C})) \tag{6.55}$$

$$\psi \mapsto d\psi - \sum_{i=1}^n L_{-1}^{(i)}\psi dz_i \tag{6.56}$$

on the trivial bundle. Here, $d\psi \in \Gamma(E \otimes T^*\mathrm{Conf}_n(\mathbb{C}))$ is the section defined as follows. Take a local frame $\{\sigma_i\}$ and expand $\psi = \psi^i\sigma_i$. Define

$$(d\psi)(p) = \sigma_i(p) \otimes d\psi_p^i \tag{6.57}$$

where $d\psi_p^i \in (T_p\mathrm{Conf}_n(\mathbb{C}))^*$ is the pushforward of ψ^i at p. Similarly, we can view $z_i : \mathrm{Conf}_n(\mathbb{C}) \to \mathbb{C}$ as projection onto a local coordinate of $\mathbb{C}$ and we can define $L_{-1}^{(i)}\psi dz_i \in \Gamma(E \otimes T^*\mathrm{Conf}_n(\mathbb{C}))$ to be the section satisfying

$$(L_{-1}^{(i)}\psi dz_i)(p) = (L_{-1}^{(i)}\psi)(p) \otimes d(z_i)_p \tag{6.58}$$

where $d(z_i)_p$ is the pushforward of z_i at p.[4] The following lemma shows that we can restrict the connection ∇ to the conformal block bundle.

Lemma 6.4.2. *[25, pp. 54-55] Let $\psi \in \mathcal{E}_{\lambda_1,\ldots,\lambda_n}(U)$. Then*

$$\nabla_{\partial_j}(\psi) \in \mathcal{E}_{\lambda_1,\ldots,\lambda_n}(U) \tag{6.59}$$

for every ∂_j in the canonical frame for the tangent bundle.

[4] We can identify $d(z_i)_p$ as an element of the cotangent space $(T_p\mathrm{Conf}_n(\mathbb{C}))^*$ through the identification $T_{z_i(p)}\mathbb{C} \cong \mathbb{C}$.

Proof. Recall that for $f \in \mathfrak{g} \otimes \mathcal{M}_{D_1,\ldots,D_n}(U)$, we can write

$$\hat{\tau}_j(f) = \sum_{n=-N}^{\infty} a_n(p_1, \ldots, p_n)(t_j)^n \tag{6.60}$$

where $t_j = z - z_j$. If we let f_{z_j} denote $\frac{\partial f}{\partial z_j}$, then we have

$$\hat{\tau}_j(f_{z_j}) = \sum_{n=-N}^{\infty} \frac{\partial a_n}{\partial z_j} t_j^n - a_n n t_j^{n-1}. \tag{6.61}$$

We can also define the operator ∂_j on $\mathfrak{g} \otimes \mathcal{O}(U) \otimes \mathbb{C}((t_j))$ that acts trivially on $\mathbb{C}((t_j))$ so that we have

$$\hat{\tau}_j(f_{z_j}) = \partial_j \hat{\tau}_j(f) - \frac{\partial}{\partial t_j} \hat{\tau}_j(f). \tag{6.62}$$

Using the result

$$[L_n, X \otimes t^m] = -mX \otimes t^{n+m} \tag{6.63}$$

we can therefore write

$$\hat{\tau}_j(f_{z_j}) = \partial_j \hat{\tau}_j(f) + [L_{-1}, \hat{\tau}_j(f)]. \tag{6.64}$$

If $i \neq j$ then

$$\hat{\tau}_i(f_{z_j}) = \partial_j \hat{\tau}_i(f). \tag{6.65}$$

We will now prove our lemma by first showing that

$$\begin{aligned} \frac{\partial}{\partial z_j} \psi(v_1 \otimes \cdots \hat{\tau}_i(f) v_i \otimes \cdots \otimes v_n) = \frac{\partial \psi}{\partial z_j}(v_1 \otimes \cdots \hat{\tau}_i(f) v_i \otimes \cdots \otimes v_n) \\ + \psi(v_1 \otimes \cdots \otimes \partial_j \hat{\tau}_i(f) v_i \otimes \cdots \otimes v_n) \end{aligned} \tag{6.66}$$

for all $v_i \in V_{\lambda_i}$ and $f \in \mathfrak{g} \otimes \mathcal{M}_{D_1,\ldots,D_n}(U)$. Here, we view each side of the equation as a map from $\mathrm{Conf}_n(\mathbb{C})$ to $\mathbb{C}$. Note also that $\frac{\partial \psi}{\partial z_j}$ denotes the covariant derivative d_{∂_j} applied to ψ. We begin by writing $\psi = \psi^i \sigma_i$ for some local frame σ_i. We have

$$\frac{\partial}{\partial z_j} \psi(v_1 \otimes \cdots \hat{\tau}_i(f) v_i \otimes \cdots \otimes v_n) = \frac{\partial}{\partial z_j} \big((\psi^i \sigma_i)(v_1 \otimes \cdots \hat{\tau}_i(f) v_i \otimes \cdots \otimes v_n) \big) \tag{6.67}$$

$$\begin{aligned} &= \left(\frac{\partial \psi^i}{\partial z_j} \right) \sigma_i(v_1 \otimes \cdots \hat{\tau}_i(f) v_i \otimes \cdots \otimes v_n) \\ &\quad + \psi^i \frac{\partial}{\partial z_j} \sigma_i(v_1 \otimes \cdots \hat{\tau}_i(f) v_i \otimes \cdots \otimes v_n). \end{aligned} \tag{6.68}$$

Now, $\partial_{z_j}(\psi(v_1 \otimes \cdots \hat{\tau}_i(f) v_i \otimes \cdots \otimes v_n))(\phi(p)) = (d_p \psi(v_1 \otimes \cdots \hat{\tau}_i(f) v_i \otimes \cdots \otimes v_n)(\partial_j(p))$ where ϕ is the chart $(z_1, \ldots, z_n)$ on $\mathrm{Conf}_n(\mathbb{C})$. Recalling that

$$d_{\partial_j} \psi(p) = d\psi^i_p(\partial_j(p)) \sigma_i(p), \tag{6.69}$$

we can therefore make the identification

$$\left(\frac{\partial\psi^i}{\partial z_j}\right)\sigma_i(v_1\otimes\cdots\hat{\tau}_i(f)v_i\otimes\cdots\otimes v_n)=\frac{\partial\psi}{\partial z_j}(v_1\otimes\cdots\hat{\tau}_i(f)v_i\otimes\cdots\otimes v_n). \tag{6.70}$$

It is also clear that

$$\psi^i\frac{\partial}{\partial z_j}\sigma_i(v_1\otimes\cdots\hat{\tau}_i(f)v_i\otimes\cdots\otimes v_n)=\psi(v_1\otimes\cdots\otimes\partial_j\hat{\tau}_i(f)v_i\otimes\cdots\otimes v_n) \tag{6.71}$$

and so we conclude that Equation (6.66) holds.

Next, we will prove that

$$\begin{aligned}&\sum_{i=1}^n(\nabla_{\partial_j}\psi)(v_1\otimes\cdots\otimes\hat{\tau}_i(f)v_i\otimes\cdots\otimes v_n)\\&\quad=\sum_{i=1}^n\left(\frac{\partial}{\partial z_j}\psi(v_1\otimes\cdots\otimes\hat{\tau}_i(f)v_i\otimes\cdots\otimes v_n)-\psi(v_1\otimes\cdots\otimes\hat{\tau}_i(f_{z_j})v_i\otimes\cdots\otimes v_n)\right)\end{aligned} \tag{6.72}$$

for all $v_i\in V_{\lambda_i}$ and $f\in\mathfrak{g}\otimes\mathcal{M}_{D_1,\ldots,D_n}(U)$. This suffices to prove the lemma since the right hand side is 0 by the assumption $\psi\in\mathcal{E}_{\lambda_1,\ldots,\lambda_n}(U)$. By Equation (6.66), we can write the right hand side of (6.72) as

$$\begin{aligned}\sum_{i=1}^n\Bigg(&\frac{\partial\psi}{\partial z_j}(v_1\otimes\cdots\hat{\tau}_i(f)v_i\otimes\cdots\otimes v_n)+\psi(v_1\otimes\cdots\otimes\partial_j\hat{\tau}_i(f)v_i\otimes\cdots\otimes v_n)\\&-\psi(v_1\otimes\cdots\otimes\hat{\tau}_i(f_{z_j})v_i\otimes\cdots\otimes v_n)\Bigg).\end{aligned} \tag{6.73}$$

Focussing on the last two terms, we can use Equations (6.64) and (6.65) to write

$$\begin{aligned}&\sum_{i=1}^n\psi(v_1\otimes\cdots\otimes\partial_j\hat{\tau}_i(f)v_i\otimes\cdots\otimes v_n)-\psi(v_1\otimes\cdots\otimes\hat{\tau}_i(f_{z_j})v_i\otimes\cdots\otimes v_n)\\&\quad=\psi(v_1\otimes\cdots\otimes\partial_j\hat{\tau}_j(f)v_j\otimes\cdots\otimes v_n)-\psi(v_1\otimes\cdots\otimes\hat{\tau}_j(f_{z_j})v_j\otimes\cdots\otimes v_n) &(6.74)\\&\quad=-\psi(v_1\otimes\cdots\otimes[L_{-1},\hat{\tau}_j(f)]v_j\otimes\cdots\otimes v_n) &(6.75)\end{aligned}$$

Using the fact that $\psi\in\mathcal{E}_{\lambda_1,\ldots,\lambda_n}$, we can write this as

$$\begin{aligned}&\sum_{i=1}^n\psi(v_1\otimes\cdots\otimes\partial_j\hat{\tau}_i(f)v_i\otimes\cdots\otimes v_n)-\psi(v_1\otimes\cdots\otimes\hat{\tau}_i(f_{z_j})v_i\otimes\cdots\otimes v_n)\\&\quad=-\sum_{i=1}^n\Big[\psi(v_1\otimes\cdots\otimes\hat{\tau}_i(f)v_i\otimes\cdots\otimes L_{-1}v_j\otimes\cdots\otimes v_n)\\&\quad-\psi(v_1\otimes\cdots\otimes[L_{-1},\hat{\tau}_j(f)]v_j\otimes\cdots\otimes v_n)\Big]\end{aligned} \tag{6.76}$$

$$= -\sum_{i\neq j}\Big[\psi(v_1\otimes\cdots\otimes\hat{\tau}_i(f)v_i\otimes\cdots\otimes L_{-1}v_j\otimes\cdots\otimes v_n)$$

$$-\psi(v_1\otimes\cdots\otimes\hat{\tau}_j(f)L_{-1}v_j\otimes\cdots\otimes v_n)-\psi(v_1\otimes\cdots\otimes[L_{-1},\hat{\tau}_j(f)]v_j\otimes\cdots\otimes v_n)\Big] \tag{6.77}$$

$$= -\sum_{i=1}^{n}(L_{-1}^{(j)}\psi)(v_1\otimes\cdots\otimes\hat{\tau}_i(f)v_i\otimes\cdots\otimes v_j\otimes\cdots\otimes v_n) \tag{6.78}$$

We have therefore established that Equation (6.72) holds and so we have proven the lemma. □

Note that this lemma establishes that we have covariant derivatives

$$\nabla_{\partial_j}:\mathcal{E}_{\lambda_1,\ldots,\lambda_n}(U)\to\mathcal{E}_{\lambda_1,\ldots,\lambda_n}(U) \tag{6.79}$$

for all j and U and so we have the restricted connection

$$\nabla_{KZ}:=\nabla|_{\mathcal{E}_{\lambda_1,\ldots,\lambda_n}}:\Gamma(\mathcal{E}_{\lambda_1,\ldots,\lambda_n})\to\Gamma(\mathcal{E}_{\lambda_1,\ldots,\lambda_n}\otimes T^*\mathrm{Conf}_n(\mathbb{C})) \tag{6.80}$$

called the KZ connection.

Theorem 6.4.3. *[25, p. 56] The KZ connection is flat.*

Proof. It suffices to prove that ∇ is flat. By definition of ∇, we can write

$$\nabla = d+\omega \tag{6.81}$$

with the identification

$$\omega = -\sum_{i=1}^{n}L_{-1}^{(i)}dz_i. \tag{6.82}$$

Viewing ω as a $\mathrm{Hom}_{\mathbb{C}}(\bigotimes_{i=1}^{n}V_{\lambda_i},\mathbb{C})$-valued one-form, we see that $d\omega=0$ since $L_{-1}^{(i)}$ does not depend on local coordinates. Now,

$$\omega\wedge\omega=\sum_{i,j}L_{-1}^{(i)}L_{-1}^{(j)}dz_i\wedge dz_j. \tag{6.83}$$

Since $dz_i\wedge dz_j=-dz_j\wedge dz_i$ and $[L_{-1}^{(i)},L_{-1}^{(j)}]=0$, we conclude that

$$\omega\wedge\omega=0. \tag{6.84}$$

□

Before deriving the Knizhnik-Zamolodchikov equations, we need to introduce some notation. Let $\{J^a\}$ be a basis for the Lie algebra $\mathfrak{g}$ that is orthonormal with respect to the Killing form. Define an operator $\Omega^{(ij)}$ on sections of the conformal block bundle by

$$(\Omega^{(ij)}\psi)(v_1\otimes\cdots\otimes v_n)=\sum_a\psi(v_1\otimes\cdots\otimes J^a v_i\otimes\cdots\otimes J^a v_j\otimes\cdots\otimes v_n). \tag{6.85}$$

Now, given the integrable highest weight $\hat{\mathfrak{g}}_i$-modules V_{λ_i}, let $\overline{V}_{\lambda_i}$ denote the corresponding finite-dimensional irreducible representations of $\overline{\mathfrak{g}}_i$ with highest weights λ_i. Here, $\overline{\mathfrak{g}}_i$ is the horizontal subalgebra of $\hat{\mathfrak{g}}_i$. Given an element $\psi : V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n} \to \mathbb{C}$ in the space of conformal blocks $\mathrm{Hom}_{\mathfrak{g}(p_1,\ldots,p_n)}(\bigotimes_{i=1}^{n} V_{\lambda_i}, \mathbb{C})$, denote by $\psi_0 : \overline{V}_{\lambda_1} \otimes \cdots \otimes \overline{V}_{\lambda_n} \to \mathbb{C}$ its restriction to $\overline{V}_{\lambda_1} \otimes \cdots \otimes \overline{V}_{\lambda_n}$.

Theorem 6.4.4. *[25, pp. 57-58] Let ψ be a flat section of the conformal block bundle. Its restriction ψ_0 satisfies the KZ equations*

$$\frac{\partial \psi_0}{\partial z_i} = \sum_{j \neq i} \frac{\Omega^{(ij)} \psi_0}{z_i - z_j}. \tag{6.86}$$

Proof. Consider the function

$$f(z) = (z - z_i)^{-1} \tag{6.87}$$

for some $k < 0$. Expanding this as a Taylor series in $t_j = z - z_j$, we can write

$$(z - z_i)^{-1} = \sum_{m=0}^{\infty} a_m^{(j)} t_j^m \tag{6.88}$$

where

$$a_m^{(j)} = \frac{-1}{(z_i - z_j)^{m+1}}. \tag{6.89}$$

□

Recall that elements of the space of conformal blocks are invariant under the diagonal action of $X \otimes f$ for any $X \in \mathfrak{g}$. This property is summarised by Equation (6.17). Using this, we see that

$$\psi(v_1 \otimes \cdots \otimes (X \otimes t_i^{-1}) v_i \otimes \cdots \otimes v_n) = -\sum_{j \neq i} \sum_{m=0}^{\infty} a_m^{(j)} \psi(v_1 \otimes \cdots \otimes (X \otimes t_j^m) v_j \otimes \cdots \otimes v_n). \tag{6.90}$$

Now, we use our assumption that $v_i \in \overline{V}_{\lambda_i}$, so that $L_n v_i = 0$ for $n > 0$ and v_i is an eigenstate of L_0 for all $v_i \in \overline{V}_{\lambda_i}$. This follows from the fact that any element of $\overline{V}_{\lambda_i}$ is obtained by acting on the highest weight vector by elements of the horizontal subalgebra, and L_n commutes with all elements of the horizontal subalgebra. By the Sugawara construction in Section 4.1, we therefore have

$$(L_{-1}^{(i)} \psi)_0 = \sum_a \sum_{j \neq i} (z_i - z_j)^{-1} \psi_0(v_1 \otimes \cdots \otimes J^a v_i \otimes \cdots \otimes J^a v_j \otimes \cdots \otimes v_n) \tag{6.91}$$

$$= \sum_{j \neq i} \frac{\Omega^{(ij)} \psi_0}{z_i - z_j}. \tag{6.92}$$

Since for all $v_i \in \overline{V}_{\lambda_i}$, $L_n v_i = 0$ for $n > 0$ and v_i is an eigenstate of L_0, we also have

$$(L_n^{(i)} \psi)_0 = 0 \text{ for all } n > 0 \tag{6.93}$$

$$(L_0^{(i)}\psi)_0 = \Delta_{\lambda_i}\psi_0 \text{ for some constant } \Delta_{\lambda_i}. \tag{6.94}$$

Finally, ψ_0 is flat and so the covariant derivative

$$\nabla_{\partial_j}\psi_0 = \frac{\partial \psi_0}{\partial z_j} - L_{-1}^{(j)}\psi_0 \tag{6.95}$$

vanishes for all j. We therefore arrive at the KZ equations

$$\frac{\partial \psi_0}{\partial z_i} = \sum_{j\neq i} \frac{\Omega^{(ij)}\psi_0}{z_i - z_j}. \tag{6.96}$$

6.5 Monodromy of the KZ Equations

For any smooth vector bundle $E \to M$ with flat connection, we can define a representation of $\pi_1(M)$ called the monodromy representation. In the case of the conformal block bundle, $\pi_1(\mathrm{Conf}_n(\mathbb{C}))$ is the pure braid group and so we obtain a representation of this group since the KZ connection is flat. This can then be used to obtain a representation of the braid group on n strands. We begin by explaining how one obtains the monodromy representation in general before providing an explicit example of such a representation.

Definition 6.5.1. [35, p. 159] Let $\pi : E \to N$ be a smooth vector bundle and $\phi : M \to N$ a smooth map. Let ∇ be a connection on E. Note that the notion of a pullback bundle is defined in Example B.0.4. The pullback connection $\phi^*\nabla$ is defined on $\phi^*(E)$ by requiring that the relation

$$(\phi^*\nabla)_X(\phi^*\sigma) = \phi^*(\nabla_{d\phi(X)}\sigma) \tag{6.97}$$

holds for all tangent vectors X and sections $\sigma \in \Gamma(E)$.[5]

At the end of Appendix B, we show that this does in fact define a connection. The case where $\gamma : [0,1] \to M$ is a smooth curve on M is particularly important. In this case, we will call a section $\sigma \in \Gamma(\gamma^*E)$ parallel if

$$(\gamma^*\nabla)_X\sigma(p) = 0 \tag{6.98}$$

for all $X \in \Gamma(T[0,1])$ and $p \in M$. Let $\{\sigma_i\}$ be a local frame, which we show exists for any smooth vector bundle in Appendix B. In the case of the tangent bundle, we denote such a local frame by $\{\partial_i\}$. Since $\nabla_{\partial_i}\sigma_j \in \Gamma(E)$, we can expand this in terms of our local frame to obtain

$$\nabla_{\partial_i}\sigma_j = \Gamma_{ij}^k\sigma_k \tag{6.99}$$

for some $\Gamma_{ij}^k \in C^\infty(M)$ called the Christoffel symbols.

[5] It is important to note that we consider tangent vectors X and not vector fields. By $d\phi(X)$, we mean $d\phi_p(X)$ where $X \in T_pM$.

Now, let $X \in \Gamma(T[0,1])$ be a vector field and let $\eta \in \Gamma(\gamma^* E)$ be an arbitrary section. By Example B.0.1, $X = \alpha(t)\partial_t$ with $t \in [0,1]$ and $\alpha \in C^\infty([0,1])$. We can also write $\eta = \eta^j(t)(\gamma^*\sigma_j)(t)$ locally for some $\eta^i \in C^\infty([0,1])$ By Equations (6.99), (6.35), and (6.36), we obtain

$$\begin{aligned}
(\gamma^*\nabla)_{\alpha(t)\partial_t}\eta &= (\gamma^*\nabla)_{\alpha(t)\partial_t}\eta^j(t)(\gamma^*\sigma_j)(t) && (6.100)\\
&= \alpha(t)(\gamma^*\nabla)_{\partial_t}\eta^j(t)(\gamma^*\sigma_j)(t) && (6.101)\\
&= \alpha(t)(\partial_t\eta^j(t)\sigma_j(t) + \eta^j(t)(\gamma^*\nabla)_{\partial_t}(\gamma^*\sigma_j)(t)) && (6.102)\\
&= \alpha(t)(\partial_t\eta^j(t)\sigma_j(t) + \eta^j(t)\widetilde{\Gamma}^k_{tj}(\gamma^*\sigma_k)(t)) && (6.103)
\end{aligned}$$

where $\widetilde{\Gamma}^k_{tj}\gamma^*\sigma_k := \nabla_{\partial_t}\gamma^*\sigma_j$ are the Christoffel symbols of the pullback bundle.

We now deduce that a section $\eta \in \Gamma(\gamma^* E)$ is parallel if we locally have

$$\partial_t\eta^j(t)\sigma_j(t) + \eta^j(t)\widetilde{\Gamma}^k_{tj}(\gamma^*\sigma_k)(t) = 0 \qquad (6.104)$$

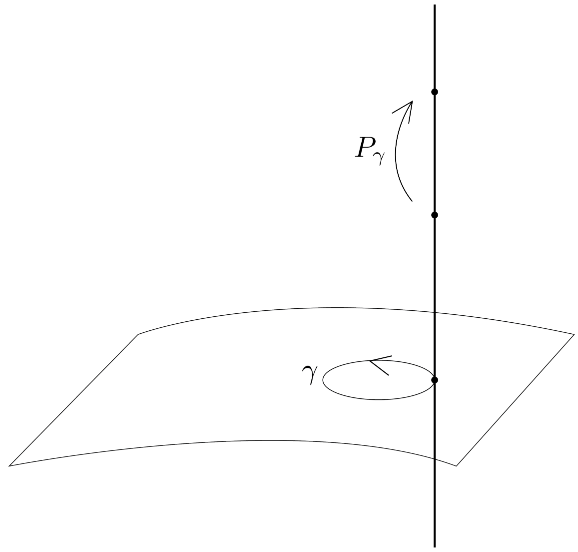
for all $t \in [0,1]$ on any open set. This is a system of first order differential equations for the components of η. If we fix $\eta(0) \in \pi^{-1}(\gamma(0))$, we obtain an initial value problem. By the Picard-Lindelöf theorem, this has a unique solution. We can now define a map called the parallel transport map.

Definition 6.5.2. [35, p. 31] Let ∇ be a connection on a smooth vector bundle $\pi : E \to M$ and let $\gamma : [0,1] \to M$ be a smooth curve on M. The parallel transport map is the map

$$\begin{aligned}
P_\gamma : E_{\gamma(0)} &\to E_{\gamma(1)} && (6.105)\\
e &\mapsto \eta_e(1) && (6.106)
\end{aligned}$$

where η_e is the unique smooth section satisfying $\eta_e(0) = e$ and whose components satisfy Equation (6.104).

Given a path in the base space of a smooth vector bundle, the parallel transport map allows us to move elements between two fibres in a way that depends on the path taken. Of particular interest to us is the case where the path is a loop. The image below gives a visualisation of the parallel transport map for a loop in the base space of a line bundle.



In the case where γ is a loop, it is not true in general that homotopic loops give the same parallel transport map. However, for certain connections they do and this allows us to study the monodromy of these vector bundles. In fact, we have the following theorem.

Theorem 6.5.1. *[21, p. 451] The connection ∇ is flat if and only if for any two homotopic loops γ_1, γ_2 on M, we have $P_{\gamma_1} = P_{\gamma_2}$.*

If γ_1 and γ_2 are loops at p, notice that uniqueness of the solution to the initial value problem given by Equation (6.104) implies

$$P_{\gamma_1\gamma_2} = P_{\gamma_2} \circ P_{\gamma_1} \tag{6.107}$$

where $\gamma_1\gamma_2$ is the loop traversing γ_1 for $t \in [0, 1/2]$ and γ_2 for $t \in [1/2, 1]$. We then see that P_γ is invertible for all loops γ. If $\gamma : [0, 1] \to M$ is a loop then define $\gamma^{-1} : [0, 1] \to M$ by $\gamma^{-1}(t) = \gamma(1 - t)$. Theorem 6.5.1 then implies

$$P_\gamma \circ P_{\gamma^{-1}} = P_{\gamma^{-1}} \circ P_\gamma = P_{id} \tag{6.108}$$

where $id(t) = p$ for all $t \in [0, 1]$. Noting that $(id)\gamma$ is homotopic to γ for any loop γ, we see that there is a group structure (with P_{id} being the identity) on the collection of parallel transport maps for a given $p \in M$. We call this the holonomy group at p [21, p. 450].[6]

Definition 6.5.3. [21, p. 450] For a smooth vector bundle $\pi : E \to M$ with flat connection, the monodromy representation of the fundamental group $\pi_1(M, p)$ is a representation ρ defined by

$$\rho : \pi_1(M, p) \to GL(\pi^{-1}(p)) \tag{6.109}$$
$$\gamma \mapsto P_\gamma. \tag{6.110}$$

This is a representation by Equation (6.107).

[6] A precise relationship between holonomy and curvature for principal bundles is given by the Ambrose-Singer theorem.

We now see that the monodromy representation of the conformal block bundle is a representation of the pure braid group $\pi_1(\mathrm{Conf}_n(\mathbb{C}))$. Kohno [25, p. 80] explains how to use this to obtain a representation of the Artin braid group in the case where $V_1 = V_2 = \cdots = V_n = V$. One defines a right action of the symmetric group S_n on $\mathrm{Hom}_{\mathbb{C}}(V^{\otimes n}, \mathbb{C})$ by

$$(\phi \cdot \sigma)(v_1 \otimes \cdots \otimes v_n) = \phi(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}) \tag{6.111}$$

for $\phi \in \mathrm{Hom}_{\mathbb{C}}(V^{\otimes n}, \mathbb{C})$ and $\sigma \in S_n$. It can then be shown that the KZ equation on the vector bundle

$$\mathrm{Conf}_n(\mathbb{C}) \times_{S_n} \mathrm{Hom}_{\mathbb{C}}(V^{\otimes n}, \mathbb{C}) \to \mathrm{Conf}_n(\mathbb{C})/S_n \tag{6.112}$$

is flat, and so we obtain our desired representation.

Example 6.5.1. We will now give an explicit representation of the braid group on 4 strands. Recall the expressions we obtained for the four-point correlators in Section 5.5. We computed six correlators and the span of these six correlators give a subspace of the vector space of solutions to the KZ equations. It was shown that the following two linearly independent correlators give a basis for this subspace. These are

$$\langle \phi(z_1)\phi(z_2)\psi(z_3)\psi(z_4)\rangle = f_4(\eta) \prod_{1\le i<j\le 4} z_{ij}^{-1/6} = \frac{c_1}{\eta^{1/3}(\eta+1)^{1/3}} \prod_{1\le i<j\le 4} z_{ij}^{-1/6} \tag{6.113}$$

$$\langle \phi(z_1)\psi(z_2)\psi(z_3)\phi(z_4)\rangle = f_5(\eta) \prod_{1\le i<j\le 4} z_{ij}^{-1/6} = \frac{c_1\eta^{2/3}}{(\eta+1)^{1/3}} \prod_{1\le i<j\le 4} z_{ij}^{-1/6} \tag{6.114}$$

where the cross ratio is defined to be

$$\eta = \frac{z_{14}z_{23}}{z_{12}z_{34}}. \tag{6.115}$$

Recall from Section 4.4 that one of the affine Ward identities was

$$f_6(\eta) = f_4(\eta) + f_5(\eta) \tag{6.116}$$

where

$$\langle \psi(z_1)\phi(z_2)\psi(z_3)\phi(z_4)\rangle = f_6(\eta) \prod_{1\le i<j\le 4} z_{ij}^{-1/6} = \frac{c_1(\eta+1)^{2/3}}{\eta^{1/3}} \prod_{1\le i<j\le 4} z_{ij}^{-1/6}. \tag{6.117}$$

Now, using the identity

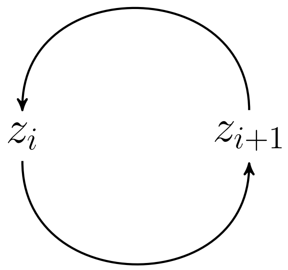
$$z_{13}z_{24} - z_{12}z_{34} = z_{14}z_{23} \tag{6.118}$$

we can see how η transforms when interchanging any two of our four coordinates. To keep track of our branches, we need to write down a path defining this interchange. For $1 \le i \le 3$ and $t \in [0,1]$, we will consider the transformations

$$z_i \mapsto \frac{1}{2}(z_i + z_{i+1}) + \frac{1}{2}(z_i - z_{i+1})e^{\pi i t} \tag{6.119}$$

$$z_{i+1} \mapsto \frac{1}{2}(z_i + z_{i+1}) + \frac{1}{2}(z_{i+1} - z_i)e^{\pi i t}. \tag{6.120}$$

This corresponds to $z_i - z_{i+1} \mapsto e^{\pi i t}(z_i - z_{i+1})$, and so the coordinates are being rotated rotated anticlockwise as shown in the diagram below.



Under the interchange $z_1 \leftrightarrow z_2$, we have

$$\begin{aligned}
\eta &\mapsto \frac{z_{24}z_{13}}{e^{\pi i}z_{12}z_{34}} && (6.121)\\
&= \frac{e^{-\pi i}(z_{12}z_{34} + z_{14}z_{23})}{z_{12}z_{34}} && (6.122)\\
&= e^{-\pi i}(1+\eta). && (6.123)
\end{aligned}$$

Under the interchange $z_2 \leftrightarrow z_3$, we have

$$\begin{aligned}
\eta &\mapsto e^{\pi i}\frac{z_{14}z_{23}}{z_{13}z_{24}} && (6.124)\\
&= e^{\pi i}\frac{z_{14}z_{23}}{z_{12}z_{34} + z_{14}z_{23}} && (6.125)\\
&= e^{\pi i} + \frac{z_{12}z_{34}}{z_{12}z_{34} + z_{14}z_{23}} && (6.126)\\
&= e^{\pi i} + \frac{1}{1+\eta} && (6.127)\\
&= e^{\pi i}\frac{\eta}{1+\eta}. && (6.128)
\end{aligned}$$

Finally, under the interchange $z_3 \leftrightarrow z_4$, we have

$$\begin{aligned}
\eta &\mapsto \frac{z_{13}z_{24}}{e^{\pi i}z_{12}z_{34}} && (6.129)\\
&= \frac{e^{-\pi i}(z_{12}z_{34} + z_{14}z_{23})}{z_{12}z_{34}} && (6.130)\\
&= e^{-\pi i}(1+\eta). && (6.131)
\end{aligned}$$

Under the interchange $z_1 \leftrightarrow z_2$ (and also $z_3 \leftrightarrow z_4$), we therefore have

$$f_4(\eta) = \frac{c_1}{\eta^{1/3}(\eta+1)^{1/3}} \mapsto \frac{c_1}{(e^{-\pi i}(1+\eta))^{1/3}(e^{-\pi i}\eta)^{1/3}} \tag{6.132}$$

$$= \frac{c_1}{e^{-2\pi i/3}(1+\eta)^{1/3}(\eta)^{1/3}} \tag{6.133}$$

$$= e^{2\pi i/3} f_4(\eta). \tag{6.134}$$

Furthermore,

$$f_5(\eta) = \frac{c_1 \eta^{2/3}}{(\eta+1)^{1/3}} \mapsto \frac{c_1(e^{-\pi i}(1+\eta))^{2/3}}{(e^{-\pi i}\eta)^{1/3}} \tag{6.135}$$

$$= e^{-\pi i/3} f_6(\eta) \tag{6.136}$$

$$= e^{-\pi i/3}(f_4(\eta) + f_5(\eta)). \tag{6.137}$$

The last equality follows from Equation (6.116). To see how the correlators transform, note that for the interchange $z_i \leftrightarrow z_{i+1}$ we have

$$\prod_{1 \le i < j \le 4} z_{ij}^{-1/6} = z_{12}^{-1/6} z_{13}^{-1/6} z_{14}^{-1/6} z_{23}^{-1/6} z_{24}^{-1/6} z_{34}^{-1/6} \mapsto e^{-\pi i/6} \prod_{1 \le i < j \le 4} z_{ij}^{-1/6}. \tag{6.138}$$

We can now represent the transformation of the basis correlators $\langle \phi(z_1)\phi(z_2)\psi(z_3)\psi(z_4)\rangle$ and
$\langle \phi(z_1)\psi(z_2)\psi(z_3)\phi(z_4)\rangle$ under the interchange $z_1 \leftrightarrow z_2$ as

$$\begin{pmatrix} \langle \phi(z_1)\phi(z_2)\psi(z_3)\psi(z_4)\rangle \\ \langle \phi(z_1)\psi(z_2)\psi(z_3)\phi(z_4)\rangle \end{pmatrix} \mapsto e^{-\pi i/6} \begin{pmatrix} e^{2\pi i/3} & 0 \\ e^{-\pi i/3} & e^{-\pi i/3} \end{pmatrix} \begin{pmatrix} \langle \phi(z_1)\phi(z_2)\psi(z_3)\psi(z_4)\rangle \\ \langle \phi(z_1)\psi(z_2)\psi(z_3)\phi(z_4)\rangle \end{pmatrix}. \tag{6.139}$$

We will define

$$A := e^{-\pi i/6} \begin{pmatrix} e^{2\pi i/3} & 0 \\ e^{-\pi i/3} & e^{-\pi i/3} \end{pmatrix} = \begin{pmatrix} e^{\pi i/2} & 0 \\ e^{-\pi i/2} & e^{-\pi i/2} \end{pmatrix}. \tag{6.140}$$

Note that this matrix also represents the transformation of the correlators under $z_3 \leftrightarrow z_4$ since the transformation of η is the same for both $z_1 \leftrightarrow z_2$ and $z_3 \leftrightarrow z_4$. We will now compute the transformation matrix corresponding to $z_2 \leftrightarrow z_3$.

Under the interchange $z_2 \leftrightarrow z_3$, we have

$$f_4(\eta) = \frac{c_1}{\eta^{1/3}(\eta+1)^{1/3}} \mapsto \frac{c_1}{(e^{\pi i}\eta/(1+\eta))^{1/3}(1/(1+\eta))^{1/3}} \tag{6.141}$$

$$= \frac{(1+\eta)^{2/3}}{e^{\pi i/3}\eta^{1/3}} \tag{6.142}$$

$$= e^{-\pi i/3} f_6(\eta) \tag{6.143}$$

$$= e^{-\pi i/3}(f_4(\eta) + f_5(\eta)) \tag{6.144}$$

where the last equality follows from Equation (6.116). Furthermore,

$$f_5(\eta) = \frac{c_1\eta^{2/3}}{(\eta+1)^{1/3}} \mapsto \frac{c_1(e^{\pi i}\eta/(1+\eta))^{2/3}}{(1/(1+\eta))^{1/3}} \tag{6.145}$$

$$= e^{2\pi i/3} \frac{c_1\eta^{2/3}}{(1+\eta)^{1/3}} \tag{6.146}$$

$$= e^{2\pi i/3} f_5(\eta). \tag{6.147}$$

We now see that the correlators $\langle\phi(z_1)\phi(z_2)\psi(z_3)\psi(z_4)\rangle$ and $\langle\phi(z_1)\psi(z_2)\psi(z_3)\phi(z_4)\rangle$ transform as

$$\begin{pmatrix} \langle\phi(z_1)\phi(z_2)\psi(z_3)\psi(z_4)\rangle \\ \langle\phi(z_1)\psi(z_2)\psi(z_3)\phi(z_4)\rangle \end{pmatrix} \mapsto e^{-\pi i/6} \begin{pmatrix} e^{-\pi i/3} & e^{-\pi i/3} \\ 0 & e^{2\pi i/3} \end{pmatrix} \begin{pmatrix} \langle\phi(z_1)\phi(z_2)\psi(z_3)\psi(z_4)\rangle \\ \langle\phi(z_1)\psi(z_2)\psi(z_3)\phi(z_4)\rangle \end{pmatrix}. \tag{6.148}$$

We then define

$$B := e^{-\pi i/6} \begin{pmatrix} e^{-\pi i/3} & e^{-\pi i/3} \\ 0 & e^{2\pi i/3} \end{pmatrix} = \begin{pmatrix} e^{-\pi i/2} & e^{-\pi i/2} \\ 0 & e^{\pi i/2} \end{pmatrix}. \tag{6.149}$$

Given our calculations, we can now construct a two-dimensional representation of the braid group on 4 strands as follows. Let V be the complex vector space spanned by the correlators $\langle\phi(z_1)\phi(z_2)\psi(z_3)\psi(z_4)\rangle$ and $\langle\phi(z_1)\psi(z_2)\psi(z_3)\phi(z_4)\rangle$. Let $\{\sigma_i\}_{1\leq i\leq 3}$ denote the generators of the braid group on 4 strands. Define

$$\rho : B_4 \to \mathrm{End}(V) \tag{6.150}$$
$$\sigma_1 \mapsto A \tag{6.151}$$
$$\sigma_2 \mapsto B \tag{6.152}$$
$$\sigma_3 \mapsto A. \tag{6.153}$$

Since $\rho(\sigma_1) = \rho(\sigma_3)$, we have $\rho(\sigma_1)\rho(\sigma_3) = \rho(\sigma_3)\rho(\sigma_1)$. Therefore, to show that ρ defines a representation, we need only show that $ABA = BAB$. We have

$$ABA = \begin{pmatrix} e^{\pi i/2} & 0 \\ e^{-\pi i/2} & e^{-\pi i/2} \end{pmatrix} \begin{pmatrix} e^{-\pi i/2} & e^{-\pi i/2} \\ 0 & e^{\pi i/2} \end{pmatrix} \begin{pmatrix} e^{\pi i/2} & 0 \\ e^{-\pi i/2} & e^{-\pi i/2} \end{pmatrix} \tag{6.154}$$
$$= \begin{pmatrix} e^{\pi i/2} & 0 \\ e^{-\pi i/2} & e^{-\pi i/2} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \tag{6.155}$$
$$= \begin{pmatrix} 0 & e^{-\pi i/2} \\ e^{-\pi i/2} & 0 \end{pmatrix}. \tag{6.156}$$

Also,

$$BAB = \begin{pmatrix} e^{-\pi i/2} & e^{-\pi i/2} \\ 0 & e^{\pi i/2} \end{pmatrix} \begin{pmatrix} e^{\pi i/2} & 0 \\ e^{-\pi i/2} & e^{-\pi i/2} \end{pmatrix} \begin{pmatrix} e^{-\pi i/2} & e^{-\pi i/2} \\ 0 & e^{\pi i/2} \end{pmatrix} \tag{6.157}$$
$$= \begin{pmatrix} e^{-\pi i/2} & e^{-\pi i/2} \\ 0 & e^{\pi i/2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \tag{6.158}$$
$$= \begin{pmatrix} 0 & e^{-\pi i/2} \\ e^{-\pi i/2} & 0 \end{pmatrix} \tag{6.159}$$

and so we conclude $ABA = BAB$ as required. We have therefore constructed a two-dimensional representation of the braid group on 4 strands by studying the monodromy of our solutions to the KZ equations.[7]

[7]This is not the complete monodromy representation since we have used our solutions with no dependence on the hypergeometric functions. One should be able to obtain a more general representation that reduces to this one for the case $c_2 = 0$, where c_2 is the second arbitrary constant in our general solution to the KZ equations.

Conclusion

In this thesis, we introduced the theory of semisimple Lie algebras and affine Lie algebras. Of particular importance was our loop algebra construction of the untwisted affine Lie algebras. We used this loop algebra construction in the study of the Wess-Zumino-Witten models. We saw that the states of these models could be realised as elements of certain highest weight representations of an untwisted affine Lie algebra.

In our study of the Wess-Zumino-Witten models, an important result was the Sugawara construction. This construction gave us a canonical way to embed the universal enveloping algebra of the Virasoro algebra into a completion of the universal enveloping algebra of any untwisted affine Lie algebra. Another important result was the derivation of the $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ Knizhnik-Zamolodchikov equations at level k. We saw that the equations we arrived at differed slightly from those commonly used in the literature. This disparity is a result of an assumption regarding commuting fields that is not completely justified.

Our main result in the study of the Wess-Zumino-Witten models was the computation of a collection of four-point correlation functions. For this, we used the Ward identities, the affine Ward identities, and the Knizhnik-Zamolodchikov equations. We then introduced vertex operator algebras to give a free field realisation of the $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ Wess-Zumino-Witten model at level 1. This allowed us to verify our solutions to the Knizhnik-Zamolodchikov equations using vertex operators.

Finally, we introduced the Knizhnik-Zamolodchikov equations on the conformal block bundle. We showed how one can obtain a representation of the fundamental group of the base space of any flat vector bundle. In the case of the conformal block bundle, this gives a representation of the braid group. By studying the monodromy of our solutions to the Knizhnik-Zamolodchikov equations, we constructed a two-dimensional representation of the Artin braid group on 4 strands.

There are further questions that can be investigated. It would be interesting to see if we could explicitly construct a representation of the braid group using our solutions to the Knizhnik-Zamolodchikov equations for arbitrary constants c_1 and c_2. These general solutions contain hypergeometric functions and so one would need some identities of hypergeometric functions to construct such a representation.

There are also many topics worth investigating that are related to this thesis. For example, the monodromy representation on the conformal block bundle has connections to quantum groups. An important result in this area is the Drinfeld-Kohno theorem, which states that the monodromy representation of the braid group is equivalent to a representation obtained from a certain universal R-matrix [21, p. 459]. One could verify this explicitly for the case of the representation we constructed.

Instead of looking at purely mathematical aspects of this area, one could continue in the direction of conformal field theory. There are many more complicated conformal field theories currently being studied, such as those of the logarithmic kind. An introduction to logarithmic conformal field theory is provided by [7]. There are other fundamental concepts in conformal field theory worth investigating including characters, modular invariance, and fusion rules. There are also more sophisticated methods of constructing conformal field theories such as the orbifold and coset constructions. This is all discussed in detail in [8].

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Appendix A

Lie Theory Results

Theorem A.0.1. *Let κ_{ad} denote the Killing form of a Lie algebra $\mathfrak{g}$ and let $\kappa = \frac{1}{2h^\vee}\kappa_{ad}$ denote the rescaled Killing form. Let $\{J^a\}$ be a basis for $\mathfrak{g}$ which is orthonormal with respect to the rescaled Killing form. Then*

$$2h^\vee \delta_{a,d} = \sum_{b,c} f_{abc} f_{cbd}. \tag{A.1}$$

Proof. We have

$$\begin{aligned}
tr(ad(J^c)[ad(J^a), ad(J^b)]) &= tr(ad(J^c)ad([J^a, J^b])) && \text{(A.2)}\\
&= \sum_d f_{abd} tr(ad(J^c)ad(J^d)) && \text{(A.3)}\\
&= \sum_d 2h^\vee f_{abd}\delta_{c,d} && \text{(A.4)}\\
&= 2h^\vee f_{abc}. && \text{(A.5)}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
tr(ad(J^c)[ad(J^a), ad(J^b)]) &= tr(ad(J^c)ad([J^a, J^b])) && \text{(A.6)}\\
&= tr(ad(J^c)ad(J^a)ad(J^b)) - tr(ad(J^c)ad(J^b)ad(J^a)) && \text{(A.7)}\\
&= tr(ad(J^c)ad(J^a)ad(J^b)) - tr(ad(J^a)ad(J^c)ad(J^b)) && \text{(A.8)}\\
&= tr(ad([J^c, J^a])ad(J^b)) && \text{(A.9)}\\
&= \sum_d f_{cad} tr(ad(J^d)ad(J^b)) && \text{(A.10)}\\
&= 2h^\vee \sum_d f_{cad}\delta_{d,b} && \text{(A.11)}\\
&= 2h^\vee f_{cab}. && \text{(A.12)}
\end{aligned}$$

So, $f_{abc} = f_{cab}$. Now, clearly $f_{abc} = -f_{bac}$. Permuting the indices then gives $f_{abc} = -f_{acb}$. We will now use this to derive our result. Note that

$$[J^a, J^d] = \sum_c f_{adc} J^c \tag{A.13}$$

$$\kappa(J^a, J^d) = tr(ad(J^a)ad(J^d)). \tag{A.14}$$

Now,

$$ad(J^a)ad(J^d)J^c = [J^a, [J^d, J^c]] \tag{A.15}$$

$$= \left[J^a, \sum_b f_{dcb} J^b\right] \tag{A.16}$$

$$= \sum_b f_{dcb} \sum_e f_{abe} J^e \tag{A.17}$$

$$= \sum_{b,e} f_{dcb} f_{abe} J^e. \tag{A.18}$$

So,

$$tr(ad(J^a)ad(J^d)) = \sum_{b,c} f_{dcb} f_{abc} = \sum_{b,c} f_{abc} f_{cbd}. \tag{A.19}$$

Here, we have used total antisymmetry of the structure constants. We conclude that

$$\kappa_{ad}(J^a, J^d) = 2h^\vee \kappa(J^a, J^d) = 2h^\vee \delta_{a,d} = \sum_{b,c} f_{abc} f_{cbd} \tag{A.20}$$

as required. □

Appendix B

Vector Bundles

Definition B.0.1. A real smooth vector bundle of rank k is a triple (E, M, π) satisfying:

1. E and M are smooth manifolds.
2. $\pi : E \to M$ is a smooth surjection.
3. $\pi^{-1}(\{p\}) \cong \mathbb{R}^k$ for all $p \in M$.
4. For every point p of M we require that there is an open neighbourhood U of M containing p and a diffeomorphism
$$\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k \tag{B.1}$$
such that for all $x \in \pi^{-1}(U)$ we have:
   (a) $(pr_U \circ \phi)(x) = \pi(x)$.
   (b) $pr_{\mathbb{R}^k} \circ \phi|_{\pi^{-1}(\{p\})} : \pi^{-1}(\{x\}) \to \mathbb{R}^k$ is a vector space isomorphism.

   Here, pr_U and $pr_{\mathbb{R}^k}$ denote the projection map onto U and $\mathbb{R}^k$ respectively. We call the map ϕ a local trivialisation of the vector bundle. If the vector space $\pi^{-1}(\{p\})$ has dimension k for all $p \in M$ then we say that the vector bundle has rank k.

Definition B.0.2. Given a vector bundle with local trivialisations (U_α, ϕ_α), we obtain functions

$$\phi_\alpha \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n \tag{B.2}$$
$$(p, v) \mapsto (p, \tau_{\alpha\beta}(p)v) \tag{B.3}$$

where $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$ is a smooth map. We call the maps $\tau_{\alpha\beta}$ the transition functions of the local trivialisations.

The following theorem [26, p. 269] gives a useful method of constructing vector bundles.

Theorem B.0.1. *Let M be a smooth manifold with an open cover $\{U_\alpha\}$ and a collection of smooth maps*

$$\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R}) \tag{B.4}$$

satisfying $\tau_{\alpha\beta}\tau_{\beta\gamma} = \tau_{\alpha\gamma}$. Then there is a canonical smooth vector bundle of rank n with local trivialisations whose transition functions are the maps $\tau_{\alpha\beta}$.

Example B.0.1. An important example of a smooth vector bundle is the tangent bundle $\pi : TM = \coprod_{p\in M} T_pM \to M$. Before we discuss the local trivialisations, note that we can view the tangent space T_pM of a manifold M of dimension n at some point p as the vector space of derivations at p, that is the vector space of linear maps $D : C^\infty(M) \to \mathbb{R}$ satisfying

$$D(fg) = D(f)g(p) + f(p)D(g) \tag{B.5}$$

for all $f, g \in C^\infty(M)$. A basis for the tangent space at p is given by the derivations ∂_i defined by $(\partial_i f)(p) = (\partial_i(f \circ \phi^{-1}))(\phi(p))$ where $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$ is a local coordinate containing p and $\partial_i(f \circ \phi^{-1})$ is partial differentiation in the usual sense.[1] Now, let (ϕ_α, U_α) be a chart on M. We can define a map

$$\widehat{\phi_\alpha} : \pi^{-1}(U_\alpha) \to \mathbb{R}^{2n} \tag{B.6}$$

$$\left(x, \sum_{i=1}^{n} v_i\partial_i\right) \mapsto (\phi_\alpha(x), v_1, \ldots, v_n). \tag{B.7}$$

This map defines a topology on TM so that $V \subseteq TM$ is open if and only if $\widehat{\phi_\alpha}(V \cap \pi^{-1}(U_\alpha))$ is open for every α. A smooth structure on TM is then defined by the charts $(\widehat{\phi_\alpha}, \pi^{-1}(U_\alpha))$. Finally, we can define transition functions $\tau_{\alpha\beta}$ by sending an element p of $U_\alpha \cap U_\beta$ to the Jacobian of the map $\phi_\alpha\phi_\beta^{-1}$ at the point $\phi_\alpha(p)$. By Theorem B.0.1, we can construct our desired bundle called the tangent bundle [26, p. 252].

Definition B.0.3. Given a vector bundle E with transition functions $\tau_{\alpha\beta}$, the dual bundle is given by Theorem B.0.1 where we take our transition functions to be $((\tau_{\alpha\beta})^\intercal)^{-1}$ and the fibres are dual to the fibres on E. Here, $((\tau_{\alpha\beta})^\intercal)^{-1}$ is the map defined by $((\tau_{\alpha\beta})^\intercal)^{-1}(p) = ((\tau_{\alpha\beta}(p))^\intercal)^{-1}$ and $(\tau_{\alpha\beta}(p))^\intercal$ denotes the transpose of the linear map $\tau_{\alpha\beta}(p)$.

Example B.0.2. The cotangent bundle $\pi : \coprod_{p\in M} T_pM^* \to M$ is defined to be dual to the tangent bundle.

Example B.0.3. Given two smooth vector bundles E and F over the same smooth manifold M, we can define the tensor product bundle $E \otimes F$ to be the vector bundle

[1] This is proven in [26]. One first shows that this is the case for $M \cong \mathbb{R}^n$ and then taking the pushforward of these basis elements under ϕ^{-1} gives the desired basis for an arbitrary smooth manifold.

whose fibres are the tensor products of the fibres of E and F. To obtain the transition functions, note that we already have the transition functions

$$\tau^E_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R}) \tag{B.8}$$
$$\tau^F_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(m, \mathbb{R}) \tag{B.9}$$

where $n, m \in \mathbb{N}$ are the ranks of E and F respectively. We can then define

$$\tau^{E\otimes F}_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(nm, \mathbb{R}) \tag{B.10}$$
$$p \mapsto \tau^E_{\alpha\beta}(p) \otimes \tau^F_{\alpha\beta}(p). \tag{B.11}$$

Definition B.0.4. A smooth section of a smooth vector bundle $\pi : E \to M$ is a smooth function $\sigma : M \to E$ satisfying $\pi \circ \sigma = id|_M$.

Example B.0.4. Given a smooth vector bundle $\pi : E \to N$ and a smooth map $\phi : M \to N$, we define $\phi^*E = \{(p, e) \in M \times E \mid \phi(p) = \pi(e)\}$. This gives us the commutative diagram below.

$$\begin{array}{ccc} \phi^*E & \xrightarrow{pr_E} & E \\ {\scriptstyle pr_M}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\phi} & N \end{array}$$

We call $pr_M : \phi^*E \to M$ the pullback bundle of E under ϕ. Notice that the commutativity of the above diagram implies that $pr_E(pr_M^{-1}(p)) = \pi^{-1}(\phi(p))$ for all $p \in M$ and so we obtain the vector space isomorphism $pr_M^{-1}(p) \cong \pi^{-1}(\phi(p))$ for all $p \in M$. The local trivialisations of ϕ^*E are obtained from E as follows. Given a local trivialisation (α, U) of E, we obtain a local trivialisation $(\beta, \phi^{-1}(U))$ of ϕ^*E with β defined as

$$\beta : pr_M^{-1}(\phi^{-1}(U)) \to \phi^{-1}(U) \times \mathbb{R}^k \tag{B.12}$$
$$(p, e) \mapsto (p, (pr_{\mathbb{R}^k} \circ \alpha)(e)). \tag{B.13}$$

Given a smooth section $\sigma : N \to E$, we obtain a smooth section $\phi^*\sigma : M \to \phi^*(E)$ called the pullback section defined by

$$\phi^*\sigma(p) = (p, (\sigma \circ \phi)(m)). \tag{B.14}$$

Definition B.0.5. Let $\pi : E \to M$ be a smooth vector bundle with local trivialisations (ϕ_i, U_i). A local frame defined on U_i is a collection of smooth sections $\sigma_i : U_i \to E$ such that $\sigma_i(p)$ is a basis for $\pi^{-1}(p)$ for every $p \in M$.

Given a smooth vector bundle $\pi : E \to M$ of rank n, we can construct a local frame as follows. Define $\sigma_i : U_i \to E$ by $\sigma_i(p) = \phi_i^{-1}(p, e_i)$ where e_i is the standard basis for $\mathbb{R}^n$ and $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$ is a local trivialisation with $p \in U_i$. Then, given a smooth section σ, we have $\phi(\sigma(p)) = (p, \sum_{i=1}^n \alpha_i e_i)$ for some $\alpha_i \in \mathbb{R}$. This implies $\sigma(p) = \phi^{-1}(p, \sum_{i=1}^n \alpha_i e_i) = \sum_{i=1}^n \alpha_i \phi^{-1}(p, e_i) = \sum_{i=1}^n \alpha_i \sigma_i(p)$.[2] For this reason, we can say that a local frame defines a basis for the space of smooth local sections in the sense that any smooth local section σ can be written in the form $\sigma = \sum_{i=1}^n \alpha_i \sigma_i$ where the coefficients α_i depend on $p \in M$. This is useful in Chapter 6 where we would like to view connections locally.

Definition B.0.6. Given a vector space V, the exterior algebra $\Lambda(V)$ is defined to be $T(V)/I$ where I is the ideal generated by elements of the form $x \otimes x$ where $x \in V$. Here, $T(V)$ is the tensor algebra on V. The exterior product $\wedge$ is defined as $x \wedge y = x \otimes y + I$. The kth exterior power of V is the subspace of $\Lambda(V)$ spanned by elements of the form $x_1 \wedge x_2 \wedge \cdots \wedge x_k$. This is denoted $\Lambda^k(V)$.

Definition B.0.7. A smooth differential k-form on a smooth manifold M is a smooth section of the kth exterior power of the cotangent bundle of M. Note that the kth exterior power of the cotangent bundle is defined to be

$$\bigsqcup_{p \in M} \Lambda^k(T_p^* M) \to M. \tag{B.15}$$

For our purposes, we wish to generalise this definition slightly.

Definition B.0.8. A Lie algebra-valued differential k-form is a smooth section of the bundle

$$(\mathfrak{g} \times M) \otimes \bigsqcup_{p \in M} \Lambda^k(T_p^* M) \to M. \tag{B.16}$$

Here, $\pi : \mathfrak{g} \times M \to M$ is a trivial vector bundle. We then take the tensor product bundle with $\bigsqcup_{p \in M} \Lambda^k(T_p^* M) \to M$.

Similarly, one can define a vector space-valued differential k-form. We have a useful characterisation of Lie algebra valued differential forms given in [29].

Theorem B.0.2. *Let G be a Lie group and $\mathfrak{g}$ its associated Lie algebra. A map*

$$h \mapsto f_h \tag{B.17}$$

where $f_h : T_h G \times \cdots \times T_h G \to \mathfrak{g}$ is an alternating multilinear map defines a Lie algebra valued differential form.

To conclude this section, we show that Equation (6.97) defines a connection on $\phi^*(E)$. Consider a local frame $\{\sigma_i\}$ of E. Then it is easy to show that the pullback sections $\{\phi^* \sigma_i\}$ give a local frame of $\phi^*(E)$. Given $(p, e) \in \phi^*(E)$, we can write $e = \sum_{i=1}^n \alpha_i(\phi(p)) \sigma_i(\phi(p))$ for some $\alpha_i \in C^\infty(N)$ since $e \in \pi^{-1}(\phi(p))$. Then we have

$$(p, e) = \left(p, \sum_{i=1}^n \alpha_i(\phi(p))(\sigma_i \circ \phi)(p) \right) \tag{B.18}$$

[2]Recall that the restriction of a local trivialisation to a fibre defines a vector space isomorphism and so the second equality holds.

$$= \sum_{i=1}^{n} (\alpha_i \circ \phi)(p)(p, (\sigma_i \circ \phi)(p)) \tag{B.19}$$

$$= \sum_{i=1}^{n} \beta_i(p)(\phi^* \sigma_i)(p) \tag{B.20}$$

where $\beta_i = \alpha_i \circ \phi \in C^\infty(M)$. Now, given any local section η of $\phi^*(E)$, we can use our local frame of $\phi^*(E)$ to write

$$\eta = \sum_{i=1}^{n} \gamma_i \phi^* \sigma_i \tag{B.21}$$

for some $\gamma_i \in C^\infty(M)$. So, given that we have defined $\phi^*\nabla$ on any pullback section $\phi^*\sigma$, we can use our local frame $\{\phi^*\sigma_i\}$, $\mathbb{R}$-linearity, and the Leibniz rule to define $\phi^*\nabla$ on any section of ϕ^*E.

Appendix C

Lie Groups

In this section, we briefly discuss Lie groups and introduce the Maurer-Cartan form. The Maurer-Cartan form will be used in Section 3.1 to define the action of the Wess-Zumino-Witten models. Hall [15] provides an excellent discussion of matrix Lie groups. For a discussion of more general Lie groups, see [23] or [6].

Definition C.0.1. A Lie group is a topological group equipped with a smooth structure such that the group multiplication and inversion are smooth maps.

Given a Lie group G with identity element e, its corresponding Lie algebra $\mathfrak{g}$ is defined to be the vector space T_eG.[1] The Lie bracket is constructed as follows [9, pp. 2-3]. First, for $X \in G$ define

$$\begin{aligned} \mathrm{Ad}_X : G &\to G && \text{(C.1)} \\ Y &\mapsto XYX^{-1}. && \text{(C.2)} \end{aligned}$$

Taking the pushforward of this map at the identity gives a map

$$T_e(\mathrm{Ad}_X) : T_eG \to T_eG. \tag{C.3}$$

Now, define

$$\begin{aligned} \mathrm{Ad} : G &\to GL(T_e(G)) && \text{(C.4)} \\ X &\mapsto T_e(\mathrm{Ad}_X). && \text{(C.5)} \end{aligned}$$

The map Ad defines a Lie bracket on $T_e(G)$ by

$$[X, Y] = \mathrm{Ad}(X)(Y). \tag{C.6}$$

We can study a Lie group from its corresponding Lie algebra. More specifically, we can recover the local information of a Lie group through the exponential map,

[1] Alternatively, one defines the corresponding Lie algebra as the space of left invariant vector fields on G. The Lie bracket is given by noting that a vector field on G can be viewed as a derivation on the ring of smooth functions on G, so that one can take the commutator of two vector fields.

which we define as[2]

$$\exp : \mathfrak{g} \to G \tag{C.7}$$
$$X \mapsto \gamma(1). \tag{C.8}$$

Here, $\gamma : \mathbb{R} \to G$ is the unique continuous group homomorphism satisfying $\gamma'(0) = X$.

Example C.0.1. The group $SL(2, \mathbb{C})$ of 2×2 complex matrices of determinant 1 is a complex Lie group whose corresponding Lie algebra is $\mathfrak{sl}_2(\mathbb{C})$. The outline of the argument is as follows. We can view $SL(2, \mathbb{C})$ as a (closed) submanifold of $GL(2, \mathbb{C})$ and so the Lie group structure of $SL(2, \mathbb{C})$ is inherited from that of $GL(2, \mathbb{C})$.[3] To see what the tangent vectors look like at the identity, we can (to first order) consider a curve

$$\gamma(t) = I + tA \tag{C.9}$$

on $SL(2, \mathbb{C})$ for some matrix A. To first order we have

$$\det \gamma(t) = 1 + tr(A)t = 1 \tag{C.10}$$

and so we can deduce that $tr(A) = 0$. We can then see that the Lie algebra of $SL(2, \mathbb{C})$ must correspond to traceless 2×2 complex matrices, which we can identity as $\mathfrak{sl}_2(\mathbb{C})$.

In Appendix B, we introduce vector bundles and the notion of a Lie algebra-valued differential 1-form. This allows us to define the following 1-form that will be used in Section 3.1.

Definition C.0.2. Let G be a Lie group. Define

$$\begin{aligned} f_g : G &\to G \\ h &\mapsto gh. \end{aligned} \tag{C.11}$$

Note that f_g is smooth for all $g \in G$ since G is a Lie group. The pushfoward of $f_{g^{-1}}$ at g defines the Maurer-Cartan form ω_g.

We can see that the Maurer-Cartan form associates to every point $g \in G$ a map

$$\omega_g : T_g G \to \mathfrak{g}. \tag{C.12}$$

By Theorem B.0.2, this is a Lie algebra-valued differential 1-form.

[2]We call this the exponential map because we can write $\exp(X) = \sum_{i=0}^{\infty} \frac{X^i}{i!}$ for certain matrix Lie groups.

[3]This is a theorem of Cartan called the closed-subgroup theorem.

Appendix D

Conformal Invariance for the Wess-Zumino-Witten Models

We would like to obtain the OPE $J^a(z)J^b(w)$. This can be done using conformal invariance. Our method of obtaining the OPE follows [33, pp. 25-27]. This is essentially the same method we used to compute the OPE given by Equation (3.77), but it requires a bit more work. In the holomorphic sector, we have a conserved charge

$$Q = \frac{1}{2\pi i}\oint_0 \kappa(\epsilon(z), J(z))dz - \frac{1}{2\pi i}\oint_0 \kappa(\overline{\epsilon}(\overline{z}), \overline{J}(\overline{z}))d\overline{z} \tag{D.1}$$

corresponding to a symmetry

$$g \mapsto \overline{f} g f \tag{D.2}$$

with f and $\overline{f}$ holomorphic and antiholomorphic respectively. Using the exponential map, we can write

$$f(z) = \exp(\epsilon(z)) \tag{D.3}$$

$$\overline{f}(\overline{z}) = \exp(\overline{\epsilon}(\overline{z})) \tag{D.4}$$

where $\epsilon(z)$ and $\overline{\epsilon}(\overline{z})$ take values in the Lie algebra. Under the symmetry

$$g \mapsto \overline{f} g f \tag{D.5}$$

we have (to first order in ϵ, $\overline{\epsilon}$, $\partial\epsilon$, and $\partial\overline{\epsilon}$)

$$J(z) \mapsto k(g^{-1} - \epsilon(z)g^{-1} - g^{-1}\overline{\epsilon}(\overline{z}))(\partial g + \partial\overline{\epsilon}(\overline{z}) + \overline{\epsilon}(\overline{z})\partial g + \partial g\epsilon(z) + g\partial\epsilon(z)). \tag{D.6}$$

So,

$$\begin{aligned}\delta_\epsilon J(z) &= -k\epsilon(z)g^{-1}\partial g + kg^{-1}\partial g\epsilon(z) + k\partial\epsilon(z) &\text{(D.7)}\\ &= [J(z), \epsilon(z)] + k\partial\epsilon(z). &\text{(D.8)}\end{aligned}$$

Now, let $\{J^a\}$ be a basis for $\mathfrak{g}$ orthonormal with respect to the Killing form. Write

$$\epsilon(z) = \sum_a \epsilon^a(z) J^a \tag{D.9}$$

$$J(z) = \sum_a J^a(z) J^a \tag{D.10}$$

for scalar fields $\epsilon(z)$ and $J^a(z)$. We have

$$\kappa(\epsilon(z), J(z)) = \sum_{a,b} \epsilon^a(z) J^b(z) \kappa(J^a, J^b) \tag{D.11}$$

$$= \sum_a \epsilon^a(z) J^a(z). \tag{D.12}$$

This allows us to write

$$\delta_\epsilon J(w) = [Q, J^a(w)] \tag{D.13}$$

$$= \frac{1}{2\pi i} \oint_0 \sum_a \epsilon^a(z) [J^a(z), J^b(w)] dz \tag{D.14}$$

$$= \sum_a \frac{1}{2\pi i} \oint_w \epsilon^a(z) \mathcal{R}\{J^a(z) J^b(w)\} dz \tag{D.15}$$

In the last line, we have used the contour trick from Section 3.3. Using Equation (D.8), we can also write

$$\delta_\epsilon J(w) = \sum_a \left(\sum_{b,c} f_{abc} J^b(w) \epsilon^c(w) + k \partial \epsilon^a(w) \right) \tag{D.16}$$

using total antisymmetry of the structure constants f_{abc} (proven in Appendix A). Comparing this with Equation (D.15), we can read off the OPE

$$J^a(z) J^b(w) \sim \frac{k \delta_{a,b}}{(z-w)^2} + \sum_c f_{abc} \frac{J^c(w)}{z-w}. \tag{D.17}$$

In an arbitrary basis for the Lie algebra $\mathfrak{g}$, the $\delta_{a,b}$ term should be replaced with $\kappa(J^a, J^b)$.

Appendix E

Canonical Quantisation of the Free Boson

We begin with studying a classical field $\phi(\theta, t)$ on a cylinder parametrised by angle θ and height t. We require that ϕ satisfies periodic boundary conditions. The Lagrangian density for the free boson is [8]

$$\mathcal{L} = \frac{g}{2}\left[(\partial_t \phi)^2 - (\partial_\theta \phi)^2\right]. \tag{E.1}$$

Here, g is a coupling constant. Recall the Euler-Lagrange equation for $L = \int_{S^1} \mathcal{L}$,

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0. \tag{E.2}$$

This is obtained by extremizing the action S for some time interval $[a, b]$. We define

$$S = \int_{S^1 \times [a,b]} \mathcal{L}. \tag{E.3}$$

The Euler-Lagrange equation gives us the classical equation of motion,

$$(\partial_t^2 - \partial_\theta^2)\phi = 0. \tag{E.4}$$

To obtain a quantum field theory, we begin by decomposing the field into a sum of modes as follows.

$$\phi(\theta, t) = \sum_{n \in \mathbb{Z}} e^{in\theta/R}\phi_n(t). \tag{E.5}$$

Here, R is the radius of the cylinder [8]. We can consider our Lagrangian density to be a function of these modes. The ϕ_n are then generalised coordinates, and we can define the generalised (conjugate) momenta by $\pi_n = \partial\mathcal{L}/\partial\dot{\phi}_n$. We then impose that these modes be operators with equal-time commutation relations. This is known as canonical quantisation.

The free boson, in the context of conformal field theory, is studied on the complex plane. We can apply a similar process of canonical quantisation after mapping the cylinder to the plane. This is done with the coordinates [32]

$$z = e^{i(\theta - t)/R} \quad \overline{z} = e^{-i(\theta - t)/R}. \tag{E.6}$$

The corresponding equations of motion result in fields $\partial\phi(z)$ and $\overline{\partial}\phi(\bar{z})$ that are holomorphic and antiholomorphic on $\mathbb{C}\backslash\{0\}$ respectively [32]. By Laurent's theorem, we can expand these as Laurent series in any punctured neighbourhood of the origin with coefficients a_n and $\overline{a_n}$ respectively. Our process of canonical quantisation then involves imposing the following commutation relations [32].

$$[\phi_n(t), \phi_m(t)] = 0 \qquad [\phi_n(t), \dot{\phi}_m(t)] = \frac{ig}{2\pi R}\delta_{n,-m} \qquad [\dot{\phi}_n(t), \dot{\phi}_m(t)] = 0. \tag{E.7}$$

We also impose that the a_n and $\overline{a_n}$ are operators. From this, we can deduce the commutation relations of the modes a_n. We do this by first expanding our fields $\partial\phi(z)$ and $\overline{\partial}\phi(\bar{z})$ in terms of modes. We write

$$\partial\phi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \qquad \overline{\partial}\phi(\bar{z}) = \sum_{n\in\mathbb{Z}} \overline{a}_n \overline{z}^{-n-1}. \tag{E.8}$$

This gives

$$a_n = \frac{1}{2\pi i}\int_0 \partial\phi(z) z^n dz = \frac{1}{2\pi i}\int_0^{2\pi R} \frac{R}{2i}\frac{1}{z}(\partial_\theta - \partial_t)\phi(\theta, t)e^{in(\theta-t)/R}\frac{i}{R}z d\theta. \tag{E.9}$$

Here, we used

$$\partial_\theta = \frac{i}{R}z\partial_z - \frac{i}{R}\overline{z}\partial_{\overline{z}} \tag{E.10}$$

$$\partial_t = -\frac{i}{R}z\partial_z + \frac{i}{R}\overline{z}\partial_{\overline{z}}. \tag{E.11}$$

So,

$$\begin{aligned} a_n &= \frac{1}{4\pi i}\int_0^{2\pi R}\left[\frac{i}{R}\sum_{m\in\mathbb{Z}} m e^{i\theta(m+n)/R}\phi_m(t) - \sum_{m\in\mathbb{Z}} e^{i\theta(m+n)/R}\dot{\phi}_m(t)\right] e^{-int/R} d\theta \quad &(E.12)\\ &= \frac{1}{4\pi i}\left[-2\pi i n\phi_{-n}(t) - 2\pi R\dot{\phi}_{-n}(t)\right] e^{-int/R} &(E.13)\\ &= -\left(\frac{n}{2}\phi_{-n}(t) + \frac{R}{2i}\dot{\phi}_{-n}(t)\right) e^{-int/R}. &(E.14)\end{aligned}$$

Now, we calculate

$$[a_n, a_m] = \left[\left(\frac{n}{2}\phi_{-n}(t) + \frac{R}{2i}\dot{\phi}_{-n}(t)\right) e^{-int/R}, \left(\frac{m}{2}\phi_{-m}(t) + \frac{R}{2i}\dot{\phi}_{-m}(t)\right) e^{-imt/R}\right] \tag{E.15}$$

$$\begin{aligned} &= \left(\frac{nm}{4}[\phi_{-n}(t), \phi_{-m}(t)] + \frac{nR}{4i}[\phi_{-n}(t), \dot{\phi}_{-m}(t)] + \frac{Rm}{4i}[\dot{\phi}_{-n}(t), \phi_{-m}(t)]\right.\\ &= \left. -\frac{R^2}{4}[\dot{\phi}_{-n}(t), \dot{\phi}_{-m}(t)]\right) e^{-i(n+m)t/R} \quad (E.16)\end{aligned}$$

$$= \left(\frac{nR}{4i}[\phi_{-n}(t), \dot{\phi}_{-m}(t)] - \frac{Rm}{4i}[\dot{\phi}_{-n}(t), \phi_{-m}(t)] \right) e^{-i(n+m)t/R} \tag{E.17}$$

$$= \frac{gn}{8\pi}\delta_{n,-m} - \frac{gm}{8\pi}\delta_{n,-m} = \frac{ng}{4\pi}\delta_{n,-m}. \tag{E.18}$$

Typically this is rescaled so that $[a_n, a_m] = n\delta_{n,-m}$. We also introduce a central element K that satisfies $[a_n, K] = 0$. We then write $[a_n, a_m] = n\delta_{n,-m}K$. From this, the elements $\{a_n\} \cup \{K\}$ generate a Lie algebra called the Heisenberg algebra. The operators a_n and the field $\partial\phi(z)$ correspond to the holomorphic (or chiral) sector of the theory. It turns out that the free boson has a holomorphic factorisation, meaning that we can mostly study these sectors independently.[1]

[1] More specifically, representations of the Lie algebra containing the operators $\overline{a_n}$ form the states in what is known as the antiholomorphic sector of the theory. The space of states for the free boson is $\bigoplus_{p \in \mathfrak{h}^*}(V_p \otimes V_p)$ where each V_p represents a particular sector.

Appendix F

Free Boson Vertex Operators

In the free boson, our Lie algebra of interest has a basis $\{a_n\}_{n\in\mathbb{Z}} \cup \{k\}$ with bracket

$$[a_n, a_m] = n\delta_{n+m,0}k \qquad [a_n, k] = 0. \tag{F.1}$$

We have a field $\partial\phi$ analogous to the fields $J^a(z)$ in the WZW models given by

$$\partial\phi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}. \tag{F.2}$$

This field is obtained by a process called canonical quantisation. This process is outlined in Appendix E. The state-field correspondence can be used to deduce the OPE

$$\partial\phi(z)\partial\phi(w) \sim \frac{1}{(z-w)^2}. \tag{F.3}$$

Let $\phi(z)$ be a field satisfying $\partial_z\phi(z) = \partial\phi(z)$. Note that this is not necessarily unique. Let ϕ_0 be the zero mode of the field $\phi(z) - a_0\log z$. Define the field

$$V_p(z) = e^{p\phi_0} e^{pa_0\log z} \prod_{n=1}^{\infty} \left[e^{pa_{-n}z^n/n} e^{-pa_n z^{-n}/n} \right]. \tag{F.4}$$

Such a field is called a vertex operator.[1]

There is a subtle issue with enlarging our CFT (or more specifically, our vertex operator algebra) so that the mode ϕ_0 is contained in the corresponding Lie algebra. The issue is that a physically meaningful theory of the free boson which includes the mode ϕ_0 must be non-chiral. Such theories are not able to be formalised within the framework of vertex operator algebras[2], and so there is no vertex operator algebra which contains the free boson and the mode ϕ_0 in a consistent manner. For an explanation of what we mean by chirality, see Appendix E. We will assume that it makes sense to talk about fields such as that previously defined. This is fine for our

[1]One usually encounters vertex operators written in the form $V_p(z) = \exp(:ip\phi(z):)$.

[2]Vertex operator algebras formalise the chiral sector of a CFT. Certain questions about how to combine the holomorphic and antiholomorphic sectors of a CFT cannot be answered within the framework of VOAs.

discussion since our ultimate goal is to compute correlation functions, which are not affected by this issue.

The state corresponding to the vertex operator previously defined is $|p\rangle$. An outline for showing this is given in [32]. We will now present the details, starting with computing $[a_n, \phi_0]$. Since ϕ_0 is the zero mode of $\phi(z) - a_0 \log z$, we can write

$$\phi_0 = \frac{1}{2\pi i} \oint_0 (\phi(w) - a_0 \log w)\, w^{-1} dw. \tag{F.5}$$

We also have $[a_n, a_0 \log w] = [a_n, a_0]\log w + a_0[a_n, \log w] = 0$ since $\log w$ is a scalar. So,

$$[a_n, \phi_0] = \frac{1}{2\pi i} \oint_0 [a_n, \phi(w)] w^{-1} dw. \tag{F.6}$$

Writing $a_n = \frac{1}{2\pi i} \oint_0 \partial\phi(z) z^m dz$ and using our contour trick, we have

$$[a_n, \phi_0] = \frac{1}{(2\pi i)^2} \oint_0 \oint_w \partial\phi(z)\phi(w) z^m w^{-1} dz dw. \tag{F.7}$$

The only possible OPE for $\partial\phi(z)\phi(w)$ which is consistent with the OPE for $\partial\phi(z)\partial\phi(w)$ is

$$\partial\phi(z)\phi(w) \sim \frac{1}{z-w}. \tag{F.8}$$

Substituting this into our expression for $[a_n, \phi_0]$ gives

$$[a_n, \phi_0] = \frac{1}{(2\pi i)^2} \oint_0 \oint_w \frac{z^m w^{-1}}{z-w} dz dw \tag{F.9}$$

$$= \frac{1}{2\pi i} \oint_0 w^{m-1} dw = \delta_{m,0}. \tag{F.10}$$

We now wish to show

$$|p\rangle = e^{p\phi_0} |0\rangle. \tag{F.11}$$

Since $[a_n, \phi_0] = 0$ for $n > 0$, it is clear that $[a_n, \phi_0^k] = 0$ for $n > 0$. As a result, $[a_n, e^{p\phi_0}] = 0$. We also know that $[a_0, \phi_0] = 1$. If we assume that $[a_0, \phi_0^k] = k\phi_0^{k-1}$, then

$$[a_0, \phi_0^{k+1}] = [a_0, \phi_0^k]\phi_0 + \phi_0^k[a_0, \phi_0] = (k+1)\phi_0^k. \tag{F.12}$$

By induction, $[a_0, \phi_0^n] = n\phi_0^{n-1}$ and so

$$[a_0, e^{p\phi_0}] = \sum_{n=1}^{\infty} \frac{p^n}{n!}[a_0, \phi_0^n] = \sum_{n=1}^{\infty} \frac{p^n}{(n-1)!}\phi_0^{n-1} = pe^{p\phi_0}. \tag{F.13}$$

Therefore,

$$a_0 e^{p\phi_0} |0\rangle = [a_0, e^{p\phi_0}] |0\rangle = pe^{p\phi_0} |0\rangle. \tag{F.14}$$

Similarly, $a_n e^{p\phi_0} |0\rangle = 0$ for $n > 0$. We conclude that

$$|p\rangle = e^{p\phi_0} |0\rangle. \tag{F.15}$$

Before deducing the OPE for $V_p(z)V_q(w)$, we need to prove some more results. First, for $n \neq m$, we have

$$[e^{\alpha a_n}, e^{\beta a_{-m}}] = \sum_{i=1}^{\infty} \frac{\alpha^i}{i!}[a_n^i, e^{\beta a_{-m}}] = 0. \tag{F.16}$$

The last equality follows from applying commutator identities to $[a_n^i, a_{-m}^j]$ and using the bracket $[a_n, a_{-m}] = 0$ for $n \neq m$. Another result of interest is

$$\left[e^{p\phi_0} e^{pa_0 \log z}, \prod_{n=1}^{\infty} e^{pa_{-n} z^n / n}\right] = 0 \tag{F.17}$$

for $n \geq 1$. This follows from commutator identities and the results

$$[e^{p\phi_0}, e^{pa_{-n} z^n/n}] = 0 \tag{F.18}$$

$$[e^{pa_0 \log z}, e^{pa_{-n} z^n/n}] = 0. \tag{F.19}$$

For $n \geq 1$, equations (F.18) and (F.19) hold and follow from commutator identities and the brackets $[\phi_0, a_{-n}] = [a_0, a_{-n}] = 0$. The third result that we will use is

$$e^{\alpha a_0} e^{\beta \phi_0} = e^{\alpha\beta} e^{\beta\phi_0} e^{\alpha a_0}. \tag{F.20}$$

This can be proven using the Zassenhaus formula [5]. Since $[a_0, \phi_0] = 1$, the formula simplifies significantly and we have

$$e^{\alpha a_0} e^{\beta \phi_0} = e^{\alpha a_0 + \beta\phi_0} e^{[\alpha a_0, \beta\phi_0]} = e^{\alpha a_0 + \beta\phi_0} e^{\alpha\beta/2} \tag{F.21}$$

$$e^{\beta\phi_0} e^{\alpha a_0} = e^{\alpha a_0 + \beta\phi_0} e^{[\beta\phi_0, \alpha a_0]} = e^{\alpha a_0 + \beta\phi_0} e^{-\alpha\beta/2}. \tag{F.22}$$

Combining these two equations gives the result. From the first two results, we can deduce

$$\lim_{z\to 0} V_p(z) |0\rangle = \lim_{z\to 0} e^{p\phi_0} e^{pa_0 \log z} \lim_{k\to\infty} \prod_{n=1}^{k} \left[e^{pa_{-n} z^n/n} e^{-pa_n z^{-n}/n}\right] |0\rangle \tag{F.23}$$

$$= \lim_{z\to 0} e^{p\phi_0} e^{pa_0 \log z} \lim_{k\to\infty} \prod_{n=1}^{k} \left[e^{pa_{-n} z^n/n}\right] \prod_{n=1}^{k} \left[e^{-pa_n z^{-n}/n}\right] |0\rangle \tag{F.24}$$

$$= \lim_{z\to 0} e^{p\phi_0} e^{pa_0 \log z} \prod_{n=1}^{\infty} \left[e^{pa_{-n} z^n/n}\right] |0\rangle \tag{F.25}$$

$$= \lim_{z\to 0} \prod_{n=1}^{\infty} \left[e^{pa_{-n} z^n/n}\right] e^{p\phi_0} e^{pa_0 \log z} |0\rangle \tag{F.26}$$

$$= \lim_{z\to 0} \prod_{n=1}^{\infty} \left[e^{pa_{-n} z^n/n}\right] |p\rangle \tag{F.27}$$

$$= |p\rangle . \tag{F.28}$$

This tells us that the state corresponding to the field $V_p(z)$ is $|p\rangle$.

We remark that this derivation ignored some subtle issues by assuming

$$\left[e^{p\phi_0}e^{pa_0\log z}, \lim_{k\to\infty}\prod_{n=1}^{k} e^{pa_{-n}z^n/n}\right] = \lim_{k\to\infty}\left[e^{p\phi_0}e^{pa_0\log z}, \prod_{n=1}^{k} e^{pa_{-n}z^n/n}\right] \tag{F.29}$$

and

$$\lim_{z\to 0}\prod_{n=1}^{\infty}\left[e^{pa_{-n}z^n/n}\right]|p\rangle = \prod_{n=1}^{\infty}\lim_{z\to 0}\left[e^{pa_{-n}z^n/n}\right]|p\rangle . \tag{F.30}$$

Finally, we can deduce the OPE for $V_p(z)V_q(w)$. Using the same method in proving Equation (F.17), we can deduce

$$\left[e^{p\phi_0}e^{pa_0\log z}, \prod_{n=1}^{k} e^{pa_{-n}z^n/n}e^{-pa_n z^{-n}/n}\right] = 0. \tag{F.31}$$

This result and Equation (F.20) allow us to compute

$$\begin{aligned}
V_p(z)|q\rangle &= V_p(z)e^{q\phi_0}|0\rangle && \text{(F.32)}\\
&= \prod_{n=1}^{\infty}\left[e^{pa_{-n}z^n/n}e^{-pa_n z^{-n}/n}\right]e^{p\phi_0}e^{pa_0\log z}e^{q\phi_0}|0\rangle && \text{(F.33)}\\
&= \prod_{n=1}^{\infty}\left[e^{pa_{-n}z^n/n}e^{-pa_n z^{-n}/n}\right]e^{pq\log z}e^{(p+q)\phi_0}e^{pa_0\log z}|0\rangle && \text{(F.34)}\\
&= z^{pq}\prod_{n=1}^{\infty}\left[e^{pa_{-n}z^n/n}e^{-pa_n z^{-n}/n}\right]|p+q\rangle . && \text{(F.35)}
\end{aligned}$$

Using Equation (F.16) gives

$$\begin{aligned}
V_p(z)|q\rangle &= z^{pq}\exp\left(p\sum_{n=1}^{\infty}\frac{z^n}{n}a_{-n}\right)|p+q\rangle && \text{(F.36)}\\
&= z^{pq}\sum_{i=0}^{\infty}\frac{p^i}{i!}\sum_{n_1,\ldots n_i\in\mathbb{N}}\frac{a_{-n_1}\cdots a_{-n_i}}{n_1\cdots n_i}z^{n_1+\cdots+n_i}|p+q\rangle . && \text{(F.37)}
\end{aligned}$$

By inspection, the OPE $V_p(z)V_q(w)$ can be computed to any order using the state-field correspondence

$$a_{-k_1}^{n_1}\cdots a_{-k_l}^{n_l}|p+q\rangle \longleftrightarrow \frac{:\left[\prod_{i=1}^{n_1}\left[\partial^{k_1}\phi(z)\right]\cdots\prod_{i=1}^{n_l}\left[\partial^{k_l}\phi(z)\right]\right]V_{p+q}(z):}{(k_1-1)!\cdots(k_l-1)!}. \tag{F.38}$$

Explicitly, we have

$$V_p(z)V_q(w) = (z-w)^{pq}\Big\{V_{p+q}(w) + p:\partial\phi(w)V_{p+q}(w):(z-w)$$

$$+\frac{1}{2}[p^2 :\partial\phi(w)\partial\phi(w)V_{p+q}(w): +p :\partial^2\phi(w)V_{p+q}(w):](z-w)^2+\cdots\Big\}. \tag{F.39}$$

We conclude this section by proving that $E(z)$ and $F(z)$ anticommute with both $\phi(w)$ and $\psi(w)$ up to singular terms and the first regular term. The generalisation of this result to arbitrary level was used in Section 4.3. The fields $E(z)$ and $F(z)$ correspond to the vertex operators $V_{\sqrt{2}}(z)$ and $V_{-\sqrt{2}}(z)$ respectively. The fields $\phi(w)$ and $\psi(w)$ correspond to the vertex operators $V_{\frac{1}{\sqrt{2}}}(w)$ and $V_{\frac{-1}{\sqrt{2}}}(w)$ respectively. One of the possibilities is trivial, and so we focus on showing that $V_{\sqrt{2}}(z)V_{\frac{-1}{\sqrt{2}}}(w) = -V_{\frac{-1}{\sqrt{2}}}(w)V_{\sqrt{2}}(z)$ up to the singular terms and the first regular term. Up to these terms, we have

$$V_{\sqrt{2}}(z)V_{\frac{-1}{\sqrt{2}}}(w) = \frac{1}{z-w}\left\{V_{\frac{1}{\sqrt{2}}}(w)+\sqrt{2}:\partial\phi(w)V_{\frac{1}{\sqrt{2}}}(w):(z-w)\right\} \tag{F.40}$$

$$= \frac{V_{\frac{1}{\sqrt{2}}}(w)}{z-w}+\sqrt{2}:\partial\phi(w)V_{\frac{1}{\sqrt{2}}}(w):. \tag{F.41}$$

We also have

$$-V_{\frac{-1}{\sqrt{2}}}(z)V_{\sqrt{2}}(w) = -\frac{1}{z-w}\left\{V_{\frac{1}{\sqrt{2}}}(w)-\frac{1}{\sqrt{2}}:\partial\phi(w)V_{\frac{1}{\sqrt{2}}}(w):(z-w)\right\} \tag{F.42}$$

$$= \frac{V_{\frac{1}{\sqrt{2}}}(w)}{w-z}+\frac{1}{\sqrt{2}}:\partial\phi(w)V_{\frac{1}{\sqrt{2}}}(w):. \tag{F.43}$$

Interchanging z and w gives

$$-V_{\frac{-1}{\sqrt{2}}}(w)V_{\sqrt{2}}(z) = \frac{V_{\frac{1}{\sqrt{2}}}(z)}{z-w}+\frac{1}{\sqrt{2}}:\partial\phi(z)V_{\frac{1}{\sqrt{2}}}(z):. \tag{F.44}$$

Expanding the fields on the right hand side as Taylor series in w, we obtain

$$-V_{\frac{-1}{\sqrt{2}}}(w)V_{\sqrt{2}}(z) = \frac{V_{\frac{1}{\sqrt{2}}}(w)}{z-w}+\partial V_{\frac{1}{\sqrt{2}}}(w)+\frac{1}{\sqrt{2}}:\partial\phi(w)V_{\frac{1}{\sqrt{2}}}(w):. \tag{F.45}$$

Using the expression $V_{\frac{1}{\sqrt{2}}}(w) = \exp(:\frac{1}{\sqrt{2}}\phi(w):)$, we can identify $\partial V_{\frac{1}{\sqrt{2}}}(w)$ with $\frac{1}{\sqrt{2}}:\partial\phi(w)V_{\frac{1}{\sqrt{2}}}(w):$. We therefore obtain

$$-V_{\frac{-1}{\sqrt{2}}}(w)V_{\sqrt{2}}(z) = \frac{V_{\frac{1}{\sqrt{2}}}(w)}{z-w}+\sqrt{2}:\partial\phi(w)V_{\frac{1}{\sqrt{2}}}(w): \tag{F.46}$$

as required.

Appendix G

The Reconstruction Theorem

We will now present the reconstruction theorem, which allows us to verify that the vertex algebras we are interested in are in fact vertex algebras.

Theorem G.0.1. ***Reconstruction Theorem.*** *Let V be a $\mathbb{Z}_{\geq 0}$-graded vector space with $|0\rangle \in V_0$ not the zero vector. Let $T : V \to V$ be a linear map of degree 1. Let $\{v^n\}_{n\in S}$ be a collection of homogeneous elements of V with $S \subseteq \mathbb{Z}$.*[1] *Assume that for each element v^n, we have a field*

$$v^n(z) = \sum_{m\in\mathbb{Z}} v^n_m z^{-m-1}. \tag{G.1}$$

Furthermore, assume that the following hold:

1. *$v^n_m |0\rangle = 0$ for all $m \geq 0, n \in S$.*
2. *$v^n_{-1} |0\rangle = v^n$ for all $n \in S$.*
3. *All pairs of fields $v^n(z)$ are local.*
4. *$T |0\rangle = 0$.*
5. *$[T, v^n(z)] = \partial_z v^n(z)$ for all $n \in S$.*
6. *The elements*
$$v^{n_1}_{i_1} v^{n_2}_{i_2} \cdots v^{n_k}_{i_k} |0\rangle \tag{G.2}$$
form an ordered basis for V with $i_j \leq i_{j+1}$, $i_j < 0$ and if $i_j = i_{j+1}$ then $n_j \leq n_{j+1}$.

Then defining $Y(\cdot, z) : V \to End V[[z]]$ to be the linear extension of

$$Y(v^{n_1}_{i_1} v^{n_2}_{i_2} \cdots v^{n_k}_{i_k} |0\rangle, z) = \frac{:\partial_z^{-i_1-1} v^{n_1}(z) \partial_z^{-i_2-1} v^{n_2}(z) \cdots \partial_z^{-i_k-1} v^{n_k}(z):}{(-i_1-1)!(-i_2-1)! \cdots (-i_n-1)!} \tag{G.3}$$

gives the structure of a vertex algebra on V.

[1] We call an element of a graded vector space $\bigoplus_{n\in\mathbb{Z}} V_n$ homogeneous if it lies in V_n for some $n \in \mathbb{Z}$.

Proof. We begin by defining a grading on V with the function

$$v^{n_1}_{i_1} v^{n_2}_{i_2} \cdots v^{n_k}_{i_k} |0\rangle \mapsto -\sum_{n=1}^{k} i_n. \tag{G.4}$$

Now, note that the field

$$\partial_z^{-i_l - 1} v^{n_l}(z) \tag{G.5}$$

has degree $-i_l$. This is easily shown by induction. For the base case, we can see that $v^{n_l}(z)$ has degree 1 since the degree of $v^{n_l}_m$ is $-m$ by our choice of grading. The inductive step follows from $[T, v^n(z)] = \partial_z v^n(z)$ and the assumption that T is homogeneous of degree 1. By Lemma 5.1.1, the field $Y(v^{n_1}_{i_1} v^{n_2}_{i_2} \cdots v^{n_k}_{i_k} |0\rangle, z)$ has degree $-\sum_{n=1}^{k} i_n$.

We now use induction to prove that $Y(A, z)|0\rangle|_{z=0} = A$. For the base case $A = v^{n_k}_{-l}|0\rangle$, we have

$$\begin{aligned} &Y(v^{n_k}_{-l}|0\rangle, z)|0\rangle|_{z=0} \\ &\quad = \frac{1}{(l-1)!}\left[\sum_{m\in\mathbb{Z}} (-m-1)(-m-2)\cdots(-m-l+1) v^{n_k}_m z^{-m-l} |0\rangle\right]\Bigg|_{z=0} &\text{(G.6)} \\ &\quad = v^{n_k}_{-l}|0\rangle. &\text{(G.7)} \end{aligned}$$

This can be seen since for $m \geq 0$, $v^{n_k}_m |0\rangle = 0$. The terms for $m < -l$ are zero since we are evaluating at $z = 0$. The terms for $m = -1, -2, \ldots, -l+1$ are zero due to the term $(-m-1)(-m-2)\cdots(-m-l+1)$. The only term that is nonzero corresponds to $m = -l$. Now, assume that the field

$$Y(A, z) = \sum_{m\in\mathbb{Z}} A_m z^{-n-1} \tag{G.8}$$

satisfies

$$Y(A, z)|0\rangle|_{z=0} = A. \tag{G.9}$$

Let $a > 0$. Note that

$$\begin{aligned} \partial_z^{a-1} v^{n_b}(z) &= \sum_{m\in\mathbb{Z}} \partial_z^{a-1} v^{n_b}_m z^{-m-1} &\text{(G.10)} \\ &= \sum_{m\in\mathbb{Z}} (-m-1)(-m-2)\cdots(-m-a+1) v^{n_b}_m z^{-m-a} &\text{(G.11)} \\ &= \sum_{m\in\mathbb{Z}} (-m-2+a)(-m-3+a)\cdots(-m) v^{n_b}_{m+1-a} z^{-m-1}. &\text{(G.12)} \end{aligned}$$

So the mth mode of $\partial_z^{a-1} v^{n_b}(z)$ is

$$(-m-2+a)(-m-3+a)\cdots(-m) v^{n_b}_{m+1-a}. \tag{G.13}$$

Then

$$Y(v^{n_b}_{-a} A, z)|0\rangle|_{z=0}$$

$$= \left. \frac{:\partial_z^{a-1} v^{n_b}(z) Y(A,z):}{(a-1)!} |0\rangle \right|_{z=0} \tag{G.14}$$

$$= \frac{1}{(a-1)!} \sum_{n\in\mathbb{Z}} \left(\sum_{m\le -1} (-m-2+a)(-m-3+a)\cdots(-m) v^{n_b}_{m+1-a} A_{n-m-1} \right.$$
$$\left. \left. + \sum_{m>-1} A_{n-m-1}(-m-2+a)(-m-3+a)\cdots(-m) v^{n_b}_{m+1-a} \right) z^{-n-1} |0\rangle \right|_{z=0} \tag{G.15}$$

By assumption,

$$(-m-2+a)(-m-3+a)\cdots(-m) v^{n_b}_{m+1-a} |0\rangle = 0 \tag{G.16}$$

for $m \geq a-1$. For $m = 0, \ldots, a-2$, the factor $(-m-2+a)(-m-3+a)\cdots(-m)$ is zero. Now we use the fact that $A_m |0\rangle = 0$ for $m \geq 0$ and $A_{-1}|0\rangle = A$ by our inductive assumption. As a result, we know that

$$\left. \frac{1}{(a-1)!} \sum_{m\le -1} (-m-2+a)(-m-3+a)\cdots(-m) v^{n_b}_{m+1-a} A_{n-m-1} z^{-n-1} |0\rangle \right|_{z=0} = 0 \tag{G.17}$$

for $n \geq 0$. Furthermore, this is zero for $n \leq -2$ as a result of evaluating at $z = 0$. So,

$$Y(v^{n_b}_{-a} A, z)|0\rangle|_{z=0}$$
$$= \frac{1}{(a-1)!} \sum_{m\le -1} (-m-2+a)(-m-3+a)\cdots(-m) v^{n_b}_{m+1-a} A_{-m-2} |0\rangle \tag{G.18}$$
$$= v^{n_b}_{-a} A_{-1} |0\rangle \tag{G.19}$$
$$= v^{n_b}_{-a} A \tag{G.20}$$

as required.

Induction can be used again to prove that $[T, Y(A,z)] = \partial_z Y(A,z)$ for all $A \in V$. The base case $A = v^{n_k}_{-a}|0\rangle$ is immediate from

$$[T, \partial_z^{a-1} v^{n_k}(z)] = \partial_z^a v^{n_k}(z). \tag{G.21}$$

We now assume that $[T, Y(A,z)] = \partial_z Y(A,z)$. Given two fields $A(z)$ and $B(z)$ we have

$$:\partial_z A(z) B(z): + :A(z)\partial_z B(z):$$
$$= \sum_{n\in\mathbb{Z}} \left(\sum_{m\le -1} -m A_{m-1} B_{n-m} + \sum_{m>-1} -m B_{n-m} A_{m-1} \right) z^{-n-2}$$
$$+ \sum_{n\in\mathbb{Z}} \left(\sum_{m\le -1} -(n-m) A_m B_{n-m-1} \right.$$

$$+\sum_{m>-1} -(n-m)B_{n-m-1}A_m\Bigg) z^{-n-2} \tag{G.22}$$

$$= \sum_{n\in\mathbb{Z}} -(n+1)\left(\sum_{m\leq -1} A_m B_{n-m-1} + \sum_{m>-1} B_{n-m-1}A_m\right) z^{-n-2} \tag{G.23}$$

$$= \partial_z : A(z)B(z) : . \tag{G.24}$$

It is also clear that the identity

$$[T, : A(z)B(z) :] =: [T, A(z)]B(z) : + : A(z)[T, B(z)] : \tag{G.25}$$

holds. These properties will be used to finish the inductive step.

$$[T, Y(v^{n_b}_{-a}A, z)] = \frac{1}{(a-1)!}\left[T, : \partial_z^{a-1} v^{n_b}(z)Y(A, z) :\right] \tag{G.26}$$

$$= \frac{1}{(a-1)!}\left\{: [T, \partial_z^{a-1} v^{n_b}]Y(A, z) : + : \partial_z^{a-1} v^{n_b}(z)[T, Y(A, z)] :\right\} \tag{G.27}$$

$$= \frac{1}{(a-1)!}\left\{: \partial_z^{a} v^{n_b}(z)Y(A, z) : + : \partial_z^{a-1} v^{n_b}(z)\partial_z Y(A, z) :\right\} \tag{G.28}$$

$$= \frac{1}{(a-1)!}\partial_z : \partial_z^{a-1} v^{n_b} Y(A, z) := \partial_z Y(v^{n_b}_{-a}A, z). \tag{G.29}$$

Finally, it remains to prove that for all $A, B \in V$, the fields $Y(A, z)$ and $Y(B, w)$ are local. This is also done using induction. The base case requires showing that $Y(v^{n_b}_{-a}|0\rangle, z)$ and $Y(v^{n_k}_{-l}|0\rangle, w)$ are local. This follows from the locality of $\partial_z^{a-1} v^{n_b}(z)$ and $\partial_z^{l-1} v^{n_k}(w)$. Note that the inductive step also follows from the locality of these two fields by Lemma 5.1.4. Using the locality of $v^{n_b}(z)$ and $v^{n_k}(w)$, we have

$$(z-w)^N[v^{n_b}(z), v^{n_k}(w)] = 0 \tag{G.30}$$

for some $N \in \mathbb{N}$. Differentiating with respect to z and multiplying by $z-w$ then gives

$$(z-w)^{N+1}[\partial_z v^{n_b}(z), v^{n_k}(w)] = 0. \tag{G.31}$$

Similarly, we can differentiate with respect to w and multiply by $z-w$. We can repeat this as many times as we like to show that $\partial_z^{a-1} v^{n_b}(z)$ and $\partial_z^{l-1} v^{n_k}(w)$ are local for all a and l. □

Appendix H

The Braid Group

The definitions in this section can be obtained from Chapter 1 of [2].

Definition H.0.1. The nth configuration space of a topological space X is defined to be

$$\mathrm{Conf}_n(X) = \{(x_1, \dots, x_n) \mid x_i \in X, x_i \neq x_j \text{ for } i \neq j\}. \tag{H.1}$$

The symmetric group S_n acts on this space by permuting the x_i. That is,

$$\sigma \cdot (x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)}). \tag{H.2}$$

Definition H.0.2. Given our action of S_n on $\mathrm{Conf}_n(X)$, we can form the quotient space

$$\mathrm{UConf}_n(X) := \mathrm{Conf}_n(X)/S_n. \tag{H.3}$$

We call this the unordered nth configuration space of X.

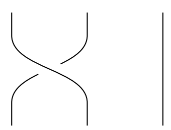
Definition H.0.3. The Artin braid group on n strands is defined to be

$$B_n = \pi_1(\mathrm{UConf}_n(\mathbb{R}^2)). \tag{H.4}$$

The pure braid group on n strands is defined to be

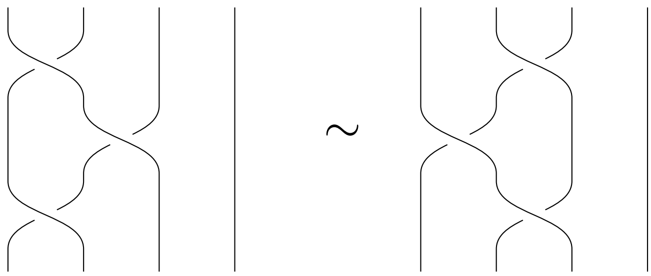
$$P_n = \pi_1(\mathrm{Conf}_n(\mathbb{R}^2)). \tag{H.5}$$

By considering representatives of the fundamental group of $\mathrm{UConf}_n(\mathbb{R}^2)$, we can draw diagrams of braids up to homotopy equivalence. The diagram below shows an example of an element of B_3.



Notice that one of the strands moves over the top of another strand. This is a different element of the braid group to the element obtained by choosing the other strand to move over the top instead. In fact, the two elements obtained in this way are inverses.

We will now discuss the relations satisfied by the elements of the braid group. First, define σ_i to be the braid given by swapping the ith and $(i+1)$th strands. It is clear that these braids generate the braid group. Now, note that $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j| \geq 2$. This is clear since for $|i-j| \geq 2$, the ith and $(i+1)$th strands do not cross the jth or $(j+1)$th strands. More interestingly, the relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ holds for $i \leq n-2$. By keeping track of the start and end points of each strand, we can see that this is the case for $i = 1$, $n = 4$.



This similarly holds for other values of i and n. We therefore obtain the presentation

$$B_n \cong \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| \geq 2, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } i \leq n-2 \rangle . \tag{H.6}$$

We generalise our definition of the braid group to arbitrary connected topological spaces as follows.

Definition H.0.4. The braid group of n strands on a connected topological space X is defined to be

$$B_n(X) = \pi_1(\mathrm{UConf}_n(X)). \tag{H.7}$$

Definition H.0.5. The pure braid group of n strands on a connected topological space X is defined to be

$$P_n(X) = \pi_1(\mathrm{Conf}_n(X)). \tag{H.8}$$