The Boson-Fermion Correspondence and its Applications

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Declaration

This thesis is an account of research undertaken between February 2014 and October 2014 at The Department of Theoretical Physics, Research School of Physics and Engineering, The Australian National University, Canberra, Australia.

Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

Tianshu Liu
October, 2014

Acknowledgements

This has been the most challenging, exhausting, yet enjoyable year of my life. I was placed in an office with views so great that they could even distract me occasionally. But more importantly, I got to know a group of amazing theoretical physicists, not only their remarkable works have which led me into the exciting world of academia, but also their diligent and conscientious attitudes have inspired and encouraged me tremendously.

First and foremost, I would like to thank my supervisor David Ridout who has introduced me to CFT, a field which interests me so much that I hope to study it further in my career. David has invested an enormous amount of his valuable time in motivating and helping me with this study. I especially appreciate his great patience in correcting my thesis over and over again, not only for the math and the physics, but also the English grammar. (I don't know how to use articles.) It must became quite painful for him towards the end. Without his kindness, guidance and persistent help, this thesis would not have been possible.

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At last, I wish to express my deepest gratitude to my family, who have been understanding, supporting and encouraging me through this hard year as they always do.

I am grateful that I have spent this unforgettable year with all of you, for there will be disappointing times in my career and these happy memories will always become supports for my dreams and my future.

Abstract

Conformal field theory is one of most essential items in the toolbox of mathematical physics. It has remarkable applications in a wide range of areas including string theory, statistical models at criticality and various aspects of mathematics.

This thesis presents two of the most fundamental examples of CFT — a free boson and a free fermion. The aim is to solve both theories by computing their primary field correlation functions. We take the vertex operator approach familiar to most theoretical physicists. In the first part, we associate a pair of free fermions to a single free boson. And in the second part, a free fermion is written in terms of the vertex operators, which are essentially exponentials of the bosonic field. Given that the correlation functions of the free boson are readily computable, we are now provided with a method of computing those of the free fermion, which is extremely difficult to achieve with other approaches. We will confirm the boson-fermion correspondence by showing the equivalence of the energy-momentum tensors and partition functions.

The boson-fermion correspondence, as a basic tool in mathematical physics, plays a very important role in many areas. As an application, we investigate the relation between the critical Ising model with a free fermion model using the same procedure as in constructing the boson-fermion correspondence. We solve the Ising model by identifying its primary correlation functions with those of a free fermion.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Overview

Quantum Field Theory (QFT) arises from the aesthetic appeal of a successful union of quantum mechanics and the special theory of relativity. The inability of relativistic quantum mechanics to model creation and annihilation of particles needs to be resolved. QFT postulates particles as excited states of an underlying physical field, which can be used for describing any number of particles. This is important in relativistic dynamics, where the particle number changes over time.

Conformal Field Theory (CFT) is a special class of QFT with angle-preserving symmetries. The earliest use of conformal maps traces back to 1569, where the Mercator projection [1] was presented by Gerardus Mercator for nautical purposes. In this projection, lines of constant course were represented as straight segments and scale increases from the Equator to the Pole. The size and shape of large objects are distorted, but the angles and shapes of small objects are preserved, which makes it ideal for measuring directions. In 1970, it was first proposed by Polyakov [2] that critical fluctuations in statistical models have conformal as well as scale invariance. The idea was then applied by Patashinski and Pokrovski in their study of the scaling ideas of second-order phase transition theory in 1979 [3]. Conformal symmetry requires the metric g_{uv} to be invariant up to a positive scaling $\Lambda(x)$, that is to say, a transformation of the coordinates from x to x' is conformal if it satisfies [4, 5, 6, 7]

$$g'_{uv}(x') = \Lambda(x) g_{uv}(x). \tag{1.1}$$

As we will show in the next chapter, such transformations preserve angles, which is where the name 'conformal' comes from.

The underlying algebra of CFT is called the Virasoro algebra [8] which is a very important complex Lie algebra. It is spanned by an infinite basis $\{L_n : n \in \mathbb{Z}\} \cup \{c\}$, where L_n are the Virasoro operators satisfying the following commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \tag{1.2}$$

$$[L_n, c] = 0, \tag{1.3}$$

and c is called the central charge.

Operators generating the Virasoro algebra were first written down in physics by M.A. Virasoro when he was studying the dual resonance model in 1970. The reducibility of Virasoro representations was solved by Kac, Feigin and Fuchs in their 1979 and 1982 papers [9, 10]. The seminal work of Belavin, Polyakov and Zamolodchikov (usually referred to as BPZ) [11, 12] is considered to be the foundation of CFT in the field of statistical mechanics. In their paper published in 1984, a primary field was first defined as a field which corresponds to a vacuum $|p\rangle$. For historical reasons, certain primary fields are known as vertex operators, which play a crucial role in the construction of the boson-fermion correspondence. They applied the Virasoro algebra to propose "minimal models" and identify several of them as scaling limits of statistical lattice models, e.g., the Ising Model. This paper also introduced important concepts such as a conformal family and conformal block. It discussed the behaviour of the energy momentum tensor under conformal transformation.

The development of conformal field theory and of string theory often went hand-in-hand. In the same year, Witten's paper [13] on string theory was published, it had an immense influence on the development of two-dimensional CFT. In his paper, Witten presented a procedure of bosonization or the transformation of local fermion fields to local boson fields. Because many quantities are much easier to calculate in the boson language than for fermions, bosonization provides a method of transferring fermion computations to bosons where they are easier. Witten also discussed the Wess-Zumino-Witten (WZW) model, which is CFT for string theory on a Lie group.

The work of the pioneers has strongly influenced the development of conformal field theory, making it one of the most active research areas in mathematical physics today.

The algebra of CFT, which is defined via operator algebras and their representation theory, is very special in two dimensions. An infinite number of parameters are needed to describe the conformal transformations. Among them, only 6 independent transformations are geometrically well-defined [14]. These include translations, rotations and dilations. The rest of the transformations correspond to non-global conformal transformations that are not everywhere regular. This richness of conformal symmetry in two dimensions allows the observable quantities of a physical system to be computed exactly without using Feynman diagrams. This property of CFT has led to its important applications to string theory and statistical models such as the Ising model, which will be explored in the thesis.

String theory is an attempt towards the unification of all forces of Nature. Two-dimensional conformal invariance comes naturally in this theory. A string is an extended one-dimensional object, its time-evolution traces out a two-dimensional manifold in space-time, which is called the "world-sheet". To prevent the appearance of states with negative probability amplitudes, one has to ensure the two-dimensional system to be reparametrization invariant on the world-sheet, which is tantamount to conformal invariance [15].

Another important example featuring conformal symmetry is the critical phenomenon displayed by a ferromagnet. The Ising model is a two-dimensional lattice with each point representing an atomic spin, which can either be up or down and is allowed to interact with its neighbouring spins. At generic temperatures, spins are either randomly distributed or in large domains. In either case, the system does not exhibit scale symmetry. However, at the critical temperature of the system, scale invariance emerges: spin distributions over any two length scales are statistically correlated. The system at this point undergoes a second order phase transition [16] and typical configurations display a fractal structure, which can be modelled using the conformal symmetry.

The boson-fermion correspondence is a basic tool in mathematical physics, it is an identification of the bosonic and the fermionic Fock spaces. The equivalence of the fermionic algebra and bosonic vertex operator construction originated in the work of Skyrme [17], and was subsequently developed by T.H.R. Streater and R. Wilde [18], I.F. Coleman [19], S. Mandelstam [20] and others. The core of the project is to construct a correspondence between a free boson and two fermions in terms of CFT and to apply the results to a physical system – the Ising model. As mentioned before, the correspondence transfers computations about one theory into another where they are easier. The procedure is remarkably useful in elucidating the properties of two dimensional field theories and is a fundamental tool in CFT and QFT.

1.2 Thesis Outline

In order to construct the boson-fermion correspondence, we first need some formalism for the fields. Chapter 2 introduces some basic concepts of CFT including defining conformal transformations as mappings which preserve angles, i.e., they leave the metric invariant up to a rescaling factor.

Chapters 3, 4 and 5 introduce the necessary concepts of the free boson and the free fermion theory in a complementary manner. First, we define bosonic and fermionic fields on a cylinder. The cylinder is then Wick rotated and mapped onto a complex plane by a change of variables, this allows us to work with an Euclidean metric upon which the two-dimensional CFT is defined. After formulating the actions of the fields and the equations of motion, we proceed to the canonical quantisation: The bosonic field is decomposed into Fourier modes with the degrees of freedom being identified as creation and annihilation operators. The formalism is similar to that used to solve the harmonic oscillators in quantum mechanics. In these chapters, we follow the standard treatment for a free boson and a free fermion as introduced in textbooks written by Di Francesco et al. [14] and Ketov [16]. This is complemented by the approaches from review articles including those from Ginsparg [5], Schellekens [6] and Gaberdiel [7], etc.

Chapter 6 is devoted to the construction of the boson-fermion correspondence. A range of mathematical techniques and concepts will be introduced and applied, including the state-field correspondence, vertex operators, operator product expansions, etc. We will find that a single free boson is equivalent to two free fermions, and because of this, we expect their energy-momentum tensors and their partition functions to coincide. We will check this explicitly in order to prove the validity of our construction. The correspondence helps us to compute the correlation functions which are the physically measurable quantities that allow a theory to be solved.

Chapter 7 adapts the results from previous chapters and applies it to a statistical model – the Ising model. A correspondence between the Ising model and the fermionic field will be explored and used to compute the correlation functions of the Ising model.

Chapter 6 and 7 constitute my original research during the Honours year.

Chapter 8 provides a summary for the thesis.

Chapter 2

Conformal Field Theory

In this chapter, we introduce some basic concepts of conformal field theory at the classical level. We relate conformal transformations to mappings which leave the metric invariant up to a rescaling. We pay special attention to the two-dimensional case, where generators for the conformal algebra are defined and used as a basis for local and global transformations. We also introduce the language of holomorphy which splits the conformal algebra into two identical but non-interacting copies. The texts [4, 5, 15] are excellent sources for the subject.

2.1 General Conformal Invariance

A set of fields $\Phi(x)$ and their action $S[\Phi]$ are the basic ingredients of any quantum field theory. Starting with a general d-dimensional manifold, we expect its metric $g_{uv}(x)$, which measures angles and lengths, to transform as a rank-2 symmetric tensor with respect to a finite coordinate transformation $x^\mu \to x'^\mu$:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu} g_{\lambda\rho}(x), \quad g_{\mu\nu} = g_{\nu\mu}. \tag{2.1}$$

We define a coordinate transformation $x \to x'$ to be conformal if it leaves the metric invariant up to a positive scale factor $\Lambda(x)$:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x). \tag{2.2}$$

Such transformations $x \to x'$ preserve the angle between any two non-zero vectors u^μ and v^ν from the vector space:

$$\begin{aligned}\cos\theta &= \frac{u^\mu v_\mu}{(u^\mu u_\mu)^{1/2}(v^\mu v_\mu)^{1/2}} \\ &= \frac{u^\mu g_{\mu\nu}(x) v^\nu}{(u^\mu g_{\mu\nu}(x) u^\nu)^{1/2}(v^\mu g_{\mu\nu}(x) v^\nu)^{1/2}}.\end{aligned} \tag{2.3}$$

The angle θ' in the transformed coordinates x' is manifestly conserved:

$$\begin{aligned}\cos\theta' &= \frac{u^\mu g'_{\mu\nu}(x') v^\nu}{(u^\mu g'_{\mu\nu}(x') u^\nu)^{1/2}(v^\mu g'_{\mu\nu}(x') v^\nu)^{1/2}} \\ &= \frac{u^\mu \Lambda(x) g_{\mu\nu}(x) v_\nu}{(u^\mu \Lambda(x) g_{\mu\nu}(x) u^\nu)^{1/2}(v^\mu \Lambda(x) g_{\mu\nu}(x) v^\nu)^{1/2}} \\ &= \cos\theta.\end{aligned} \tag{2.4}$$

Studying conformal transformations at the infinitesimal level allows us to determine all elements of the conformal algebra. Up to its first order, we can write an infinitesimal coordinate transformation as $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$, which implies

$$dx'^\mu = dx^\mu + \partial_\rho \epsilon^\mu(x) dx^\rho. \tag{2.5}$$

To come up with the constraint for an infinitesimal transformation to be conformal, we exploit the fact that the area element is a physically measurable quantity, therefore is conserved under coordinate transformations, that is,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g'_{\mu\nu} dx'^\mu dx'^\nu. \tag{2.6}$$

We start with the transformed coordinates x' and expand the infinitesimal area in terms of the original coordinates x:

$$\begin{aligned} ds^2 = g'_{\mu\nu} dx'^\mu dx'^\nu &= g'_{\mu\nu}(dx^\mu + \partial_\rho \epsilon^\mu dx^\rho)(dx^\nu + \partial_\sigma \epsilon^\nu dx^\sigma) \\ &= g'_{\mu\nu} dx^\mu dx^\nu + \partial_\rho \epsilon^\mu g'_{\mu\nu} dx^\rho dx^\nu + \partial_\sigma \epsilon^\nu g'_{\mu\nu} dx^\mu dx^\sigma \\ &= (g'_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) dx^\mu dx^\nu. \end{aligned} \tag{2.7}$$

Comparing eq. (2.6) with eq. (2.7), we arrive at an equation which tells us how the metric transforms as we translate the coordinate by an infinitesimal amount:

$$g_{\mu\nu} = g'_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu. \tag{2.8}$$

For the purpose of the thesis, we are only interested in a flat Euclidean space whose metric is $g_{\mu\nu} = \delta_{\mu\nu}$. From eq. (2.2) and (2.8), we are able to derive a relation between the dimension of the space d with the scaling factor of the metric $\Lambda(x)$:

$$\begin{aligned} g_{\mu\nu} &= \Lambda(x) g_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \\ \implies (1 - \Lambda(x)) g_{\mu\nu} &= \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu. \end{aligned} \tag{2.9}$$

Multiplying the metric $g^{\mu\nu}$ on both sides of eq. (2.9) gives

$$\begin{aligned} (1 - \Lambda) d &= \partial_\mu \epsilon^\mu + \partial_\nu \epsilon^\nu \\ \implies (1 - \Lambda) &= \frac{2}{d} (\partial \cdot \epsilon), \end{aligned} \tag{2.10}$$

where $d = g^{\mu\nu} g_{\mu\nu}$ for a flat space. When $d > 2$, the conformal algebra is finite-dimensional. It is spanned by infinitesimal translations, rotations, dilations and the special conformal transformations (which are nothing but translations followed by inversions). All of these transformations are globally well-defined, which means that they integrate to transformations of the conformal group [15].

2.2 Conformal Invariance in Two Dimensions

We are interested in the $d = 2$ case, where the powerful symmetries of the CFT make it very special. We will show that there are infinitely many local conformal transformations in two dimensions, though not all of them are everywhere well-defined. This enables us to compute the exact solutions for two-dimensional conformal field theories without using any approximation methods such as perturbation theory and Feynman diagrams.

With $d = 2$, eq. (2.10) becomes

$$1 - \Lambda = \partial \cdot \epsilon. \tag{2.11}$$

Combining eq. (2.9) and eq. (2.11) in the Euclidean metric $g = diag\{1, 1\}$, we arrive at the following set of constraints:

$$\partial_1 \epsilon_1 - \partial_2 \epsilon_2 = 0, \quad \partial_1 \epsilon_2 + \partial_2 \epsilon_1 = 0. \tag{2.12}$$

To change coordinates from $\mathbb{R}^2$ to $\mathbb{C}$, we write

$$z = x^1 + ix^2, \quad \bar{z} = x^1 - ix^2, \tag{2.13}$$

and in terms of z and $\bar{z}$, eq. (2.12) becomes

$$\partial_{\bar{z}} \epsilon(z, \bar{z}) = 0, \tag{2.14a}$$
$$\partial_z \bar{\epsilon}(z, \bar{z}) = 0. \tag{2.14b}$$

Eq. (2.14a) suggests that the function ϵ is independent of $\bar{z}$, we call functions $\epsilon(z)$ satisfying eq. (2.14a) holomorphic and $\bar{\epsilon}(\bar{z})$ which satisfy eq. (2.14b) anti-holomorphic. Such functions can be expanded as Fourier series in terms of z or $\bar{z}$ without singularities (except at zero and infinity).

The conformal algebra contains all infinitesimal transformations which preserve angles. This algebra is spanned by an infinite number of differential operators which we call generators:

$$\ell_n = -z^{n+1}\partial, \quad \bar{\ell}_n = -\bar{z}^{n+1}\bar{\partial}, \qquad n \in \mathbb{Z}, \tag{2.15}$$

where we denote ∂_z and $\partial_{\bar{z}}$ by ∂ and $\bar{\partial}$, respectively. The sets $\{\ell_n\}$ and $\{\bar{\ell}_n\}$ form two independent pieces of the conformal algebra whose elements satisfy these commutation relations:

$$[\ell_m, \ell_n] = (m - n)\ell_{m+n}, \quad [\bar{\ell}_m, \bar{\ell}_n] = (m - n)\bar{\ell}_{m+n}, \quad [\bar{\ell}_m, \ell_n] = 0. \tag{2.16}$$

Comparing with eq. (1.2), we find these are exactly the Virasoro algebra with a central charge of 0. In general, ℓ_n and $\bar{\ell}_n$ describe non-integrable local (or infinitesimal) conformal transformations. The infinite-dimensional conformal algebra has a finite-dimensional subalgebra spanned by $\{\ell_{-1}, \ell_0, \ell_1\}$ and its anti-holomorphic copy $\{\bar{\ell}_{-1}, \bar{\ell}_0, \bar{\ell}_1\}$. Certain linear combinations of these generators describe infinitesimal global conformal transformations:

$$\begin{aligned}
P_1 &= -(\ell_{-1} + \bar{\ell}_{-1}), && \text{translation in } x^1 \text{ direction}\\
P_2 &= -i(\ell_{-1} - \bar{\ell}_{-1}), && \text{translation in } x^2 \text{ direction}\\
D &= -(\ell_0 + \bar{\ell}_0), && \text{dilation}\\
M_{12} &= -i(\ell_0 - \bar{\ell}_0), && \text{rotation}\\
K_1 &= -(\ell_{-1} + \bar{\ell}_{-1}), && \text{special conformal transformation in } x^1\\
K_1 &= -i(\ell_{-1} - \bar{\ell}_{-1}), && \text{special conformal transformation in } x^2,
\end{aligned} \tag{2.17}$$

These transformations are "global" because they can be integrated into a finite-dimensional global conformal group.

The holomorphic algebra spanned by the ℓ_n and its anti-holomorphic copy are identical and commuting, which means that the overall conformal algebra is the direct sum of the two algebras. We will study the holomorphic algebra only. All results are identical for the anti-holomorphic copy, and it is usually easy to put the two results together to get the compete physical result.

Chapter 3

The Free Boson

This chapter gives a detailed account of a free bosonic field and its compactification. A free boson is described as a closed string on a cylinder, which is then mapped into a complex plane on which the CFT is defined. The bosonic fields are expanded into Fourier modes and their degrees of freedom are identified as the quantum operators. We then compactify the boson by constraining a boundary condition such that the vacuum momenta become discrete. Operator products are redefined by introducing the concept of 'normal ordering' in order to solve an infinite eigenvalue problem. At last, we study the quantised conformal generators and use them to identify the underlying algebra of free bosonic theory as the Virasoro algebra. The chapter follows the standard treatments for the free boson which can be found in [4, 14, 16].

3.1 The Free Bosonic Field

A free boson is one of the best and the simplest illustrations of two-dimensional CFT. It is described by a scalar field $\varphi(x,t)$ in terms of the space coordinate x and time t. In string theory, the bosonic field is represented by a closed string of radius R, it satisfies a periodic boundary condition in the space direction:

$$\varphi(x,t) = \varphi(x + 2\pi R, t). \tag{3.1}$$

The cylinder traced out by the closed string is called the *world-sheet* and it is mapped by φ to a *space-time* $\mathbb{R}$. Note that space-time is really 26-dimensional in bosonic string theory [21], but this is an unnecessary complication to our calculation which we ignore.

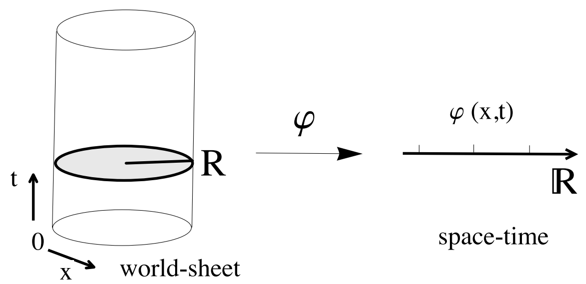

Figure 3.1: $\varphi : S^1 \times \mathbb{R} \to \mathbb{R}$ maps the cylindrical world-sheet to space-time on a real line.

The *action* of the bosonic string is given by [4]

$$S[\varphi] = \frac{1}{2g} \int_{S^1 \times \mathbb{R}} \partial_\mu \varphi \, \partial^\mu \varphi \, dx \, dt, \tag{3.2}$$

where g is a coupling constant. With coordinates $\{x, t\}$, we assign a lorentzian metric $g = diag\{1, -1\}$ to the cylinder. By Hamilton's principle of stationary action, we expect $S[\varphi]$ to be invariant under an infinitesimal field transformation $\varphi \to \varphi + \epsilon$. From this we are able to derive the classical equation of motion for a free boson:

$$\partial_t^2 \varphi = \partial_x^2 \varphi, \tag{3.3}$$

which is the familiar classical wave equation, its solutions describe propagating waves around the cylinder.

The next step is to quantise the classical theory using canonical quantisation. In this procedure, we determine the degrees of freedom, find their conjugate momenta and impose the canonical commutation relations. Since the bosonic field $\varphi(x, t)$ is periodic in x, we are able to Fourier decompose the space component of the field in terms of complex exponentials:

$$\varphi(x, t) = \sum_{n \in \mathbb{Z}} \varphi_n(t) e^{inx/R}. \tag{3.4}$$

The time-dependent modes $\varphi_n(t)$ are identified as the degrees of freedom of the field. The corresponding conjugate momenta can be derived from eq. (3.2) to be

$$\pi_n(t) = -\frac{L}{g} \dot{\varphi}_{-n}(t). \tag{3.5}$$

At last, we impose the following equal-time canonical commutation relations on $\varphi_n(t)$ and $\pi_n(t)$:

$$[\varphi_m(t), \varphi_n(t)] = 0, \quad [\pi_m(t), \pi_n(t)] = 0, \quad [\varphi_m(t), \pi_n(t)] = i\delta_{m,n}. \tag{3.6}$$

As introduced in chapter 2, CFT is defined over the Euclidean complex plane, not a cylinder. Fortunately, the cylinder can be easily mapped into a complex plane through Wick rotation (letting $t = i\tau$) and a change of variables:

$$z = e^{(\tau + ix)/R} = e^{i(x-t)/R}, \qquad \bar{z} = e^{(\tau - ix)/R} = e^{-i(x+t)/R}. \tag{3.7}$$

In terms of the newly defined variables z and $\overline{z}$, eq. (3.3) becomes

$$\partial_z \overline{\partial}_{\bar{z}} \varphi(z, \overline{z}) = 0. \tag{3.8}$$

This new equation of motion implies that $\partial\varphi(z, \bar{z})$ is holomorphic and $\overline{\partial}\varphi(z, \bar{z})$ is anti-holomorphic. Their Fourier decomposition in terms z and $\bar{z}$ are therefore

$$\partial\varphi(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}, \quad \overline{\partial}\varphi(\overline{z}) = \sum_{n \in \mathbb{Z}} \overline{a}_n \overline{z}^{-n-1}, \tag{3.9}$$

where a_n and $\overline{a}_n$ are identified as time-independent quantum operators whose commutation relations can be deduced from eq. (3.6):

$$[a_m, a_n] = m\delta_{m+n,0}, \quad [\overline{a}_m, \overline{a}_n] = m\delta_{m+n,0}, \quad [\overline{a}_m, a_n] = 0. \tag{3.10}$$

The operators a_n are partitioned into three groups:

$$a_n(n \in \mathbb{Z}) \; = \begin{cases} a_n \text{ where } n > 0 : \text{annihilation operators,} \\ a_n \text{ where } n < 0 : \text{creation operators,} \\ a_0 : \text{momentum operator.} \end{cases}$$

For each a_n, we propose an adjoint or hermitian conjugate $a_n^\dagger$ and define it as

$$a_n^\dagger = a_{-n}. \tag{3.11}$$

To construct a quantum state space, we first introduce a vacuum state $|p\rangle$, where $p \in \mathbb{R}$ is the momentum of the vacuum. This state describes a ground state of a closed string. It satisfies

$$a_n|p\rangle = 0 \text{ for } n > 0 \quad \text{ and } \quad a_0|p\rangle = p|p\rangle. \tag{3.12}$$

The quantum state space spanned by vacuum $|p\rangle$ and all its excited states (states obtained by acting with creation operators on the vacuum, e.g., $a_{-1}|p\rangle$, $a_{-3}^2 a_{-2}|p\rangle$) is called the *Fock space* and is denoted by $\mathcal{F}_p$. Its standard basis is given by

$$\left\{ \cdots a_{-3}^{j_3} a_{-2}^{j_2} a_{-1}^{j_1} |p\rangle \;\middle|\; j_i \in \mathbb{Z}^+ \cup \{0\}, \; \sum_i j_i < \infty \text{ and } p \in \mathbb{R} \right\}, \tag{3.13}$$

where we only allow finitely many creation operators to act on the vacuum.

3.2 Compactified Boson

The momentum p of eq. (3.13) is continuous, that is, p can take any value in $\mathbb{R}$. One can also construct a quantum theory in which the range of p is discrete. As in quantum mechanics, this can be achieved by *compactifying* the space. Observe that the equation of motion for a free boson eq. (3.3) is invariant under field translation $\varphi \to \varphi + C$ for some constant C. Let $C = 2\pi mR$, where m is an integer, we can adopt a more general boundary condition for the $\varphi(x,t)$:

$$\varphi(x + 2\pi R, t) = \varphi(x,t) + 2\pi mR, \quad m \in \mathbb{Z}. \tag{3.14}$$

Note that we have chosen the radius of this period to be the same as that of the world-sheet cylinder in order to obtain integer values of momenta p. This equation maps an uncompactified scalar field $\varphi(x,t)$ on a real line $\mathbb{R}$ into a circle of radius R. Points differing by $2\pi R$ are identified as the same point. Therefore any point on the uncompactified boson field is either in or is related to a point in the fundamental domain $0 \le \varphi(x) < 2\pi R$ by eq. (3.14). The compactification process is illustrated in Figure 3.2.

The compactification brings us a very important result: Only integer values of p are allowed. Recall eq. (3.9) for the Fourier decomposition of $\partial\varphi$, its integration with respect to z gives us the following expression for $\varphi(z)$:

$$\varphi(z) = \varphi_0 + a_0 \log z - \sum_{n \neq 0} \frac{a_n}{n} z^{-n}, \tag{3.15}$$

where a_0 is the momentum operator satisfying eq. (3.12), and classically φ_0 represents the

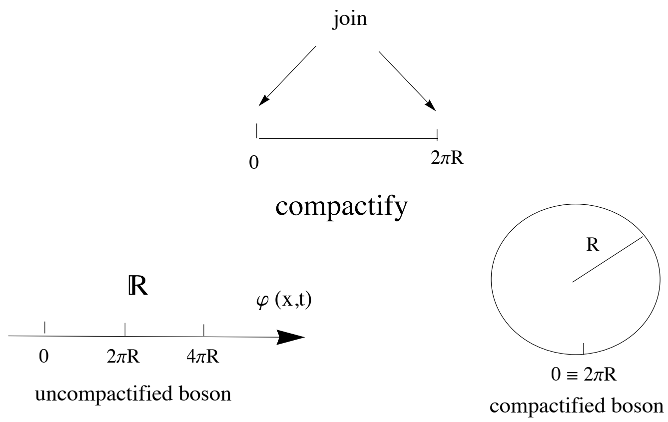

Figure 3.2: Define a fundamental domain $0 \leq \varphi(x) < 2\pi R$ for the uncompactified field, then map the whole domain onto it using eq. (3.14). The compactified boson then takes values in a circle of radius R, whose no two points are identified.

$\cdots$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-1}$	$\mathcal{F}_0$	$\mathcal{F}_1$	$\mathcal{F}_2$	$\cdots$
	$\vert -2\rangle$	$\vert -1\rangle$	$\vert 0\rangle$	$\vert 1\rangle$	$\vert 2\rangle$	
	$a_{-1}\vert -2\rangle$	$a_{-1}\vert -1\rangle$	$a_{-1}\vert 0\rangle$	$a_{-1}\vert 1\rangle$	$a_{-1}\vert 2\rangle$	
$\cdots$	$\vdots$	$a_{-2}\vert -1\rangle\ a_{-1}^2\vert -1\rangle$	$a_{-2}\vert 0\rangle\ a_{-1}^2\vert 0\rangle$	$a_{-2}\vert 1\rangle\ a_{-1}^2\vert 1\rangle$	$\vdots$	$\cdots$
		$\vdots$	$a_{-3}\vert 0\rangle\ a_{-1}a_{-2}\vert 0\rangle\ a_{-1}^3\vert 0\rangle$	$\vdots$		
			$\vdots$			

Table 3.1: The quantised bosonic Fock Spaces generated by vacua with integer momenta. The first few excited states of are shown, states on the same line of a single Fock space have the same energy. We denote the collection of all integer Fock spaces by $\mathbb{F}_0$.

position of the centre of mass of the string. φ_0 commutes with all a_n except for a_0:

$$[a_0, \varphi_0] = 1, \tag{3.16}$$

which agrees with Heisenberg's uncertainty principle in natural units ($\hbar = 1$) up to a factor of i. Eq. (3.15) implies $\varphi(z) \sim a_0 \log z$. A change of variables gives

$$\varphi(x, t) \sim a_0 x. \tag{3.17}$$

We now impose condition (3.14), which gives

$$a_0(x + 2\pi R) = a_0 x + 2\pi m R \implies a_0 = m \in \mathbb{Z}. \tag{3.18}$$

$a_0 \in \mathbb{Z}$ means that the momenta of vacuum states, i.e., the eigenvalue of a_0, are integers. Looking back at eq. (3.13), we illustrate the collection of Fock spaces generated by integer vacuum states in Table 3.1. We denote it by $\mathbb{F}_0$, which can be written as $\mathbb{F}_0 = \oplus_{p \in \mathbb{Z}} \mathcal{F}_p$.

3.3 The Energy-Momentum Tensor

Noether's theorem states that every continuous symmetry of a field theory corresponds to a conserved current j_a^μ, which satisfies [14]

$$\delta S = \int \partial_\mu j_a^\mu \omega^a d^d x, \tag{3.19}$$

where $\{\omega_a\}$ is a set of parameters defining a general infinitesimal transformation:

$$x'^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a}. \tag{3.20}$$

The invariance in action implies that the current j is conserved:

$$\delta S = 0 \qquad \Longrightarrow \qquad \partial_\mu j_a^\mu = 0. \tag{3.21}$$

The conserved current corresponding to the infinitesimal transformations we considered in chapter 2, $x^\mu \to x^\mu + \epsilon^\mu(x)$, is called the energy-momentum tensor $T^{\mu\nu}$. It is related to δS by

$$\delta S = \int T^{\mu\nu} \partial_\mu \epsilon_\nu d^d x. \tag{3.22}$$

$T^{\mu\nu}$ measures how the density of energy and momentum varies in space-time, it is one of the most important tools in CFT. To derive an explicit expression for $T^{\mu\nu}$ in terms of $\varphi(z)$ and $\varphi(\overline{z})$, we need to compute the free boson action under the following infinitesimal transformations:

$$z' = z + \epsilon(z, \overline{z}), \qquad \overline{z}' = \overline{z} + \overline{\epsilon}(z, \overline{z}). \tag{3.23}$$

From eq. (3.2) and (3.7), we find the transformed action S' in terms of z and $\overline{z}$ is given by

$$\begin{aligned} S' &= \frac{1}{g} \int \partial' \varphi' \, \overline{\partial}' \varphi' dz' d\overline{z}' \\ &= \frac{1}{g} \int \left[\partial\varphi \partial\overline{\varphi} - \overline{\partial}\epsilon \, \partial\varphi\partial\varphi - \partial\overline{\epsilon} \, \overline{\partial}\varphi\overline{\partial}\overline{\varphi} \right] dz d\overline{z} \\ &= S + \int \left[-\frac{1}{g} \partial\varphi\partial\varphi \, \overline{\partial}\epsilon - \frac{1}{g} \overline{\partial}\varphi\overline{\partial}\varphi \, \partial\overline{\epsilon} \right] dz d\overline{z}. \end{aligned} \tag{3.24}$$

Comparing with eq. (3.22), we find components T^{zz} and $T^{\overline{z}\overline{z}}$ of the EM-tensor vanish. The holomorphic ($T^{\overline{z}z}$) and anti-holomorphic ($T^{z\overline{z}}$) components are given by

$$T^{\overline{z}z} = -\frac{1}{g} \partial\varphi\partial\varphi, \qquad T^{z\overline{z}} = -\frac{1}{g} \overline{\partial}\varphi\overline{\partial}\varphi, \tag{3.25}$$

which are then renormalised and written as

$$T(z) = -\frac{g}{2} T^{\overline{z}z} = \frac{1}{2} \partial\varphi(z)\partial\varphi(z), \tag{3.26a}$$

$$\overline{T}(\overline{z}) = -\frac{g}{2} T^{z\overline{z}} = \frac{1}{2} \overline{\partial}\varphi(\overline{z})\overline{\partial}\varphi(\overline{z}). \tag{3.26b}$$

Classically, we can Fourier expand the holomorphic EM-tensor in eq. (3.26a) using eq. (3.9) as

$$T(z) = \frac{1}{2}\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\alpha_r\alpha_s z^{-r-s-2} = \sum_{n\in\mathbb{Z}}\left[\frac{1}{2}\sum_{r\in\mathbb{Z}}\alpha_r\alpha_{n-r}\right]z^{-n-2}, \tag{3.27}$$

where the degrees of freedom α_i are constants instead of quantum operators because we are in the classical picture. The term in the square bracket is defined to be the *classical Virasoro mode* L_n:

$$L_n = \frac{1}{2}\sum_{r\in\mathbb{Z}}\alpha_r\alpha_{n-r}. \tag{3.28}$$

To quantise the Virasoro operators, it is tempting to replace each constant α_i by a quantum operator a_i with the commutation relations we introduced in eq. (3.10). However, we would quickly run into problems as we try to compute, for example, the energy of a vacuum state. The L_0 mode is called the *energy operator*, that is, when L_0 acts on a quantum state, it returns the energy of the state as its eigenvalue. The 'naively' quantised energy operator is given by

$$L_0 = \frac{1}{2}\sum_{r\in\mathbb{Z}}a_r a_{-r}, \tag{3.29}$$

which acts on vacuum state $|p\rangle$ and gives

$$\begin{aligned}
L_0|p\rangle &= \frac{1}{2}\sum_{m\in\mathbb{Z}}a_m a_{-m}|p\rangle \\
&= \frac{1}{2}p^2|p\rangle + \sum_{m\in\mathbb{Z}^+}a_m a_{-m}|p\rangle \\
&= \frac{1}{2}p^2|p\rangle + \sum_{m\in\mathbb{Z}^+}\Big(a_{-m}a_m + [a_m, a_{-m}]\Big)|p\rangle \\
&= \frac{1}{2}p^2|p\rangle + \sum_{m\in\mathbb{Z}^+}m|p\rangle \\
&= \infty,
\end{aligned} \tag{3.30}$$

where we have used results from eq. (3.10) and (3.12). Unexpectedly, the vacuum energy diverges. This motivates us to give L_n a more sensible definition through 'normal ordering' such that finite results can be obtained in such calculations.

3.4 Normal Ordering

Normal ordering is a special arrangement of the operators in their product such that the corresponding eigenvalue on a state is finite. It puts creation operators to the left of annihilation operators. Mathematically, normal ordering is denoted by a set of colons ': ... :' and is defined as

$$: a_m a_n : = \begin{cases} a_m a_n, & \text{if } m \leq -1, \\ a_n a_m, & \text{otherwise.} \end{cases}$$

With this new notion of operator products, we define the *quantised Virasoro operator* L_n as

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} : a_k a_{n-k} : = \frac{1}{2}\sum_{k\leq -1} a_k a_{n-k} + \frac{1}{2}\sum_{k\geq 0} a_{n-k} a_k. \tag{3.31}$$

Similar to the quantum operators a_n, we propose the adjoint of a Virasoro operator L_n to be

$$L_n^\dagger = L_{-n}, \tag{3.32}$$

which satisfies

$$\langle p|L_{-n} = 0 \quad \text{and} \quad \langle p|L_0 = [L_0^\dagger|p\rangle]^\dagger = [L_0|p\rangle]^\dagger, \quad \text{for all } n > 0 \text{ and } p \in \mathbb{Z}. \tag{3.33}$$

We can check that the modified L_0 indeed gives a finite vacuum energy as we expect:

$$\begin{aligned} L_0|p\rangle &= \frac{1}{2}\sum_{m\in\mathbb{Z}} : a_m a_{-m} : |p\rangle \\ &= \frac{1}{2}(a_0^2 + \sum_{m\in\mathbb{Z}^+} a_{-m}a_m + \sum_{m\in\mathbb{Z}^-} a_m a_{-m})|p\rangle \\ &= \frac{1}{2}p^2|p\rangle. \end{aligned} \tag{3.34}$$

Because quantum fields can be Fourier expanded in terms of quantum operators a_n, the product of two or more fields in the quantised theory should also be normally ordered. For example, the quantised energy-momentum tensor field $T(z)$ must be defined as

$$T(z) = \frac{1}{2}\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} : a_r a_s : z^{-r-s-2} = \frac{1}{2} : \partial\varphi(z)\partial\varphi(z) : . \tag{3.35}$$

A natural question arising from the Virasoro operators is how they commute with each other. To find out, let us first compute their commutation relation with a_n:

$$\begin{aligned} [L_m, a_n] &= \left[\frac{1}{2}\sum_{k\in\mathbb{Z}} : a_k a_{m-k} : ,\, a_m\right] \\ &= \frac{1}{2}\sum_{k\leq -1}[a_k a_{m-k}\,,a_n] + \frac{1}{2}\sum_{k\geq 0}[a_{m-k}a_k\,,a_n] \\ &= \frac{1}{2}\sum_{k\leq -1}(a_k[a_{m-k}\,,a_n] + [a_k\,,a_n]a_{m-k}) + \frac{1}{2}\sum_{k\geq 0}(a_{m-k}[a_k\,,a_n] + [a_{m-k}\,,a_n]a_k) \\ &= \frac{1}{2}\big((m-m-n)a_{m+n} + (-n)a_{m+n}\big) \\ &= -na_{m+n} \end{aligned} \tag{3.36}$$

Before we use this result to compute $[L_m, L_n]$, let us again pay attention to the energy operator L_0. One can deduce from eq. (3.36) that if $|\psi\rangle$ is an eigenstate of L_0 with eigenvalue h, then $a_{-n}|\psi\rangle$ and $L_{-n}|\psi\rangle$, where $n \in \mathbb{Z}$, are also eigenstates of L_0, both of eigenvalue $h + n$:

$$\begin{aligned} L_0(a_{-n}|\psi\rangle) &= [L_0, a_{-n}]|\psi\rangle + a_{-n}L_0|\psi\rangle \\ &= na_{-n}|\psi\rangle + a_{-n}h|\psi\rangle \end{aligned}$$

$$= (n+h)a_{-n}|\psi\rangle; \tag{3.37}$$

$$\begin{aligned} L_0(L_{-n}|\psi\rangle) &= [L_0, L_{-n}]|\psi\rangle + L_{-n}L_0|\psi\rangle \\ &= nL_{-n}|\psi\rangle + aL_{-n}h|\psi\rangle \\ &= (n+h)L_{-n}|\psi\rangle, \end{aligned} \tag{3.38}$$

where h is known as the *conformal dimension* (or energy) of the state $|\psi\rangle$. These results will be constantly used for calculations in later chapters.

We can now go back to the commutator of two arbitrary Virasoro operators:

$$\begin{aligned} [L_m, L_n] &= \frac{1}{2}\sum_{r\in\mathbb{Z}}[: a_r a_{m-r} :, \, L_n] \\ &= \frac{1}{2}\sum_{r\le -1}[a_r a_{m-r}, L_n] + \frac{1}{2}\sum_{r\ge 0}[a_{m-r}a_r, L_n] \\ &= \frac{1}{2}\sum_{r\le -1} a_r[a_{m-r}, L_n] + \frac{1}{2}\sum_{r\le -1}[a_r, L_n]a_{m-r} + \frac{1}{2}\sum_{r\ge 0} a_{m-r}[a_r, L_n] + \frac{1}{2}\sum_{r\ge 0}[a_{m-r}, L_n]a_r \\ &= \frac{1}{2}\sum_{r\le -1}(m-r)a_r a_{m-r+n} + \frac{1}{2}\sum_{r\ge 0}(m-r)a_{m-r+n}a_r \\ &\qquad + \frac{1}{2}\sum_{r\le -1} r a_{n+r}a_{m-r} + \frac{1}{2}\sum_{r\ge 0} r a_{m-r}a_{n+r}. \end{aligned} \tag{3.39}$$

When $m+n \neq 0$, a_r and a_{m-r+n} commute with a_{n+r} and a_{m-r} respectively by eq. (3.10). The first two terms of eq. (3.39) can be combined and so can the last two terms:

$$\begin{aligned} [L_m, L_n] &= \frac{1}{2}\sum_{r\in\mathbb{Z}}(m-r)a_{m-r+n}a_r + \frac{1}{2}\sum_{r\in\mathbb{Z}}(n-r)a_{m-(r-n)}a_{n+(r-n)} \\ &= (m-n)\frac{1}{2}\sum_{r\in\mathbb{Z}} a_{m+n-r}a_r && (3.40a) \\ &= (m-n)\frac{1}{2}\sum_{r\in\mathbb{Z}} : a_{m+n-r}a_r : \\ &= (m-n)L_{m+n}, && (3.40b) \end{aligned}$$

where a_{m+n-r} commutes with a_r in eq. (3.40a), we are free to put them in normal order. When $m+n=0$, we have

$$\begin{aligned} [L_m, L_n] &= [L_m, L_{-m}] \\ &= \frac{1}{2}\sum_{r\le -1}(m-r)a_r a_{-r} + \frac{1}{2}\sum_{r\ge 0}(m-r)a_{-r}a_r \\ &\qquad + \frac{1}{2}\sum_{r\le -1} r a_{-m+r}a_{m-r} + \frac{1}{2}\sum_{r\ge 0} r a_{m-r}a_{-m+r} \\ &= \frac{1}{2}\sum_{r\le -1}(m-r)a_r a_{-r} + \frac{1}{2}\sum_{r\ge 0}(m-r)a_{-r}a_r + \frac{1}{2}\sum_{r\le -1} r a_{-m+r}a_{m-r} \\ &\qquad + \Big(\frac{1}{2}\sum_{r\ge m+1}(m-r)a_{-r}a_r + \frac{1}{2}\sum_{r=0}^{m}(m-r)a_{-r}a_r + \frac{1}{2}\sum_{r=0}^{m} r(m-r)\Big) \end{aligned}$$

$$
\begin{aligned}
&= 2m\Big(\frac{a_0^2}{2} + \sum_{r\geq 1} a_{-r}a_r\Big) + \frac{1}{2}\sum_{r=0}^{\infty} r(m-r) \\
&= 2mL_0 + \frac{1}{2}\sum_{r=0}^{\infty} r(m-r).
\end{aligned} \tag{3.41}
$$

The second term in eq. (3.41) can be proved by induction to be $(m^3 - m)/12$. Combining eq. (3.40) and eq. (3.41) into a single equation, we arrive at the commutation relations between Virasoro operators:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}. \tag{3.42}$$

3.5 Virasoro Algebra

Virasoro operators are very important in understanding the structure of conformal field theories. The algebra formed by Virasoro operators L_n and their commutation relation eq. (3.42) is called the Virasoro algebra, It is the quantum analogue of the classical conformal algebra derived in eq. (2.16)

More generally, the Virasoro algebra, denoted by $\mathfrak{Vir}$, is a special Lie algebra spanned by an infinite basis $\{L_n : n \in \mathbb{Z}\} \cup \{c\}$. The Lie bracket is defined to be

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{3.43}$$

$$[L_m, c] = 0. \tag{3.44}$$

where the second term of eq. (3.43) in known as the central term, it contains a *central charge* c, which is a very important parameter in CFT. The operation is non-commutative and non-associative, it satisfies the three axioms for a Lie algebra: bilinearity, antisymmetry and the Jacobi identity.

In CFT, we identify the Virasoro operators L_n as generators of the local conformal transformations in the quantum theory. Similarly to the classical case, the subalgebra spanned by $\{L_1, L_0, L_{-1}\}$ and its anti-holomorphic copy generate global conformal transformations. For example, just like $-(\ell_0 + \bar{\ell}_0)$ generates classical infinitesimal dilations (see eq. (2.17)), $-(L_0 + \bar{L}_0)$ generates dilations in the quantised theory.

Comparing eq. (3.42) with (3.43), we find that the (holomorphic) conformal algebra for a free boson is nothing but the Virasoro algebra central charge of 1. In fact, all CFTs are described by a Virasoro algebra but with different central charges. We will see shortly that $c = \frac{1}{2}$ for the free fermion theory.

Chapter 4

Vertex Algebras

In this chapter, we extend our study of the bosonic field by introducing a powerful algebraic structure – the vertex operator algebra. Vertex operators are fields which correspond to vacuum states. They are of great importance in the construction of the boson-fermion correspondence. We introduce two important tools which lie at the heart of CFT – the state-field correspondence and the operator product expansion (OPE). In the last section, we study the correlation functions of fields, which are the physically measurable quantities of quantum field theories. A CFT is said to be solved once all correlation functions involving three vertex operators are computed. The main ideas of this chapter can be found in sources such as [4, 5, 6]

4.1 The State-Field Correspondence

The *state-field correspondence* is one of the basic requirements in the definition of vertex algebras. Mathematically, it allows us to relate a quantum field to a quantum state, which is a very powerful aspect of the CFT. The state-field correspondence in the field-to-state direction is defined as

$$\lim_{z\to 0} \psi(z)|0\rangle = |\psi\rangle. \tag{4.1}$$

where '$|0\rangle$' is introduced as the *true vacuum* of the theory, that is, $|0\rangle$ has no energy or momentum:

$$a_0|0\rangle = L_0|0\rangle = 0. \tag{4.2}$$

A simple calculation shows that $|0\rangle$ is also annihilated by L_{-1}:

$$L_{-1}|0\rangle = \frac{1}{2}\left[\sum_{r\leq -1} a_r a_{-r-1}|0\rangle + \sum_{r\geq 0} a_{-r-1}a_r|0\rangle\right] = 0. \tag{4.3}$$

This is called the *translation invariance* of the vacuum.

Eq. (4.1) states that given any quantum field $\psi(z)$, we act with it on $|0\rangle$, then remove the z-dependence by taking its limit as z goes to zero. The resultant quantum state $|\psi\rangle$ is defined as the state corresponding to $\psi(z)$. Note that the limit must be well-defined for the state to exist. Recall in eq. (3.7), z is related to the time τ by $z = e^{(\tau+ix)/R}$. As z goes to zero, τ approaches minus infinity. Hence the state $|\psi\rangle$ in eq. (4.1) is a state in the infinite past. In scattering theory, $|\psi\rangle$ is considered to be the state before the scattering process and is referred to as the *asymptotic in-state*. In this thesis, we will only use the field-to-state direction of the correspondence stated in eq. (4.1). Its inversion exists [22, 23] but is rather tedious and requires strong constraints on the fields, which we will not concern

Fields	States	Fields	States
$\mathbb{1}(z)$	$\vert 0\rangle$	$\partial\varphi(z)$	$a_{-1}\vert 0\rangle$
$\partial^2\varphi(z)$	$a_{-2}\vert 0\rangle$	$T(z)$	$L_{-2}\vert 0\rangle$
$\partial T(z)$	$L_{-3}\vert 0\rangle$	$V_p(z)$	$\vert p\rangle$
$:\partial\varphi(z)\partial\varphi(z):$	$a_{-1}^2\vert 0\rangle$	$\partial\phi(z)$	$L_{-1}\vert\phi\rangle$

Table 4.1: Commonly used state-field correspondence results. $\mathbb{1}(z)$ is the identity field and $V_p(z)$ is the vertex operator of momentum p. The last correspondence is true for all fields $\phi(z)$.

ourselves with.

Let us illustrate this idea with an example: To find out which quantum state corresponds to the holomorphic field $\partial\varphi(z)$, we expand the field into its Fourier modes and apply eq. (4.1):

$$\begin{aligned}
\lim_{z\to 0}\partial\varphi(z)|0\rangle &= \lim_{z\to 0}\Big[\sum_{n\in\mathbb{Z}} a_n z^{-n-1}|0\rangle\Big] \\
&= \lim_{z\to 0}\Big[\sum_{n\leq -1} z^{-n-1}a_n|0\rangle\Big] + \lim_{z\to 0}\Big[\sum_{n\geq 1} z^{-n-1}a_n|0\rangle\Big] + \lim_{z\to 0}\Big[z^{-1}a_0|0\rangle\Big] \\
&= \lim_{z\to 0}\Big[\sum_{n\leq -1} z^{-n-1}a_n|0\rangle\Big] \\
&= \lim_{z\to 0}\Big(a_{-1}|0\rangle + z a_{-2}|0\rangle + z^2 a_{-3}|0\rangle + ...\Big) \\
&= a_{-1}|0\rangle. \qquad (4.4)
\end{aligned}$$

The holomorphic field $\partial\varphi(z)$ thus corresponds to the quantum state $a_{-1}|0\rangle$. Some of the important results from the state-field correspondence are listed in Table 4.1. The following generalised results can be derived from some simple calculations:

$$\frac{1}{(j-1)!}\partial^j\varphi(z) \longleftrightarrow a_{-j}|0\rangle, \qquad (4.5)$$

$$\frac{1}{(j-1)!}\frac{1}{(k-1)!}:\partial^j\varphi(z)\partial^k\varphi(z): \longleftrightarrow a_{-j}a_{-k}|0\rangle, \qquad (4.6)$$

$$\frac{1}{(j-2)!(k-2)!}:\partial^{(k-2)}T(z)\partial^{(j-2)}T(z): \longleftrightarrow L_{-j}L_{-k}|0\rangle. \qquad (4.7)$$

4.2 Vertex Operators

One of the correspondences in Table 4.1: $V_p(z)\longleftrightarrow|p\rangle$ involves one of the most important concepts of CFT – the vertex operators $V_p(z)$, which are related to the vacua $|p\rangle$ by the state-field correspondence:

$$\lim_{z\to 0} V_p(z)|0\rangle = |p\rangle. \qquad (4.8)$$

We call fields which correspond to vacua primary, and fields which correspond to descendants of vacua (excited states) secondary. The vertex operators $V_p(z)$ are therefore free boson primaries.

Let us take a closer look at the vertex operators. We claim that

$$V_p(z) =: e^{p\varphi(z)} := \sum_{k=0}^{\infty} \frac{p^k}{k!} : \overbrace{\varphi(z) \cdots \varphi(z)}^{k \text{ times}} : . \tag{4.9}$$

We first realise from eq. (4.8) and (4.9) that $V_0(z)$ is the identity field: $V_0(z) =: e^0(z) := \mathbb{1}(z)$. Recall in eq. (3.15), we had an explicit expression for $\varphi(z)$ in terms of a_n and φ_0. Substituting it into eq. (4.9), we arrive at the definition of vertex operators:

$$V_p(z) = \left[e^{p\varphi_0} e^{pa_0 \log z}\right] \prod_{n=1}^{\infty} \left[e^{pa_{-n} z^n/n} e^{-pa_n z^{-n}/n}\right], \tag{4.10}$$

where we have extended our notion of normal ordering by placing φ_0 to the left of a_0.

Applying the state-field correspondence on both sides of eq. (4.10) gives

$$\lim_{z\to 0} V_p(z)|0\rangle = \lim_{z\to 0} \left[e^{p\varphi_0} e^{pa_0 \log z}\right] \prod_{n=1}^{\infty} \left[e^{pa_{-n} z^n/n} e^{-pa_n z^{-n}/n}\right] |0\rangle \tag{4.11a}$$

$$\begin{aligned} &= \lim_{z\to 0} e^{p\varphi_0} e^{pa_0 \log z} |0\rangle \\ &= \lim_{z\to 0} e^{p\varphi_0} (1 + p \log z \, a_0 + \cdots) |0\rangle \\ &= e^{p\varphi_0} |0\rangle, \end{aligned} \tag{4.11b}$$

where the last exponential term of line (4.11a) can be Taylor expanded as $e^{-pa_n z^{-n}/n} = 1 - pz^{-n}/na_n + \frac{1}{2}(pz^{-n}/n)^2 a_n^2 - \cdots$. Since a_n is an annihilator, this term acts as an identity field on $|0\rangle$. In the second last exponential term, $e^{pa_{-n}z^n/n}$ approaches 1 as z goes to 0 since the exponent vanishes. Using eq. (4.8), we find

$$|p\rangle = e^{p\varphi_0}|0\rangle. \tag{4.12}$$

We are now left to confirm claim eq. (4.9) by showing that $e^{p\varphi_0}|0\rangle$ indeed satisfies eq. (3.12) as a vacuum state should:

$$a_n e^{p\varphi_0}|0\rangle = [a_n, e^{p\varphi_0}]|0\rangle + e^{p\varphi_0} a_n |0\rangle = 0, \tag{4.13}$$

$$\begin{aligned} a_0 e^{p\varphi_0}|0\rangle &= [a_0, 1 + p\varphi_0 + \frac{(p\varphi_0)^2}{2!} + \frac{(p\varphi_0)^3}{3!} + \cdots]|0\rangle \\ &= (0 + p + \frac{p^2}{2} 2\varphi_0 + \frac{p^3}{3!} 3\varphi_0^2 + \cdots)|0\rangle \\ &= p(1 + p\varphi_0 + \frac{(p\varphi_0)^2}{2!} + \cdots)|0\rangle \\ &= pe^{\varphi_0}|0\rangle, \end{aligned} \tag{4.14}$$

where in eq. (4.13), since a_n commutes with φ_0 for $n > 0$, it also commutes with a function of φ_0, and in eq. (4.14), we have Taylor expanded the exponential term and applied the commutator in eq. (3.16).

4.3 Operator Product Expansions

The concept of operator product expansion (OPE) arises from a problem we encounter when trying to commute two fields of different arguments. For example, the commutator of two holomorphic fields $\partial\varphi(z)$ and $\partial\varphi(w)$ diverges:

$$\begin{aligned}[\partial\varphi(z), \partial\varphi(w)] &= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}[a_n, a_m]z^{-n-1}w^{-m-1} \\ &= \sum_{n\in\mathbb{Z}} nz^{-n-1}w^{n-1} \qquad (4.15\text{a}) \\ &= \frac{1}{zw}\sum_{n\in\mathbb{Z}} n\Big(\frac{w}{z}\Big)^n \quad \to \infty, \qquad (4.15\text{b})\end{aligned}$$

where we have applied the result from eq (3.10): $[a_n, a_m] = m\delta_{m+n,0}$ to arrive at line (4.15a).

To fix the problem, we introduce the concept of '*radial ordering*' which is explicitly defined as

$$\mathcal{R}\{\phi(z)\psi(w)\} = \begin{cases}\phi(z)\psi(w) & \text{if } |z| > |w|, \\ \psi(w)\phi(z) & \text{if } |z| < |w|\end{cases} \tag{4.16}$$

for any two bosonic fields $\phi(z)$ and $\psi(w)$. One can see from the definition that radial ordering places the field with an argument that is larger in magnitude to the left of the other field. The change of variables $z = e^{(\tau+ix)/R}$ implies that $|z|$ is proportional to the exponential of time τ. Therefore, radial ordering requires fields to act on an asymptotic in-state at $\tau \to -\infty$ in chronological order. For example, if fields $\psi_i(z_i)$ are radially ordered on $|0\rangle$ in $\psi_n(z_n)\cdots\psi_2(z_2)\psi_1(z_1)|0\rangle$, we require $\psi_1(z_1)$ to act at an earlier time than $\psi_2(z_2)$, which in turn is earlier than $\psi_3(z_3)\ldots$ Hence radial ordering is also known as *time ordering* by quantum field theorists. It can be easily checked that this new notion of product indeed solves the divergence problem:

$$\mathcal{R}\{[\partial\varphi(z), \partial\varphi(w)]\} = \mathcal{R}\{\partial\varphi(z)\partial\varphi(w)\} - \mathcal{R}\{\partial\varphi(w)\partial\varphi(z)\} = 0. \tag{4.17}$$

Radial ordering can be even more useful than this: It allows us to express the product of two quantum fields as a Laurent series in powers of the difference of their arguments. As an example, let us consider the OPE of the two holomorphic fields $\partial\varphi(z)$ and $\partial\varphi(w)$. First, we assume that $|z| > |w|$, then

$$\begin{aligned}\mathcal{R}\{\partial\varphi(z)\partial\varphi(w)\} &= \partial\varphi(z)\partial\varphi(w) \\ &= \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} a_r a_s z^{-r-1}w^{-s-1} \\ &= \sum_{n\in\mathbb{Z}}\sum_{r\in\mathbb{Z}} a_r a_{n-r} z^{-r-1}w^{-n-1+r} \qquad (\text{replacing } s \to n-r) \qquad (4.18\text{a}) \\ &= \sum_{n\neq 0}\sum_{r\in\mathbb{Z}} :a_r a_{n-r}: z^{-r-1}w^{-n-1+r} + \sum_{r\leq -1} :a_r a_{-r}: z^{-r-1}w^{-1+r} \\ &\quad + \sum_{r\geq 0}(a_{-r}a_r + r)z^{-r-1}w^{-1+r}\end{aligned}$$

$$= \sum_{s\in\mathbb{Z}}\sum_{r\in\mathbb{Z}} : a_r a_s : z^{-r-1}w^{-s-1} + \frac{1}{z^2}\sum_{r=1}^{\infty} r\Big(\frac{w}{z}\Big)^{r-1} \tag{4.18b}$$

$$= \frac{1}{(z-w)^2} + : \partial\varphi(z)\partial\varphi(w) :, \tag{4.18c}$$

where we have taken out the $n = 0$ case from eq. (4.18a) and split it into two sums where $r \leq -1$ and $r \geq 0$. $a_r a_{-r}$ is normally ordered for $r \leq -1$. For $r \geq 0$, we rearrange a_r and a_{-r} in normal order using the relation $a_r a_{-r} = a_{-r}a_r + r$. The second term of line (4.18b) is the derivative of a geometric series. The two terms in eq. (4.18c) are referred to as the singular and the regular terms respectively.

The exact same argument applies for the case where $|w| > |z|$, which will give us the same result. In shorthand notation, we usually omit "$\mathcal{R}\{\cdots\}$" and the regular term of eq. (4.18c), and write the OPE as

$$\partial\varphi(z)\partial\varphi(w) \sim \frac{1}{(z-w)^2}. \tag{4.19}$$

This method allows us to compute the OPE between any two fields, some of the commonly used results are summarised in Table 4.2.

Looking back at eq. (4.18c), we can Taylor expand $\partial\varphi(z)$ in the normally ordered term at $z = w$:

$$\mathcal{R}\{\partial\varphi(z)\partial\varphi(w)\} = \frac{1}{(z-w)^2} + : \partial\varphi(w)\partial\varphi(w) : + : \partial^2\varphi(w)\partial\varphi(w) : (z-w) + \cdots \tag{4.20}$$

Diving both sides by $(z - w)$ and contour integrating with respect to z about w using Cauchy's residue theorem [24], we arrive at

$$\oint_w \frac{\mathcal{R}\{\partial\varphi(z)\partial\varphi(w)\}}{z-w}\frac{dz}{2\pi i} =: \partial\varphi(w)\partial\varphi(w) : . \tag{4.21}$$

This is the formal definition of normal ordering of two holomorphic fields. Similar results can be generalised from the OPE of any two fields.

Let us now examine the OPE of two vertex operators in detail. First consider the following expressions:

$$V_p(z)|q\rangle = V_p(z)e^{q\varphi_0}|0\rangle \tag{4.22a}$$

$$= \prod_{n=1}^{\infty} e^{pa_{-n}z^n/n} e^{-pa_n z^{-n}/n} e^{p\varphi_0} e^{pa_0 \log z} e^{q\varphi_0}|0\rangle \tag{4.22b}$$

$$= \prod_{n=1}^{\infty} e^{pa_{-n}z^n/n} e^{pq\log z}|p+q\rangle \tag{4.22c}$$

$$= z^{pq} \exp\Big[p\sum_{n=1}^{\infty}\frac{a_{-n}}{n}z^n\Big]|p+q\rangle \tag{4.22d}$$

$$= z^{pq}\sum_{k=0}^{\infty}\frac{p^k}{k!}\sum_{n_1,\dots,n_k=1}^{\infty}\frac{a_{-n_1}\cdots a_{-n_k}}{n_1\cdots n_k}z^{n_1+\cdots+n_k}|p+q\rangle \tag{4.22e}$$

$$= z^{pq}|p+q\rangle + z^{pq+1}pa_{-1}|p+q\rangle + \frac{1}{2}z^{pq+2}(p^2a_{-1}^2)|p+q\rangle + \cdots , \tag{4.22f}$$

where we have used eq. (4.12) and (4.10) for line (4.22a) and (4.22b) respectively. The second exponential term in line (4.22b) commutes everything to its right, we move it through to the rightmost position, then Taylor expand it as we did in eq. (4.11a). Eq. (4.22e) is obtained by Taylor expanding each exponential term in eq. (4.22d) and multiplying them all together. Applying the state-field correspondence on both sides of eq. (4.22f), we arrive at an expression for the OPE of two arbitrary vertex operators:

$$\begin{aligned} V_p(z)V_q(w) =& V_{p+q}(w)(z-w)^{pq} + p :\partial\varphi(w)V_{p+q}(w): (z-w)^{pq+1} \\ &+ \frac{1}{2}[\, p^2 :\partial\varphi(w)\partial\varphi(w)V_{p+q}(w): + p :\partial^2\varphi(w)V_{p+q}(w):](z-w)^{pq+2} + \cdots . \end{aligned} \tag{4.23}$$

Up to this point, we have introduced two types of ordering techniques: normal ordering and radial ordering. Fields with the same argument are normally ordered to make sense in quantum theory. Normal ordering places creation operators to the right of the annihilation operators, this guarantees that the quantum states have finite eigenvalues with respect to quantum operators. Radial ordering arranges fields according to the magnitude of their arguments, this allows us to expand the fields' product into singular terms and a normally ordered term. The last bit of knowledge we introduce in this section is a method of relating the two kinds of ordering – Wick's theorem, which provides a method for computing the OPE of more than two fields.

Before introducing Wick's theorem, we need to define an operation called *contraction* [14]: To contract two fields of different arguments means to replace them with the singular term in their OPE. For example, to contract the first and the last holomorphic field in the fields product $\partial\varphi(z_1)\partial\varphi(z_2)\partial\varphi(z_3)\partial\varphi(z_4)$, we have:

$$\partial\varphi(z_1)\partial\varphi(z_2)\partial\varphi(z_3)\partial\varphi(z_4) = \frac{1}{(z_1-z_4)^2}\partial\varphi(z_2)\partial\varphi(z_3). \tag{4.24}$$

Wick's theorem [14, 25] states that the radially-ordered product of an arbitrary number of fields $\mathcal{R}\{\varphi(z_1)\varphi(z_2)\cdots\varphi(z_n)\}$ is given by the sum of all possible contractions of the fields in the normally-ordered product $:\varphi(z_1)\varphi(z_2)\cdots\varphi(z_n):$. Note that Wick's theorem only holds for free fields, that is, fields whose singular term of their OPE is in the form of $\frac{C}{(z-w)^h}$, where C is a constant and h is the sum of conformal dimensions of the two fields. Let us compute the OPE of $\mathcal{R}\{\partial\varphi(z_1)\partial\varphi(z_2)\partial\varphi(z_3)\partial\varphi(z_4)\}$ as an example:

$$\begin{aligned} &\mathcal{R}\{\partial\varphi(z_1)\partial\varphi(z_2)\partial\varphi(z_3)\partial\varphi(z_4)\} \\ &\quad =: \partial\varphi(z_1)\partial\varphi(z_2)\partial\varphi(z_3)\partial\varphi(z_4): \ \text{(zero contraction term)} \\ &\qquad + :\partial\varphi(z_1)\partial\varphi(z_2)\partial\varphi(z_3)\partial\varphi(z_4): \ + 5 \text{ other single contraction terms} \\ &\qquad + :\partial\varphi(z_1)\partial\varphi(z_2)\partial\varphi(z_3)\partial\varphi(z_4): \ + 2 \text{ other double contraction terms} \\ &\quad =: \partial\varphi(z_1)\partial\varphi(z_2)\partial\varphi(z_3)\partial\varphi(z_4): \\ &\qquad + \frac{:\partial\varphi(z_3)\partial\varphi(z_4):}{(z_1-z_2)^2} + \frac{:\partial\varphi(z_2)\partial\varphi(z_4):}{(z_1-z_3)^2} + \frac{:\partial\varphi(z_2)\partial\varphi(z_3):}{(z_1-z_4)^2} \end{aligned}$$

Fields	Singular terms in OPE
$\partial\varphi(z)\partial\varphi(w)$	$(z-w)^{-2}$
$V_p(z)V_q(w)$	$(z-w)^{pq}V_{p+q}(w)+\cdots$
$T(z)\partial\varphi(w)$	$\partial\varphi(w)(z-w)^{-2}+\partial^2\varphi(w)(z-w)^{-1}$
$T(z)V_p(w)$	$\frac{p^2}{2}V_p(w)(z-w)^{-2}+\partial V_p(w)(z-w)^{-1}$
$T(z)T(w)$	$\frac{1}{2}(z-w)^{-4}+2T(w)(z-w)^{-2}+\partial T(w)(z-w)^{-1}$
$\partial\varphi(z)V_p(w)$	$pV_p(w)(z-w)^{-1}$

Table 4.2: Singular terms of important OPEs to this thesis. The last four are computed by Wick's theorem.

$$+\frac{:\partial\varphi(z_1)\partial\varphi(z_4):}{(z_2-z_3)^2}+\frac{:\partial\varphi(z_1)\partial\varphi(z_3):}{(z_2-z_4)^2}+\frac{:\partial\varphi(z_1)\partial\varphi(z_2):}{(z_3-z_4)^2}$$
$$+\frac{1}{(z_1-z_2)^2(z_3-z_4)^2}+\frac{1}{(z_1-z_3)^2(z_2-z_4)^2}+\frac{1}{(z_1-z_4)^2(z_2-z_3)^2}. \quad (4.25)$$

A modification of Wick's theorem allows us to compute the OPEs of sets of normally-ordered fields such as $:\partial\varphi(z)\partial\varphi(z):$ with $:\partial\varphi(w)\partial\varphi(w):$. Contractions between fields of the same argument are always ignored since they are not defined, we should only consider contractions between fields in different sets. For example, the OPE of the energy-momentum tensor $T(z)$ with the holomorphic field $\partial\varphi(w)$ is given by:

$$T(z)\partial\varphi(w)=\frac{1}{2}:\partial\varphi(z)\partial\varphi(z)::\partial\varphi(w):$$
$$\sim\frac{1}{2}:\partial\varphi(z)\partial\varphi(z)::\partial\varphi(w):+\frac{1}{2}:\partial\varphi(z)\partial\varphi(z)::\partial\varphi(w): \quad (4.26a)$$
$$\sim\frac{\partial\varphi(z)}{(z-w)^2} \quad (4.26b)$$
$$\sim\frac{\partial\varphi(w)}{(z-w)^2}+\frac{\partial^2\varphi(w)}{(z-w)}, \quad (4.26c)$$

where in line (4.26a) we have ignored the zero-contraction term because it leads to a regular term in the OPE. $\partial\varphi(z)$ of eq. (4.26b) is Taylor expanded at $z=w$: $\partial\varphi(z)=\partial\varphi(w)+(z-w)\partial^2\varphi(w)+\cdots$ to give eq. (4.26c).

Given eq. (4.19) and (4.23), a wide range of OPEs can now be computed using Wick's theorem without painful mode expansion techniques. In Table 4.2, we list the most important OPEs to this thesis, most of them are computed by Wick's theorem.

Note that the OPE of the EM-tensor $T(z)$ with a primary field $\phi(w)$ has the following form:

$$T(z)\phi(w)\sim\frac{h\phi(w)}{(z-w)^2}+\frac{\partial\phi(w)}{(z-w)}, \quad (4.27)$$

where h is the conformal dimension of the primary field $\phi(z)$. Recall that vertex operators $V_p(z)$ are primaries, its OPE with $T(z)$ (fourth expression of Table 4.2) satisfies eq. (4.27) with $h=\frac{p^2}{2}$. Another interesting OPE to look at is that of the energy-momentum tensor with itself:

$$T(z)T(w)\sim\frac{c/2}{(z-w)^4}+\frac{h_T T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w}, \quad (4.28)$$

where $c = 1$ is the central charge of the theory and $h_T = 2$ is the conformal dimension of $T(z)$. We can see from eq. (4.27) and (4.28) that OPEs not only tell us how to compute the product of two fields, but also carry important information about the nature of the field and the theory.

4.4 Correlation Functions

The most important aspect of quantum field theories is the *correlation functions* or *correlators*. Similarly to the scalar products of states in quantum mechanics, correlation functions represent the physically measurable quantities. The quantum field theory is solved once all its correlation functions are calculated. This, however, is usually done by approximation except for some special subclasses of QFT such as CFT, in which the powerful symmetry allows correlation functions to be computed exactly in principle.

The correlation function of a product of n arbitrary fields (*n-point function*) is defined as

$$\langle 0|\phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n)|0\rangle, \tag{4.29}$$

where $|0\rangle$ is the true vacuum of the theory and $\langle 0|$ is its adjoint. Note that $|0\rangle$ is killed by both a_0 and annihilation operators: $a_{n\geq 0}|0\rangle = 0$ whereas $\langle 0|$ is killed by creation operators: $\langle 0|a_{-n} = [a^\dagger_{-n}|0\rangle]^\dagger = [a_n|0\rangle]^\dagger = 0$.

Let us now consider a few examples: The 1-point function of the holomorphic field $\partial\varphi(z)$ can computed by expanding it into Fourier modes:

$$\begin{aligned}\langle 0|\partial\varphi(z)|0\rangle &= \sum_{n\in\mathbb{Z}}\langle 0|a_n|0\rangle z^{-n-1} \\ &= \sum_{n>0}\langle 0|a_{-n}|0\rangle z^{+n-1} + \sum_{n\geq 0}\langle 0|a_n|0\rangle z^{-n-1} \\ &= 0. \end{aligned}\tag{4.30}$$

The 1-point function of the identity field is trivial:

$$\langle 0|\mathbb{1}(z)|0\rangle = \langle 0|0\rangle = 1, \tag{4.31}$$

where we have defined the normalised scalar product of a Fock space vacuum to be 1. In fact, the identity field is the only field whose 1-point correlator does not vanish.

The correlation function of any normally ordered fields must vanish. Let us take the two-point function of the holomorphic fields $\partial\varphi(z)$ as an example: We first separate the creation modes of a field from its annihilation modes:

$$\partial\varphi(z) = \partial\varphi_-(z) + \partial\varphi_+(z), \tag{4.32}$$

where

$$\partial\varphi_-(z) = \sum_{n\leq -1} a_n z^{-n-1}, \qquad \partial\varphi_+(z) = \sum_{n\geq 0} a_n z^{-n-1}. \tag{4.33}$$

The 2-point function then becomes

$$\begin{aligned}\langle 0| :\partial\varphi(z)\partial\varphi(w): |0\rangle &= \langle 0| :\partial\varphi_-(z)\partial\varphi(w): |0\rangle + \langle 0| :\partial\varphi_+(z)\partial\varphi(w): |0\rangle \\ &= \langle 0|\partial\varphi_-(z)\partial\varphi(w)|0\rangle + \langle 0|\partial\varphi(w)\partial\varphi_+(z)|0\rangle && (4.34a) \\ &= 0. && (4.34b)\end{aligned}$$

In line (4.34a), by the definition of normal ordering, we place $\partial\varphi_-(z)$ to the left of $\partial\varphi(w)$ because it contains creation modes only, which kill $\langle 0|$. Likewise, the second term also vanishes. Generalising this idea to the n-point function of holomorphic fields, we can always decompose one of the fields into creation and annihilation modes. Normally ordering the modes allows them to kill either $\langle 0|$ or $|0\rangle$, the correlator becomes zero.

This result can be used to compute the correlation functions of radially-ordered products. For example, recall eq. (4.18c) for the OPE of two holomorphic fields, their correlation function is given by

$$\begin{aligned}\langle 0|\mathcal{R}\{\partial\varphi(z)\partial\varphi(w)\}|0\rangle &= \langle 0|\frac{1}{(z-w)^2} + :\partial\varphi(z)\partial\varphi(w):|0\rangle \\ &= \frac{1}{(z-w)^2}\langle 0|0\rangle + \langle 0|:\partial\varphi(z)\partial\varphi(w):|0\rangle \\ &= \frac{1}{(z-w)^2}.\end{aligned} \tag{4.35}$$

In short hand notation, we write the correlation function of radially-ordered fields as $\langle\phi_1(z_1)\cdots\phi_n(z_n)\rangle$.

In the final part of this section, we devote special attention to the correlation functions of vertex operators. Because they are built from a free boson, correlators of vertex operators can be "easily" computed. The 1-point function of a vertex operator is trivial. As we have mentioned, the only non-vanishing 1-point function is that of the identity field $\mathbb{1}(z)$ which is $V_0(z)$ by eq. (4.9). We can therefore write the 1-point vertex operator correlator as

$$\langle V_p(z)\rangle = \delta_{p,0}. \tag{4.36}$$

The 2-point correlation function can be computed using the definition of a vertex operator eq. (4.10):

$$\begin{aligned}&\langle V_p(z)V_q(w)\rangle \\ &= \Big\langle [e^{p\varphi_0}e^{pa_0\log z}]\prod_{n=1}^{\infty}[e^{pa_{-n}\frac{z^n}{n}}e^{-pa_n\frac{z^{-n}}{n}}][e^{q\varphi_0}e^{pa_0\log w}]\prod_{m=1}^{\infty}[e^{qa_{-m}\frac{w^m}{m}}e^{-qa_m\frac{w^{-m}}{m}}]\Big\rangle \\ &= \Big\langle [e^{p\varphi_0}e^{pa_0\log z}e^{q\varphi_0}e^{qa_0\log w}]\prod_{k=1}^{\infty}[e^{pa_{-k}\frac{z^k}{k}}e^{-pa_k\frac{z^{-k}}{k}}e^{qa_{-k}\frac{w^k}{k}}e^{-qa_k\frac{w^{-k}}{k}}]\Big\rangle \\ &= \Big\langle [e^{p\varphi_0}e^{pq\log z}e^{q\varphi_0}e^{pa_0\log z}e^{qa_0\log w}]\prod_{k=1}^{\infty}[e^{pa_{-k}\frac{z^k}{k}}e^{-pq(\frac{w}{z})^k\frac{1}{k}}e^{qa_{-k}\frac{w^k}{k}}e^{-pa_k\frac{z^{-k}}{k}}e^{-qa_k\frac{w^{-k}}{k}}]\Big\rangle,\end{aligned} \tag{4.37}$$

where we have used the following relations to place the terms in normal order:

$$e^{\alpha a_0}e^{\beta\varphi_0} = e^{\alpha\beta}e^{\beta\varphi_0}e^{\alpha a_0}, \qquad e^{\alpha a_n}e^{\beta a_{-n}} = e^{\alpha\beta n}e^{\beta a_{-n}}e^{\alpha a_n}. \tag{4.38}$$

These relations are proved in Appendix 4A. Simplifying and separating the product in eq. (4.37) gives

$$\langle V_p(z)V_q(w)\rangle = z^{pq}\langle 0|e^{(p+q)\varphi_0}\Big[e^{(p\log z+q\log w)a_0}\Big]\Big[\prod_{k=1}^{\infty}[e^{(p\frac{z^k}{k}+q\frac{w^k}{k})a_{-k}}\Big]$$

$$\times \Big[\prod_{n=1}^{\infty} e^{-pq(\frac{w}{z})^n \frac{1}{n}}\Big]\Big[\prod_{m=1}^{\infty} e^{-(p\frac{z^{-m}}{m}+q\frac{w^{-m}}{m})a_m}\Big]|0\rangle \tag{4.39a}$$

$$= z^{pq}\langle p+q|\Big[\prod_{n=1}^{\infty} e^{-pq(\frac{w}{z})^n \frac{1}{n}}\Big]|0\rangle$$

$$= \langle p+q|0\rangle z^{pq} e^{-pq\sum_{n=1}^{\infty}(\frac{w}{z})^n \frac{1}{n}} \tag{4.39b}$$

$$= \frac{\delta_{p+q,0}}{(z-w)^{p^2}}. \tag{4.39c}$$

Note that we have Taylor expanded the second, third and the fifth exponential terms in eq. (4.39a). They work as the identity field when acting upon either $|0\rangle$ or $\langle 0|$. We used the same technique in computing the state-field correspondence of a vertex operator (see discussion under eq. (4.11b)). The sum $\sum_{n=1}^{\infty}(\frac{w}{z})^n\frac{1}{n}$ in the exponent of eq. (4.39b) is convergent since we have let $|z| > |w|$ when computing the OPE $V_p(z)V_q(w)$. The sum is recognised as the Taylor expansion of $-\log(1-\frac{w}{z})$.

The three and four-point function of the vertex operators are calculated in exactly the same way, and the results are

$$\langle 0|V_{p_1}(z_1)V_{p_2}(z_2)V_{p_3}(z_3)|0\rangle = \delta_{p_1+p_2+p_3,0}(z_1-z_2)^{p_1p_2}(z_2-z_3)^{p_2p_3}(z_1-z_3)^{p_1p_3}, \tag{4.40}$$

$$\langle 0|V_{p_1}(z_1)V_{p_2}(z_2)V_{p_3}(z_3)V_{p_4}(z_4)|0\rangle = \delta_{p_1+p_2+p_3+p_4,0}$$
$$(z_1-z_2)^{p_1p_2}(z_1-z_3)^{p_1p_3}(z_1-z_4)^{p_1p_4}(z_2-z_3)^{p_2p_3}(z_2-z_4)^{p_2p_4}(z_3-z_4)^{p_3p_4}. \tag{4.41}$$

A natural generalisation of eq. (4.36), (4.39c), (4.40) and (4.41) gives us the n-point function of the vertex operators:

$$\langle 0|V_{p_1}(z_1)V_{p_2}(z_2)\cdots V_{p_n}(z_n)|0\rangle = \delta_{p_1+p_2+\cdots+p_n,0}\prod_{i<j}(z_i-z_j)^{p_ip_j}. \tag{4.42}$$

Note that for the correlator to be non-zero, the total momentum of the vertex operators must add up to zero. This means that if we start with an asymptotic in-state $|0\rangle$ and act with vertex operators on it, the momentum of the out-state must be conserved with respect to the in-state. This is known as the *neutrality condition* [14].

4.A A proof of the Relations in (4.38)

The following relations stated in eq. (4.38) have been used for computing the correlation functions of two vertex operators:

$$e^{\alpha a_0} e^{\beta \varphi_0} = e^{\alpha\beta} e^{\beta\varphi_0} e^{\alpha a_0}, \tag{4.43}$$

$$e^{\alpha a_n} e^{\beta a_{-n}} = e^{\alpha\beta n} e^{\beta a_{-n}} e^{\alpha a_n}. \tag{4.44}$$

This appendix presents a detailed derivation of these relations.

To prove eq. (4.43), let us first consider the following commutator:

$$[a_0, e^{\beta\varphi_0}] = \left[a_0, \left(1 + \beta\varphi_0 + \frac{\beta^2}{2!}\varphi_0^2 + \frac{\beta^3}{3!}\varphi_0^3 + \cdots\right)\right] \tag{4.45a}$$

$$= [a_0, 1] + \beta[a_0, \varphi_0] + \frac{\beta^2}{2!}[a_0, \varphi_0^2] + \frac{\beta^3}{3!}[a_0, \varphi_0^3] + \cdots, \tag{4.45b}$$

where the terms in eq. (4.45b) can be calculated inductively:

$$\begin{aligned}
[a_0, 1] &= 0 \\
\beta[a_0, \varphi_0] &= \beta \\
\frac{\beta^2}{2!}[a_0, \varphi_0^2] &= \frac{\beta^2}{2!}(\varphi_0[a_0, \varphi_0] + [a_0, \varphi_0]\varphi_0) = \beta^2\varphi_0 \\
\frac{\beta^3}{3!}[a_0, \varphi_0^3] &= \frac{\beta^3}{3!}(\varphi_0[a_0^2, \varphi_0] + [a_0^2, \varphi_0]\varphi_0) = \frac{\beta^3}{2!}\varphi_0^2 \\
&\vdots \\
\frac{\beta^k}{k!}[a_0, \varphi_0^k] &= \frac{\beta^k}{k!}(\varphi_0[a_0^{k-1}, \varphi_0] + [a_0^{k-1}, \varphi_0]\varphi_0) = \frac{\beta^k}{(k-1)!}\varphi_0^{k-1}.
\end{aligned}$$

where we have used eq. (3.16) for the commutation relation of a_0 with φ_0. Summing over all above equations as k goes to infinity gives:

$$\begin{aligned}
[a_0, e^{\beta\varphi_0}] &= \beta + \beta^2\varphi_0 + \frac{\beta^3}{2!}\varphi_0^2 + \cdots + \frac{\beta^k}{(k-1)!}\varphi_0^{k-1} + \cdots \\
&= \beta\left(1 + \beta\varphi_0 + \frac{\beta^2}{2!}\varphi_0^2 + \cdots + \frac{\beta^{k-1}}{(k-1)!}\varphi_0^{k-1} + \cdots\right) \\
&= \beta e^{\beta\varphi_0},
\end{aligned} \tag{4.47}$$

which implies

$$a_0 e^{\beta\varphi_0} = e^{\beta\varphi_0} a_0 + \beta e^{\beta\varphi_0} = e^{\beta\varphi_0}(a_0 + \beta). \tag{4.48}$$

Acting with a_0 on both sides gives

$$\begin{aligned}
a_0^2\, e^{\beta\varphi_0} &= a_0 e^{\beta\varphi_0}(a_0 + \beta) \\
&= e^{\beta\varphi_0}(a_0 + \beta)(a_0 + \beta) \\
&= e^{\beta\varphi_0}(a_0 + \beta)^2,
\end{aligned} \tag{4.49}$$

where we have applied eq. (4.48) for eq. (4.49). Again, inductively, one can show that

$$a_0^k e^{\beta\varphi_0} = e^{\beta\varphi_0}(\beta + a_0)^k. \tag{4.50}$$

We now consider the left-hand side of eq. (4.43), expanding the first exponential term into power series gives

$$e^{\alpha a_0} e^{\beta \varphi_0} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} a_0^k \, e^{\beta \varphi_0} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} (a_0^k \, e^{\beta \varphi_0}). \tag{4.51}$$

We now apply eq. (4.50) which gives

$$\begin{aligned} e^{\alpha a_0} e^{\beta \varphi_0} &= \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} e^{\beta \varphi_0} (\beta + a_0)^k \\ &= e^{\beta \varphi_0} e^{\alpha(\beta + a_0)} \\ &= e^{\alpha \beta} e^{\beta \varphi_0} e^{\alpha a_0}. \end{aligned} \tag{4.52}$$

The first relation is proved.

The proof for the second relation is extremely similar, we will only summarise the key points here. We should first consider the following commutation relation:

$$[a_n, e^{\beta a_{-n}}] = \beta n e^{\beta a_{-n}}, \tag{4.53}$$

which can be proven by expanding the exponential $e^{\beta a_{-n}}$ in terms of its power series, and computing each term using the commutation relation $[a_n, a_{-n}] = n$ (from eq. (3.10)). We then write the commutator as $a_n e^{\beta a_{-n}} = e^{\beta a_{-n}} a_n + \beta n e^{\beta a_{-n}}$ and act a_n repetitively on both sides as we did in eq. (4.49) and (4.50). Combining all such terms with appropriate coefficients will lead us to

$$e^{\alpha a_n} e^{\beta a_{-n}} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} e^{\beta a_{-n}} (\beta n + a_n)^k, \tag{4.54}$$

which reduces to

$$e^{\alpha a_n} e^{\beta a_{-n}} = e^{\alpha \beta n} e^{\beta a_{-n}} e^{\alpha a_n} \tag{4.55}$$

upon some manipulations.

Chapter 5

The Free Fermion

The free fermionic theory is another example of a two dimensional CFT. A free fermion is different from a free boson in the sense that it admits two possible boundary conditions in the space direction. This divides fermionic states into two groups – those in the Neveu-Schwarz sector and those in the Ramond sector. In this chapter, we analyse free fermions in a similar way as we did with the free bosons in chapter 3. We will derive the underlying algebra of the free fermion theory and describe Fock spaces for both sectors. This chapter follows the standard treatment for a free fermion which can be found in texts [4, 15].

5.1 The Free Fermionic Field

Fermions are characterised by antisymmetry under particle exchange in a many-particle state: $\psi_1\psi_2 = -\psi_2\psi_1$. In string theory, a free fermion is described by a closed string of spin $\frac{1}{2}$. The corresponding fermionic field $\Psi(x,t)$ consists of two spinor fields $\psi(x,t)$ and $\overline{\psi}(x,t)$ in the spin-$\frac{1}{2}$ representation of $\mathfrak{su}(2)$. Each spinor field may satisfy one of the following boundary conditions in the space direction of the cylindrical world-sheet:

$$\psi(x+L,t) = \begin{cases} +\psi(x,t) & \text{(periodic, Ramond sector)} \\ -\psi(x,t) & \text{(antiperiodic, Neveu-Schwarz sector).} \end{cases} \tag{5.1}$$

Periodic fermions form the Ramond sector whereas the anti-periodic ones form the Neveu-Schwarz sector. Each sector is independent for $\psi(x,t)$ and $\overline{\psi}(x,t)$.

In terms of the complex coordinates defined in eq. (3.7), the action of the fermionic string is given by [15]

$$S[\psi,\overline{\psi}] = \frac{1}{2\pi}\int (\psi\overline{\partial}\psi + \overline{\psi}\partial\overline{\psi})\, dz\, d\overline{z}, \tag{5.2}$$

from which we are able to derive the equations of motion using the stationary action principle:

$$\partial\overline{\psi} = 0, \quad \overline{\partial}\psi = 0. \tag{5.3}$$

These equations show that $\psi(z)$ is holomorphic and $\overline{\psi}(\overline{z})$ is anti-holomorphic. The Fourier expansion of the holomorphic field $\psi(z)$ is given by:

$$\psi(z) = \begin{cases} \sum_{n\in\mathbb{Z}} b_n z^{-n-1/2} & \text{(Ramond sector)} \\ \sum_{n\in\mathbb{Z}+1/2} b_n z^{-n-1/2} & \text{(Neveu-Schwarz sector).} \end{cases} \tag{5.4}$$

The corresponding anti-holomorphic mode expansion is analogous. The quantum operators b_n along with an identity operator 1 span an infinite-dimensional Lie super-algebra.

From canonical quantisation, we are able to derive the following anti-commutation relations for the b_n:

$$\{b_m, b_n\} = b_m b_n + b_n b_m = \delta_{m+n,0}. \tag{5.5}$$

5.2 The Neveu-Schwarz sector

Let us consider the two sectors separately: In the Neveu-Schwarz (NS) sector, the algebra is spanned by $\{b_n \mid n \in \mathbb{Z} + \frac{1}{2}\} \cup \{1\}$. We partition $\{b_n\}$ into two sets:

$$b_n(n \in \mathbb{Z} + \frac{1}{2}) \ = \begin{cases} b_n \text{ where } n > 0 : \text{annihilation operators,} \\ b_n \text{ where } n < 0 : \text{creation operators.} \end{cases} \tag{5.6}$$

We propose an adjoint for each b_n:

$$b_n^\dagger = b_{-n}. \tag{5.7}$$

Comparing with the bosonic quantum operators defined in section 3.1, there is no momentum operator b_0 in the Neveu-Schwarz sector because n is a half-integer. In fact, the fermionic space-time is so abstract that the notion of momentum cannot be defined for the fermionic theory, even in the Ramond sector.

There exists only one vacuum state in the NS-sector, this is the true vacuum $|0_{NS}\rangle$. Vacua $|p\rangle$ we had in the bosonic theory do not exist in the fermionic theory because there is no zero-mode whose eigenvalue distinguishes them. Taking $|0_{NS}\rangle$ as the true vacuum means that the state-field correspondence is applicable, for example:

$$\begin{aligned} \lim_{z\to 0} \psi(z)|0_{NS}\rangle &= \lim_{z\to 0} \sum_{n\in\mathbb{Z}+\frac{1}{2}} b_n|0_{NS}\rangle z^{-n-1/2} \\ &= \lim_{z\to 0} [b_{-1/2}|0_{NS}\rangle + b_{-3/2}|0_{NS}\rangle z + \cdots] \\ &= b_{-1/2}|0_{NS}\rangle. \end{aligned} \tag{5.8}$$

Similarly, we find

$$\partial\psi(z) \longleftrightarrow b_{-3/2}|0_{NS}\rangle \quad \text{and} \quad \mathbb{1}(z) \longleftrightarrow |0_{NS}\rangle. \tag{5.9}$$

The Fock space $\mathcal{F}_{NS}$ spanned by $|0_{NS}\rangle$ and all its excited state has the following standard basis:

$$\Big\{ \cdots b_{-5/2}^{j_{5/2}} b_{-3/2}^{j_{3/2}} b_{-1/2}^{j_{1/2}} |0_{NS}\rangle \ \Big| \ j_r \in \{0,1\} \text{ and } \sum_r j_r < \infty \Big\}. \tag{5.10}$$

Note that for any quantum state, the power of each b_n cannot be greater than 1. This is because eq. (5.5) gives

$$\{b_n, b_n\} = 2b_n^2 = \delta_{n,0} = 0 \text{ for all } n \in \mathbb{Z} + 1/2. \tag{5.11}$$

This is consistent with the Pauli exclusion principle, which states that no two identical fermions can occupy the same quantum state simultaneously. This is implied by $b_n^2 = 0$.

The energy-momentum tensor $T(z)$ of the free fermion theory corresponds to a state defined in the Neveu-Schwarz sector. The classical $T(z)$ can be derived using Noether's theorem to be

$$T(z) = -\frac{1}{2}\psi(z)\partial\psi(z). \tag{5.12}$$

To quantise the theory, we define the normal ordering of two quantum operators b_n in the NS-sector as

$$: b_m b_n : = \begin{cases} b_m b_n, & \text{if } m \leq -\frac{1}{2}, \\ -b_n b_m, & \text{if } m \geq +\frac{1}{2}. \end{cases} \tag{5.13}$$

Note that we pick up a minus sign when swapping two fermionic operators. We now expand the classical $T(z)$ using eq. (5.4) and normally order the quantum operators according to the above definition. The quantised tensor field is therefore given by

$$\begin{aligned} T(z) = -\frac{1}{2} : \psi(z)\partial\psi(z) : &= -\frac{1}{2} : \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r z^{-r-\frac{1}{2}} \sum_{s\in\mathbb{Z}+\frac{1}{2}} (-s-\frac{1}{2}) b_s z^{-s-\frac{3}{2}} : \\ &= \sum_{n\in\mathbb{Z}} \Big[\frac{1}{2} \sum_{r\in\mathbb{Z}+1/2} (n-r+\frac{1}{2}) : b_r b_{n-r} : \Big] z^{-n-2}. \end{aligned} \tag{5.14}$$

where the term in the square bracket is defined as the Virasoro mode for the NS-sector:

$$L_n = \frac{1}{2} \sum_{k\in\mathbb{Z}+\frac{1}{2}} (k+\frac{1}{2}) : b_{n-k} b_k : \quad \text{where } n \in \mathbb{Z}, \tag{5.15}$$

hence

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}. \tag{5.16}$$

Having defined the normal ordering for b_n, we are able to find the state corresponding to $T(z)$ using eq. (5.8), (5.9) and eq. (5.12):

$$T(z) \longleftrightarrow -\frac{1}{2}\, b_{-1/2} b_{-3/2} |0_{NS}\rangle. \tag{5.17}$$

As one would expect, the conformal dimension of $T(z)$ is 2 in both bosonic and fermionic theories (recall eq. (4.28)).

The expansion in eq. (5.15) allows us to compute the commutator of two Virasoro operators using the exact method as we did in eq. (3.39) to (3.42). The result is given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{24}(m^3 - m)\delta_{m+n,0}. \tag{5.18}$$

Comparing with eq. (3.43), we find this is identical to the Virasoro algebra with a central charge of $\frac{1}{2}$. This will be reconfirmed using the OPE of EM-tensors at the end of the section. The zero-mode of the Virasoro algebra L_0 is again defined as the energy operator, it allows us to confirm that $|0_{NS}\rangle$ is indeed a true vacuum as we claimed:

$$\begin{aligned} L_0|0_{NS}\rangle &= \frac{1}{2} \sum_{r\in\mathbb{Z}+1/2} (-r+\frac{1}{2}) : b_r b_{-r} : |0_{NS}\rangle \\ &= \frac{1}{2}\Big[\sum_{r\leq -1/2} (-r+\frac{1}{2}) b_r b_{-r} |0_{NS}\rangle + \sum_{r\geq +1/2} (-r+\frac{1}{2}) b_{-r} b_r |0_{NS}\rangle \Big] \end{aligned}$$

Quantum states	Energy
$\lvert 0_{NS}\rangle$	0
$b_{-1/2}\lvert 0_{NS}\rangle$	$\frac{1}{2}$
$b_{-3/2}\lvert 0_{NS}\rangle$	$\frac{3}{2}$
$b_{-1/2}b_{-3/2}\lvert 0_{NS}\rangle$	2
$b_{-5/2}\lvert 0_{NS}\rangle$	$\frac{5}{2}$
$b_{-1/2}b_{-5/2}\lvert 0_{NS}\rangle$	3
$\cdots$	$\cdots$

Table 5.1: Lowest energy states in the Neveu-Schwarz Fock space $\mathcal{F}_{NS}$.

$$= 0. \tag{5.19}$$

$|0_{NS}\rangle$ has an energy (or conformal dimension) of 0. The first few quantum states of $\mathcal{F}_{NS}$ and their energies are presented in Table 5.1.

As the last part of the section, let us compute a few important OPEs. A more complete definition of radial-ordering discussed in section 4.3 involves the *parity* of fields: The parity of a field $A(z)$ is given by

$$\bar{A} = \begin{cases} 0, & \text{if } A(z) \text{ is bosonic,} \\ 1, & \text{if } A(z) \text{ is fermionic.} \end{cases} \tag{5.20}$$

Radial ordering, incorporating parity, is defined as

$$\mathcal{R}\{A(z)B(w)\} = \begin{cases} A(z)B(w), & \text{if } |z| > |w|, \\ (-1)^{\bar{A}\bar{B}} B(w)A(z), & \text{if } |z| < |w|. \end{cases} \tag{5.21}$$

The most basic OPE is that of the fermionic field $\psi(z)$ with itself, and this can be derived using mode expansion as we did in eq. (4.18). However, we introduce a new method here for both novelty and simplicity. We propose the following ansatz:

$$\mathcal{R}\{\psi(z)\psi(w)\} = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \phi_n(w)(z-w)^{-n-1/2}. \tag{5.22}$$

where $\phi_n(w)$ are fields which we need to compute explicitly. Without loss of generality, we assume $|z| > |w|$. Applying the state-field correspondence on both sides, we arrive at

$$\sum_{n\in\mathbb{Z}+\frac{1}{2}} b_n b_{-1/2}|0_{NS}\rangle z^{-n-1/2} = \sum_{n\in\mathbb{Z}+\frac{1}{2}} |\phi_n\rangle z^{-n-1/2}, \tag{5.23}$$

where we have mode expanded $\psi(z)$ and applied the state-field correspondence result from eq. (5.8) to the left-hand side. The state $|\phi_n\rangle$ on the right-hand side corresponds to the field $\phi_n(w)$. Identifying terms with the same power of z on both sides gives

$$|\phi_n\rangle = b_n b_{-1/2}|0_{NS}\rangle. \tag{5.24}$$

When $n > \frac{1}{2}$, b_n anti-commutes with $b_{-1/2}$,

$$|\phi_n\rangle = b_n b_{-1/2}|0_{NS}\rangle = b_{-1/2}b_n|0_{NS}\rangle = 0 \implies \phi_{n>1/2} = 0. \tag{5.25}$$

When $n = \frac{1}{2}$, $\{b_{1/2}, b_{-1/2}\} = 1$, so

$$|\phi_{1/2}\rangle = b_{1/2}b_{-1/2}|0_{NS}\rangle = (-b_{-1/2}b_{1/2} + 1)|0_{NS}\rangle = |0_{NS}\rangle \implies \phi_{1/2} = \mathbb{1}(w). \tag{5.26}$$

When $n < \frac{1}{2}$, the power of $(z - w)$ on the left-hand side of eq. (5.22) is positive. They correspond to the regular terms of the OPE. Combining all three cases, we find the OPE of two fermionic fields to be

$$\psi(z)\psi(w) \sim \frac{1}{z - w}. \tag{5.27}$$

The OPE of $\psi(z)$ with its derivative fields can be computed by taking derivatives of eq. (5.27), for example,

$$\partial\psi(z)\psi(w) = \partial_z(\psi(z)\psi(w)) \sim \partial_z\Big(\frac{1}{z - w}\Big) = -\frac{1}{(z - w)^2}. \tag{5.28}$$

Wick's Theorem introduced in section 4.3 should be modified to compute the OPEs of three or more fermionic fields: A minus sign to the power of the number of fermionic fields in between the fields being contracted. For example,

$$\begin{aligned}
\mathcal{R}\{T(z)\psi(w)\} &= -\frac{1}{2}\mathcal{R}\{:\psi(z)\partial\psi(z):\,\psi(w)\} \\
&= -\frac{1}{2}:\psi(z)\partial\psi(z):\psi(w) - \frac{1}{2}:\psi(z)\partial\psi(z):\psi(w) \\
&\sim \frac{\frac{1}{2}\psi(w)}{(z - w)^2} + \frac{\partial\psi(w)}{z - w}.
\end{aligned} \tag{5.29}$$

We stated in section 4.3 that the OPE of $T(z)$ with a primary field takes the form in eq. (4.27). This is also true for the free fermion theory. $\psi(z)$ is a Virasoro primary, its corresponding state $b_{-1/2}|0_{NS}\rangle$ is annihilated by all L_n with $n > 0$. Comparing eq. (5.29) with (4.27), we find the conformal dimension of $\psi(z)$ to be $\frac{1}{2}$, which is expected from its corresponding state $b_{-1/2}|0_{NS}\rangle$:

$$L_0 b_{-1/2}|0_{NS}\rangle = \frac{1}{2}b_{-1/2}|0_{NS}\rangle. \tag{5.30}$$

We can also use Wick's theorem to compute the OPE of two EM-tensors. The result is given by

$$T(z)T(w) \sim \frac{1/4}{(z - w)^4} + \frac{2T(w)}{(z - w)^2} + \frac{\partial T(w)}{z - w}. \tag{5.31}$$

Comparing this with eq. (4.28), we confirm that the central charge for free fermion theory is indeed $\frac{1}{2}$. This is consistent with what we found out using the commutation relation in eq. (5.18). The conformal dimension of the tensor field is 2, which also agrees with what we obtained from the state-field correspondence in eq. (5.17).

5.3 The Ramond Sector

A fermonic field in the Ramond sector is rather strange compared to its behaviour in the NS-sector. Recall that the fermion field is expanded as

$$\psi(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-1/2}, \tag{5.32}$$

when acting in the Ramond sector. To study the Fock spaces in the Ramond sector, we introduce a vacuum state $|0_R\rangle$. Surprisingly, its energy with respect to L_0 is not 0 but $\frac{1}{16}$, that is, $|0_R\rangle$ is not a true vacuum. We prove this by explicitly computing a correlator in the Ramond sector $\langle 0_R|\partial\psi(z)\psi(w)|0_R\rangle$:

$$\begin{aligned}
\langle 0_R|\partial\psi(z)\psi(w)|0_R\rangle &= \partial_z\langle 0_R|\psi(z)\psi(w)|0_R\rangle \\
&= \partial_z\langle 0_R|\sum_{m,n\in\mathbb{Z}} b_n b_m z^{-n-1/2} w^{-m-1/2}|0_R\rangle \\
&= \partial_z\Big(\sum_{n\in\mathbb{Z}}\langle 0_R|b_n b_{-n} z^{-n-1/2} w^{n-1/2}|0_R\rangle\Big) \\
&= \partial_z\Big(\frac{1/2}{\sqrt{zw}} + \frac{1}{\sqrt{zw}}\sum_{n=1}^{\infty}\big(\frac{w}{z}\big)^n\Big) \\
&= \partial_z\Big(\frac{\sqrt{z/w}+\sqrt{w/z}}{2(z-w)}\Big) \\
&= \frac{-\sqrt{z/w}-\sqrt{w/z}}{2(z-w)^2} + \frac{1}{4(z-w)}\Big(\frac{1}{\sqrt{zw}} - \sqrt{\frac{w}{z^3}}\Big).
\end{aligned} \tag{5.33}$$

Now consider the OPE in eq. (5.28) with the first regular term written out:

$$\begin{aligned}
\partial\psi(z)\psi(w) &= -\frac{1}{(z-w)^2} + :\partial\psi(w)\psi(w): + \mathcal{O}(z-w) \\
&= -\frac{1}{(z-w)^2} + 2T(w) + \mathcal{O}(z-w).
\end{aligned} \tag{5.34}$$

Inserting this into the correlator of eq. (5.33), we get

$$\begin{aligned}
\langle 0_R|\partial\psi(z)\psi(w)|0_R\rangle &= -\langle 0_R|\frac{1}{(z-w)^2}|0_R\rangle + 2\sum_{n\in\mathbb{Z}}\langle 0_R|L_n z^{-n-2}|0_R\rangle + \mathcal{O}(z-w) \\
&= -\frac{1}{(z-w)^2} + 2\langle 0_R|L_0 z^{-2}|0_R\rangle + \mathcal{O}(z-w),
\end{aligned} \tag{5.35}$$

where we have applied eq. (5.16) to write $T(z)$ in terms of L_n. When $n > 1$, the L_n annihilate $|0_R\rangle$, and when $n < 1$, they annihilate $\langle 0_R|$. Hence the only term left in eq. (5.33) is the L_0 mode. Comparing limits of eq. (5.33) with (5.35) as z approaches w, we arrive at

$$2\langle 0_R|L_0|0_R\rangle = \frac{1}{8} \implies L_0|0_R\rangle = \frac{1}{16}|0_R\rangle. \tag{5.36}$$

The energy (conformal dimension) of the Ramond vacuum $|0_R\rangle$ is $\frac{1}{16}$.

The anti-commutation relation eq. (5.5) shows that b_0 exists in the Ramond sector and satisfies

$$b_n^2 = \frac{1}{2}\delta_{n,0} \quad\implies\quad b_0^2 = \frac{1}{2}. \tag{5.37}$$

States generated by $\|0_R\rangle$	States generated by $b_0\|0_R\rangle$	Energy
$\|0_R\rangle$	$b_0\|0_R\rangle$	$\frac{1}{16}$
$b_{-1}\|0_R\rangle$	$b_{-1}b_0\|0_R\rangle$	$\frac{1}{16}+1$
$b_{-2}\|0_R\rangle$	$b_{-2}b_0\|0_R\rangle$	$\frac{1}{16}+2$
$b_{-3}\|0_R\rangle,\ b_{-2}b_{-1}\|0_R\rangle$	$b_{-3}b_0\|0_R\rangle,\ b_{-2}b_{-1}b_0\|0_R\rangle$	$\frac{1}{16}+3$
$b_{-4}\|0_R\rangle,\ b_{-3}b_{-1}\|0_R\rangle$	$b_{-4}b_0\|0_R\rangle,\ b_{-3}b_{-1}b_0\|0_R\rangle$	$\frac{1}{16}+4$
$b_{-5}\|0_R\rangle,\ b_{-4}b_{-1}\|0_R\rangle,\ b_{-3}b_{-2}\|0_R\rangle$	$b_{-5}b_0\|0_R\rangle,\ b_{-4}b_{-1}b_0\|0_R\rangle,\ b_{-3}b_{-2}b_0\|0_R\rangle$	$\frac{1}{16}+5$
$\cdots$	$\cdots$	$\cdots$

Table 5.2: The Ramond Fock space $\mathcal{F}_R$. Every quantum state has its degenerate copy of the same energy.

	NS-sector	R-sector
Space boundary condition	anti-periodic	periodic
Mode expansion	$\sum_{n\in\mathbb{Z}+\frac{1}{2}} b_n z^{-n-1/2}$	$\sum_{n\in\mathbb{Z}} b_n z^{-n-1/2}$
Vacuum energy	$L_0\|0_{NS}\rangle = 0$	$L_0\|0_R\rangle = \frac{1}{16}\|0_R\rangle$
Fock space vacuum	singly-degenerate	doubly-degenerate
State-field correspondence	defined with $\|0_{NS}\rangle$	-

Table 5.3: Chapter 5 summary: a comparison of the Neveu-Schwarz and the Ramond sector.

As we have mentioned previously, momentum is not a well-defined notion for fermions. Hence b_0 is not a momentum operator but a creation operator. However, it does not create an excited state of $|0_R\rangle$. Both $|0_R\rangle$ and $b_0|0_R\rangle$ are vacuum states of the Ramond Fock space $\mathcal{F}_R$, the vacuum is said to be *doubly-degenerate*. The standard basis for $\mathcal{F}_R$ is given by

$$\left\{\cdots b_{-3}^{j_3} b_{-2}^{j_2} b_{-1}^{j_1} b_0^{j_0}|0_R\rangle \;\middle|\; j_r \in \{0,1\} \text{ and } \sum_r j_r < \infty\right\}. \tag{5.38}$$

In Table 5.2, we present the first few quantum states of the Ramond Fock space.

Because $|0_R\rangle$ is not a true vacuum, we cannot define the state-field correspondence with it. For example, if we attempt to find the state corresponding to $\psi(z)$ in the Ramond sector, we quickly run into a divergence problem:

$$\begin{aligned}\lim_{z\to 0}\psi(z)|0_R\rangle &= \lim_{z\to 0}\sum_{n\in\mathbb{Z}} b_n|0_R\rangle z^{-n-1/2} \\ &= \lim_{z\to 0}[b_0|0_R\rangle z^{-1/2} + b_{-1}|0_R\rangle z^{1/2} + \cdots] \\ &= \infty.\end{aligned} \tag{5.39}$$

As a summary, let us compare the key points of the chapter in the two sectors and present the results in Table 5.3.

Chapter 6

The Boson-Fermion Correspondence

One of the goals of this thesis is to solve the free fermion theory which is achieved by computing the correlation functions of its primary fields. This turns out to be extremely difficult for the Ramond sector with our existing level of mathematics. However, we have calculated the correlators of bosonic primaries $V_p(z)$ in chapter 4. This motivates us to construct a correspondence between the bosons and the free fermions such that the correlators of fermionic primaries can be written in terms of those of the bosonic theory. The vertex operators introduced in chapter 4 play a crucial role in this correspondence, we will show that their appropriate linear combinations can be identified as free fermions. We confirm the validity of our construction by identifying the energy-momentum tensors and the partition functions on both sides. In the last section, we will investigate how Ramond fields can be expressed in terms of vertex operators and use these relations to calculate their correlation functions. This chapter is mainly based on my original research.

6.1 Fermionization

Before constructing the correspondence, let us review some important conclusions which we arrived at from previous chapters: In the bosonic theory, the free boson $\partial\varphi(z)$ and the vertex operator $V_p(z)$ have conformal dimensions of 1 and $\frac{p^2}{2}$, respectively. The underlying Virasoro algebra has a central charge of 1. In the fermionic theory, the free fermion has a conformal dimension of $\frac{1}{2}$, its corresponding state is in the Neveu-Schwarz sector. The free fermion algebra has a central charge of $\frac{1}{2}$. This suggests that a free boson ($c = 1$) can be equivalently expressed as two free fermions ($c = \frac{1}{2} + \frac{1}{2}$). Their vacuum representations should be related by

$$\mathcal{F}_B = \mathcal{F}_{NS}^{(1)} \otimes \mathcal{F}_{NS}^{(2)}. \tag{6.1}$$

We also require the conformal dimension of the field in the fermionic theory to match up to that of a free boson, which is 1. This can be done by taking the tensor product of two free fermions of conformal dimension $\frac{1}{2}$, that is, $\partial\varphi(z)$ can be written as the normally ordered product of two independent free fermions up to some constant α:

$$\partial\varphi(z) = \alpha : \psi_1(z)\psi_2(z) : . \tag{6.2}$$

The constant α can be found by imposing the following OPEs on eq. (6.2):

$$\partial\varphi(z)\partial\varphi(w) \sim \frac{1}{(z-w)^2}, \tag{6.3}$$

$$\psi_i(z)\psi_j(z) \sim \frac{\delta_{i,j}}{z-w}. \tag{6.4}$$

Note that in eq. (6.4), the OPE of the two fermions vanishes if their indices are different because the two copies of the fermionic algebras are non-interacting.

Substituting eq. (6.2) into eq. (6.3) and computing the OPE using Wick's theorem gives

$$\begin{aligned}
\partial\varphi(z)\partial\varphi(w) &= \alpha^2 :\psi_1(z)\psi_2(z)::\psi_1(w)\psi_2(w): && \text{(6.5a)}\\
&\sim \alpha^2\Big(-\frac{:\psi_2(z)\psi_2(w):}{z-w} - \frac{:\psi_1(z)\psi_1(w):}{z-w} - \frac{1}{(z-w)^2}\Big) && \text{(6.5b)}\\
&\sim \alpha^2\Big(-\frac{:\psi_2(w)\psi_2(w):}{z-w} - \frac{:\psi_1(w)\psi_1(w):}{z-w} - \frac{1}{(z-w)^2}\Big) && \text{(6.5c)}\\
&\sim \frac{-\alpha^2}{(z-w)^2}, && \text{(6.5d)}
\end{aligned}$$

where we have Taylor expanded the z-fields at $z = w$ in eq. (6.5b). The products of identical fields in eq. (6.5c) vanish by the antisymmetry of fermionic fields: $:\psi_i(w)\psi_i(w): = -:\psi_i(w)\psi_i(w): = 0$. Comparing eq. (6.5d) with eq. (6.3), we arrive at

$$-\frac{\alpha^2}{(z-w)^2} = \frac{1}{(z-w)^2} \qquad \Longrightarrow \qquad \alpha = \pm\, i. \tag{6.6}$$

Therefore, a free boson in terms of two free fermions is expressed as

$$\boxed{\partial\varphi(z) = \pm\, i :\psi_1(z)\psi_2(z):}\,. \tag{6.7}$$

The two expressions for $\partial\varphi(z)$ in eq. (6.7) cannot be distinguished through any physical measurements, that is, their correlators are identical. Conventionally, we only consider the expression with a plus sign. This procedure is called the *fermionization* of a free boson.

6.2 Bosonization

It is intuitive that a free boson can be equivalently expressed as two independent fermions. Surprisingly, the correspondence can even go the other way: It is possible to express the free fermions in terms of the boson. This relation is special to CFT and is called the *bosonization* of free fermions. We construct this correspondence by matching up the conformal dimensions of the fields in the two theories. A free fermion $\psi(z)$ has a conformal dimension of $\frac{1}{2}$. The only fields in the bosonic theory which have the same conformal dimension are the vertex operators $V_1(z)$ and $V_{-1}(z)$ (recall $h_{V_{p(z)}} = \frac{p^2}{2}$). We therefore claim that $\psi_1(z)$ and $\psi_2(z)$ can be expressed as linear combinations of $V_1(z)$ and $V_{-1}(z)$:

$$\psi_1(z) = \alpha_1 V_1(z) + \beta_1 V_{-1}(z), \tag{6.8a}$$

$$\psi_2(z) = \alpha_2 V_1(z) + \beta_2 V_{-1}(z), \tag{6.8b}$$

where α_i and β_i are constants to be determined by the OPE in eq. (6.4). The OPEs of the vertex operators involved in this calculation are computed using eq. (4.23), they are given by

$$V_1(z)V_1(w) = V_{-1}(z)V_{-1}(w) \sim 0, \tag{6.9a}$$

$$V_1(z)V_{-1}(w) = V_{-1}(z)V_1(w) \sim \frac{1}{z-w}. \tag{6.9b}$$

Substituting eq. (6.8a) and (6.8b) into eq. (6.4) gives

$$\begin{aligned}\psi_1(z)\psi_1(w) &= \alpha_1^2 V_1(z)V_1(w) + \alpha_1\beta_1 V_1(z)V_{-1}(w) + \alpha_1\beta_1 V_{-1}(z)V_1(w) + \beta_1^2 V_{-1}(z)V_{-1}(w)\\ &\sim \frac{2\alpha_1\beta_1}{z-w},\end{aligned} \tag{6.10a}$$

$$\psi_2(z)\psi_2(w) \sim \frac{2\alpha_2\beta_2}{z-w}, \tag{6.10b}$$

$$\psi_1(z)\psi_2(w) = \psi_2(z)\psi_1(w) \sim \frac{\alpha_1\beta_2 + \alpha_2\beta_1}{z-w}. \tag{6.10c}$$

Comparing these results with eq. (6.4), we arrive at the following constraints on the constants:

$$\alpha_1\beta_1 = \frac{1}{2}, \qquad \alpha_2\beta_2 = \frac{1}{2}, \qquad \alpha_1\beta_2 + \alpha_2\beta_1 = 0. \tag{6.11}$$

One of the two sets of physically equivalent solutions is given by

$$\alpha_1 = \frac{1}{\sqrt{2}}, \quad \beta_1 = \frac{1}{\sqrt{2}}, \quad \alpha_2 = \frac{i}{\sqrt{2}}, \quad \beta_2 = -\frac{i}{\sqrt{2}}. \tag{6.12}$$

We conclude that the two free fermions are expressed in terms of the bosonic fields as

$$\boxed{\psi_1(z) = \frac{1}{\sqrt{2}}V_1(z) + \frac{1}{\sqrt{2}}V_{-1}(z); \qquad \psi_2(z) = \frac{i}{\sqrt{2}}V_1(z) - \frac{i}{\sqrt{2}}V_{-1}(z)}. \tag{6.13}$$

6.3 The Energy-Momentum Tensor

One of the consequences we can expect from the correspondence is that

$$T_B(z) = T_{F1}(z) + T_{F2}(z), \tag{6.14}$$

where $T_B(z)$ and $T_{Fi}(z)$ are the energy-momentum tensors of a free boson and a free fermion, respectively. Their expressions in terms of $\partial\varphi(z)$ and $\psi(z)$ were stated in eq. (3.35) and (5.12) to be

$$T_B(z) = \frac{1}{2} :\partial\varphi(z)\partial\varphi(z):, \tag{6.15}$$

$$T_{Fi}(z) = -\frac{1}{2} :\psi_i(z)\partial\psi_i(z):. \tag{6.16}$$

Let us first confirm eq. (6.14) using fermionization. The regular term $:\partial\varphi(z)\partial\varphi(w):$ can be written as its OPE subtracted by its singular term.

$$:\partial\varphi(z)\partial\varphi(w): = \partial\varphi(z)\partial\varphi(w) - \frac{1}{(z-w)^2} \tag{6.17a}$$

$$= i^2 :\psi_1(z)\psi_2(z)::\psi_1(w)\psi_2(w): - \frac{1}{(z-w)^2} \tag{6.17b}$$

$$\begin{aligned} &= \frac{:\psi_2(z)\psi_2(w):}{z-w} + \frac{:\psi_1(z)\psi_1(w):}{z-w} \\ &\quad + \frac{1}{(z-w)^2} - \frac{1}{(z-w)^2} + \mathcal{O}(z-w), \end{aligned} \tag{6.17c}$$

where we have applied eq. (6.7) to (6.17a) and used Wick's theorem in (6.17b). Multiplying both sides of eq. (6.17c) by $\frac{1}{2}$, then expanding the w-fields at $w = z$ gives

$$\begin{aligned} \frac{1}{2}:\partial\varphi(z)\partial\varphi(w): &= \frac{1}{2}\Big(\frac{:\psi_2(z)\psi_2(z): + (w-z):\psi_2(z)\partial\psi_2(z):}{z-w} \\ &\quad + \frac{:\psi_1(z)\psi_1(z): + (w-z):\psi_1(z)\partial\psi_1(z):}{z-w}\Big) + \mathcal{O}(z-w) \\ &= -\frac{1}{2}:\psi_1(z)\partial\psi_1(z): - \frac{1}{2}:\psi_2(z)\partial\psi_2(z): + \mathcal{O}(z-w). \end{aligned} \tag{6.18}$$

Setting $w = z$ gives

$$\frac{1}{2}:\partial\varphi(z)\partial\varphi(z): \; = -\frac{1}{2}:\psi_1(z)\partial\psi_1(z): - \frac{1}{2}:\psi_2(z)\partial\psi_2(z):, \tag{6.19}$$

which is exactly eq. (6.14) in terms of EM-tensors. This demonstrates that the energy-momentum tensor of a free boson is indeed equivalent to the sum of two free fermionic EM-tensors.

We can even re-derive eq. (6.14) using bosonization. Before doing this, it is necessary to compute the derivative of a vertex operator for later use. The results in Table 4.1 show that the state corresponding to the field $\partial V_p(z)$ is $L_{-1}|p\rangle$. Let us expand it in terms of a_n using eq. (3.30):

$$\begin{aligned} L_{-1}|p\rangle &= \frac{1}{2}\sum_{r<-1} a_r a_{-r-1}|p\rangle + \frac{1}{2}a_{-1}a_0|p\rangle + \frac{1}{2}a_{-1}a_0|p\rangle + \frac{1}{2}\sum_{r>0} a_{-r-1}a_r|p\rangle \\ &= a_{-1}a_0|p\rangle \\ &= pa_{-1}|p\rangle, \end{aligned} \tag{6.20}$$

which, again by Table 4.1, corresponds to the field $p:\partial\varphi(z)V_p(z):$. Therefore, we find the derivative of a vertex operator is given by

$$\partial V_p(z) = p:\partial\varphi(z)V_p(z):. \tag{6.21}$$

Let us now go back to show that given the correspondence relations, eq. (6.7) and (6.13) imply that $T_{F1}(z) + T_{F2}(z)$ equals $T_B(z)$:

$$T_{F1}(z) + T_{F2}(z) = \frac{1}{2}:\partial\psi_1(z)\psi_1(z): + \frac{1}{2}:\partial\psi_2(z)\psi_2(z): \tag{6.22a}$$

$$= \frac{1}{2}(\frac{1}{\sqrt{2}})^2 : \big(: \partial\varphi(z)V_1(z) : - : \partial\varphi(z)V_{-1}(z) :\big)\big(V_1(z) + V_{-1}(z)\big) :$$
$$+ \frac{1}{2}(\frac{i}{\sqrt{2}})^2 : \big(: \partial\varphi(z)V_1(z) : + : \partial\varphi(z)V_{-1}(z) :\big)\big(V_1(z) - V_{-1}(z)\big) : \tag{6.22b}$$
$$= \frac{1}{2}\big(:: \partial\varphi(z)V_1(z) : V_{-1}(z) : - :: \partial\varphi(z)V_{-1}(z) : V_1(z) : \big), \tag{6.22c}$$

where in eq. (6.22b), we have applied eq. (6.21) with $p = \pm 1$:

$$\partial V_1(z) =: \partial\varphi(z)V_1(z) :, \qquad \partial V_{-1}(z) = - : \partial\varphi(z)V_{-1}(z) : . \tag{6.23}$$

Looking at eq. (6.22c), we need to pay special attention to the normal ordering of three or more fields. Conventionally, the normally-ordered product of n fields is defined inductively as

$$: \psi_1(z_1)\psi_2(z_2)\cdots\psi_n(z_n) : = : \psi_1(z_1) : \psi_2(z_2)\cdots\psi_n(z_n) :: . \tag{6.24}$$

In the case where $n = 3$, eq. (6.24) becomes

$$: \psi_1(z_1)\psi_2(z_2)\psi_3(z_3) : = : \psi_1(z_1) : \psi_2(z_2)\psi_3(z_3) :: \tag{6.25}$$

Normal ordering is not associative, that is, for arbitrary fields $A(z)$, $B(z)$ and $C(z)$,

$$:: A(z)B(z) : C(z) : \neq : A(z) : B(z)C(z) :: . \tag{6.26}$$

Instead, they satisfy the *rearrangement lemma* [26, 33] which states that

$$:: A(z)B(z) : C(z) : = : A(z) : B(z)C(z) :: + \sum_{\ell>0} \frac{1}{\ell\,!} : \partial^\ell A\,[BC]_\ell :$$
$$+ (-1)^{\bar{A}\bar{B}} \sum_{\ell>0} : \partial^\ell B\,[AB]_\ell :, \tag{6.27}$$

where $[AB]_\ell$ represents the field associated with factor $(z-w)^{-\ell}$ in the OPE of $A(z)B(w)$. For example,

$$V_1(z)V_{-1}(w) \sim \frac{1}{z-w} \quad \Longrightarrow \quad \begin{array}{l} [V_1V_{-1}]_{\ell=1} = 1 \\ [V_1V_{-1}]_{\ell>1} = 0. \end{array} \tag{6.28}$$

The notation $\bar{A}$ in eq. (6.27) was introduced in eq. (5.20) as the parity of the field $A(z)$. Note that $\overline{\partial\varphi} = 0$ because it is bosonic, and $\overline{V}_1 = \overline{V}_{-1} = 1$ because their linear combinations are related to the free fermions. Rewriting both terms of eq. (6.22c) using the rearrangement lemma and eq. (6.25), we arrive at

$$:: \partial\varphi(z)V_1(z) : V_{-1}(z) : = \frac{1}{2}\big(: \partial\varphi(z) : V_1(z)V_{-1}(z) :: + \partial^2\varphi(z)\big), \tag{6.29a}$$
$$:: \partial\varphi(z)V_{-1}(z) : V_1(z) : = \frac{1}{2}\big(: \partial\varphi(z) : V_{-1}(z)V_1(z) :: + \partial^2\varphi(z)\big). \tag{6.29b}$$

Substitute these relations into eq. (6.22c), which then becomes

$$\begin{aligned}
T_{F1}(z) + T_{F2}(z) &= \frac{1}{4}\big(\, : \partial\varphi(z) : V_1(z)V_{-1}(z) :: - : \partial\varphi(z) : V_{-1}(z)V_1(z) :: \big) && (6.30a)\\
&= -\frac{1}{4}\big(\, : \partial\varphi(z) : V_1(z)V_1(z) :: - : \partial\varphi(z) : V_1(z)V_{-1}(z) :: + \\
&\qquad : \partial\varphi(z) : V_{-1}(z)V_1(z) :: - : \partial\varphi(z) : V_{-1}(z)V_{-1}(z) :: \big)\\
&= -\frac{1}{4} : \partial\varphi(z) : \big(V_1(z) + V_{-1}(z)\big)\big(V_1(z) - V_{-1}(z)\big) ::\\
&= \frac{i}{2} : \partial\varphi(z) : \psi_1(z)\psi_2(z) ::\\
&= \frac{1}{2} : \partial\varphi(z)\partial\varphi(z) :\\
&= T_B(z), && (6.30b)
\end{aligned}$$

where we have added the terms $: \partial\varphi(z) : V_1(z)V_1(z) ::$ and $: \partial\varphi(z) : V_{-1}(z)V_{-1}(z) ::$ to eq. (6.30a) because $V_1(z)V_1(z) = V_{-1}(z)V_{-1}(z) = 0$ by the anti-symmetries of $V_1(z)$ and $V_{-1}(z)$.

In this section, we have verified the validity of the correspondence relations by showing that the energy-momentum tensors in both theories are equivalent.

6.4 The Partition Function

One of the most important applications of CFT is the study of statistical systems at criticality. The Ising model at critical temperatures which we will investigate in the next chapter is one such example. The macroscopic properties of statistical models are described by their *partition functions*. Recall that with the Boltzmann distribution, the partition function $\mathcal{Z}$ is defined as the normalisation factor for the probabilities:

$$P_i = \frac{1}{\mathcal{Z}} e^{-E_i/kT}. \tag{6.31}$$

Explicitly, $\mathcal{Z}$ it is given by summing $e^{-E_i/kT}$ over the entire microstate space:

$$\mathcal{Z} = \sum_i e^{-E_i/kT}. \tag{6.32}$$

The aim of this section is to equate the character of a free boson to that of two free fermions, which indicates their partition functions are also equal. Theories with equivalent partition functions are described by the same macroscopic physics, so the validity of our correspondence is therefore confirmed.

To define characters in CFT, we first need to introduce some mathematical concepts. The *vacuum representation* of a theory is the Fock space generated by its true vacuum. For example, the vacuum representation of an uncompactified boson is $\mathcal{F}_0$, which is generated by the true bosonic vacuum $|0\rangle$ and operators $\{a_n : n \in \mathbb{Z}\}$. Applying the state-field correspondence to each element of $\mathcal{F}_0$ gives us an analogous field space. The two forms of $\mathcal{F}_0$ are presented in Table 6.1 for comparison.

Having the vacuum representation of an uncompactified boson is not quite enough, because we want to work with a representation which is compatible with both the bosonic and the fermionic theories. This can be achieved by performing a '*simple current extension*'

State-form	Field-form
$\|0\rangle$	$\mathbb{1}(z)$
$a_{-1}\|0\rangle$	$\partial\varphi(z)$
$a_{-2}\|0\rangle,\ a_{-1}^2\|0\rangle$	$\partial^2\varphi(z),\ :\partial\varphi(z)\partial\varphi(z):$
$a_{-3}\|0\rangle,\ a_{-2}a_{-1}\|0\rangle,\ a_{-1}^3\|0\rangle$	$\partial^3\varphi(z),\ :\partial^2\varphi(z)\partial\varphi(z):,\ :\partial\varphi(z)\partial\varphi(z)\partial\varphi(z):$
$\cdots$	$\cdots$

Table 6.1: The vacuum representation of uncompactified bosonic theory in terms of states and fields. This is known as the Fock space or vertex algebra $\mathcal{F}_0$.

[27, 34] on $\mathcal{F}_0$ which allows us to obtain a larger representation that describes fermions as well. A *simple current* $\chi(z)$ is a primary field whose OPE with any other primary field gives just one (in the regular and singular terms) primary field. For example, $V_p(z)$ is a primary field, its OPE with the other primary fields $V_q(z)$ is given by eq. (4.23) whose first term is the only one that contains a primary field, $V_{p+q}(z)$. Therefore, all the $V_p(z)$ are simple currents.

In the free boson theory, the OPE of a simple current $\chi(z)$ with $\partial\varphi(z)$ must take the following form:

$$\partial\varphi(z)\chi(w) \sim \frac{p\chi(w)}{z-w}. \tag{6.33}$$

where p is the momentum of the simple current. Recall the last result in Table 4.2, we see that $V_p(z)$ indeed satisfies this relation:

$$\partial\varphi(z)V_p(w) \sim \frac{pV_p(w)}{z-w}. \tag{6.34}$$

In our correspondence eq. (6.13), we have written the free fermions as linear combinations of $V_1(z)$ and $V_{-1}(z)$. They will be the simple currents which allow us to extend our existing vertex algebra $\mathcal{F}_0$. Recall that $V_1(z)$ and $V_{-1}(z)$ correspond to states $|1\rangle$ and $|-1\rangle$, which in turn will generate Fock spaces $\mathcal{F}_1$ and $\mathcal{F}_{-1}$, respectively. The vertex algebra can be further extended by acting with the simple currents on these spaces. For example, the OPE of the simple current $V_1(z)$ with the $\mathcal{F}_1$-element $V_1(w)$,

$$V_1(z)V_1(w) = (z-w)V_2(w) + \text{descendants of } V_2(w), \tag{6.35}$$

contains the primary field $V_2(w)$, which generates the representation $\mathcal{F}_2$. The OPE of $V_2(z)$ with $V_1(z)$ contains $V_3(z)$ which in turn generates $\mathcal{F}_3\cdots$. One can observe that at last, all Fock spaces generated by vacua with integral momenta will become part of the extended vertex algebra. As introduced in chapter 4, the collection of all extended $\mathcal{F}_p$ where $p\in\mathbb{Z}$ is denoted by $\mathbb{F}_0$, where this is associated to a compactified free boson. The idea of simple current extensions is illustrated in Figure 6.1.

We pause now to define a 'representation'. Mathematically, a representation of an algebra $\mathfrak{g}$ defined on a vector space V is a linear map from $\mathfrak{g}$ to the space of linear operators on V. Let us consider the representations of the extended algebra $\mathbb{F}_0$ in the fermionic theory. A sound candidate for a representation is the collection of all Fock spaces generated by any vacuum $|q\rangle$ where $q\in\mathbb{R}$. Let us denote this collection by $\mathbb{F}_q$ where $q\in[0,1)$ because $\mathbb{F}_q = \mathcal{F}_q\oplus\mathcal{F}_{q\pm1}\oplus\mathcal{F}_{q\pm2}\oplus\cdots$.

$$\begin{tikzcd}[row sep=small]
\cdots & \mathcal{F}_{-2} & \mathcal{F}_{-1} & \mathcal{F}_0 & \mathcal{F}_1 & \mathcal{F}_2 & \cdots \\
\cdots & V_{-2}(z) \arrow[l, "V_{-1}"'] & V_{-1}(z) \arrow[l, "V_{-1}"'] & V_0(z) \arrow[l, "V_{-1}"'] \arrow[r, "V_1"] & V_1(z) \arrow[r, "V_1"] & V_2(z) \arrow[r, "V_{-1}"] & \cdots \\
 & :\partial\varphi V_{-2}: & :\partial\varphi V_{-1}: & \partial\varphi & :\partial\varphi V_1: & :\partial\varphi V_2: & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots &
\end{tikzcd}$$

Figure 6.1: Simple current extension of $\mathcal{F}_0$ by $V_1(z)$ and $V_{-1}(z)$. The vacuum representation of an uncompactified free boson is extended into a collection of Fock spaces with integral momenta — $\mathbb{F}_0$.

States	Energies
$\|0_{NS}\rangle \otimes \|0_{NS}\rangle$	0
$b^{(1)}_{-1/2}\|0_{NS}\rangle \otimes \|0_{NS}\rangle$, $\|0_{NS}\rangle \otimes b^{(2)}_{-1/2}\|0_{NS}\rangle$	$\frac{1}{2}$
$b^{(1)}_{-1/2}\|0_{NS}\rangle \otimes b^{(2)}_{-1/2}\|0_{NS}\rangle$	1
$b^{(1)}_{-3/2}\|0_{NS}\rangle \otimes \|0_{NS}\rangle$, $\|0_{NS}\rangle \otimes b^{(2)}_{-3/2}\|0_{NS}\rangle$	$\frac{3}{2}$
$b^{(1)}_{-1/2}\|0_{NS}\rangle \otimes b^{(2)}_{-3/2}\|0_{NS}\rangle$, $b^{(1)}_{-3/2}\|0_{NS}\rangle \otimes b^{(2)}_{-1/2}\|0_{NS}\rangle$	2

Table 6.2: The first few quantum states of $|0_{NS}\rangle \otimes |0_{NS}\rangle$, this is the algebra of an extended free boson in the fermionic theory.

Consider the case where $|q\rangle$ is being acted on by the simple current $V_1(z)$:

$$\begin{aligned}
V_1(z)|q\rangle &= \lim_{w\to 0} V_1(z)V_q(w)|0\rangle \\
&= \lim_{w\to 0} \left[(z-w)^q\, V_{q+1}(w)|0\rangle + \mathcal{O}((z-w)^{q+1})\right] \qquad (6.36a)\\
&= z^q|q+1\rangle + \mathcal{O}(z^{q+1}), \qquad (6.36b)
\end{aligned}$$

where we have applied the result from eq. (4.23) to write $V_1(z)V_p(w)$ in terms of its OPE in eq. (6.36a). Since $V_1(z)$ is fermionic, its Fourier expansion acting on $|q\rangle$ is given by

$$V_1(z)|q\rangle = \sum_n v_n z^{-n-1/2}|q\rangle, \quad \text{where} \quad \begin{cases} n \in \mathbb{Z}+\frac{1}{2} & \text{for Neveu-Schwarz sector,} \\ n \in \mathbb{Z} & \text{for Ramond sector.} \end{cases} \qquad (6.37)$$

Identifying eq. (6.36b) with eq. (6.37) by equating the powers of z in both expressions, we find that $q = 0$ and $\frac{1}{2}$ for the Neveu-Schwarz and the Ramond sector, respectively. Therefore, $\mathbb{F}_0$ and $\mathbb{F}_{1/2}$ are the only representations of an extended free boson when we interpret $V_{\pm 1}(z)$ as fermions.

According to our correspondence, a free boson is equivalent to the tensor product of two free fermions. The vertex algebra of two free fermions is therefore generated by the tensor product of two true vacua in the fermionic theory — $|0_{NS}\rangle \otimes |0_{NS}\rangle$. Let us denote the corresponding quantum space as $NS \otimes NS$ of which the first few quantum states are presented in Table 6.2. This algebra has at least four representations, they are $NS \otimes NS$, $NS \otimes R$, $R \otimes NS$ and $R \otimes R$.

The character of a representation is the trace (or the sum of all eigenvalues with their multiplicities) of the operator q^H over the entire state space, where $q = e^{-1/kT}$ and $H = L_0 - \frac{C}{24}$ is the Hamiltonian. It is the restriction of the partition function to the representation instead of the whole quantum state space. Note that we have included a correction term $-\frac{C}{24}$ in the Hamiltonian because L_0 describes the energy operator on the complex plane but we use the Hamiltonian on the cylinder in our expression for characters:

$$ch[\mathcal{F}_p](q) = tr_{\mathcal{F}_p}(q^{L_0 - \frac{C}{24}}) = \sum_{\mathcal{F}_p} q^{h - \frac{c}{24}}, \tag{6.38}$$

where h is the eigenvalue of L_0 and C is a constant operator, its eigenvalue c is the central charge of the algebra.

Let us now compute the character of a free boson on one of the representations of the compactified bosonic algebra — $\mathbb{F}_0$. Recall in eq. (3.13), the basis of a free boson Fock space $\mathcal{F}_p$ is given by

$$\{\cdots a_{-3}^{n_3} a_{-2}^{n_2} a_{-1}^{n_1} |p\rangle\}, \tag{6.39}$$

where n_i is a non-negative integer and $p \in \mathbb{Z}$ for $\mathbb{F}_0$. Since $L_0 a_{-n}|p\rangle = (n+p)a_{-n}|p\rangle$ (result from eq. (3.37)), the eigenvalue of a general basis state with respect to L_0 in eq. (6.39) on $\mathcal{F}_p$ is given by

$$\frac{1}{2}p^2 + n_1 + 2n_2 + 3n_3 + \cdots = \frac{1}{2}p^2 + \sum_{j=1}^{\infty} j n_j, \tag{6.40}$$

which corresponds to the following term of the bosonic character on $\mathbb{F}_0$:

$$q^{-\frac{1}{24}}\, q^{\frac{p^2}{2}}\, q^{n_1}\, q^{2n_2}\, q^{3n_3} \cdots . \tag{6.41}$$

The overall character is given by summing over eq. (6.41) for all basis vectors in all Fock spaces $\mathcal{F}_p$ where p is an integer. This is done by taking all possible values of n_j from 0 to infinity for all j:

$$\begin{aligned}
ch[\mathbb{F}_0](q) &= q^{-\frac{1}{24}} \sum_{p\in\mathbb{Z}} q^{\frac{p^2}{2}} \Big[\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \cdots \Big] \Big[q^{n_1} q^{2n_2} q^{3n_3} \cdots \Big] \\
&= q^{-\frac{1}{24}} \sum_{p\in\mathbb{Z}} q^{\frac{p^2}{2}} \Big(\sum_{n_1=0}^{\infty} q^{n_1} \Big) \Big(\sum_{n_2=0}^{\infty} q^{2n_2} \Big) \Big(\sum_{n_3=0}^{\infty} q^{3n_3} \Big) \cdots \qquad (6.42a) \\
&= q^{-\frac{1}{24}} \sum_{p\in\mathbb{Z}} q^{\frac{p^2}{2}} \Big(\frac{1}{1-q} \Big) \Big(\frac{1}{1-q^2} \Big) \Big(\frac{1}{1-q^3} \Big) \cdots \\
&= \Big[\sum_{p\in\mathbb{Z}} q^{\frac{p^2}{2}} \Big] \Big[q^{-\frac{1}{24}} \prod_{i=1}^{\infty} \frac{1}{1-q^i} \Big], \qquad (6.42b)
\end{aligned}$$

where the series in each bracket of eq. (6.42a) is recognised to be geometric. In eq. (6.42b), we split the result into the product of two factors, which are related to two functions familiar to number-theorists [28, 29, 30]: the Dedekind eta function $\eta(q) = q^{\frac{1}{24}} \prod_{i=1}^{\infty}(1-q^i)$ and the third Jacobi theta function $\theta_3(z,q) = \sum_{n\in\mathbb{Z}} z^n q^{n^2/2}$ with $z = 1$. Let us omit the z argument and write $\theta_i(z,q)$ as $\theta_i(q)$. Eq. (6.42b) can then be written as

$$ch[\mathbb{F}_0](q) = \frac{\theta_3(q)}{\eta(q)}. \tag{6.43}$$

Analogously, the character of the other representation $\mathbb{F}_{1/2}$ is given by:

$$ch[\mathbb{F}_{1/2}](q) = \Big[\sum_{p\in\mathbb{Z}+\frac{1}{2}} q^{\frac{p^2}{2}}\Big]\Big[q^{-\frac{1}{24}}\prod_{i=1}^{\infty}\frac{1}{1-q^i}\Big], \tag{6.44}$$

which simplifies to

$$ch[\mathbb{F}_{1/2}](q) = \frac{\theta_2(q)}{\eta(q)} \tag{6.45}$$

by introducing the second Jacobi theta function [28, 30, 31] $\theta_2(z,q) = \sum_{n\in\mathbb{Z}+1/2} z^n q^{n^2/2}$ with $z = 1$.

Following the characters of a free boson, we will investigate the characters of the free fermions in the next part of the section. Let us consider the character of a single free fermion first. A free fermion in the Neveu-Schwarz sector is spanned by the basis $\{\cdots b_{-5/2}^{j_{5/2}} b_{-3/2}^{j_{3/2}} b_{-1/2}^{j_{1/2}}|0_{NS}\rangle\}$, where j_i is either 0 or 1. A similar calculation as in eq. (6.40) to eq. (6.42) leads us to

$$\begin{aligned}
ch[NS](q) &= q^{-\frac{1}{48}}\Big[\sum_{j_1=0}^{1}\sum_{j_2=0}^{1}\sum_{j_3=0}^{1}\cdots\Big]\Big[q^{\frac{1}{2}j_1}q^{\frac{3}{2}j_2}q^{\frac{5}{2}j_3}\cdots\Big] \\
&= q^{-\frac{1}{48}}\Big(\sum_{j_1=0}^{1} q^{\frac{1}{2}j_1}\Big)\Big(\sum_{j_2=0}^{1} q^{\frac{3}{2}j_2}\Big)\Big(\sum_{j_3=0}^{1} q^{\frac{5}{2}j_3}\Big)\cdots && (6.46\text{a}) \\
&= q^{-\frac{1}{48}}(1+q^{\frac{1}{2}})(1+q^{\frac{3}{2}})(1+q^{\frac{5}{2}})\cdots \\
&= q^{-\frac{1}{48}}\prod_{i=1}^{\infty}(1+q^{i-\frac{1}{2}}), && (6.46\text{b})
\end{aligned}$$

where we have applied eq. (6.38) with a fermionic central charge of $\frac{1}{2}$. Eq. (6.46b) can be simplified further using *Jacobi's triple product identity* [28, 29] which splits $\theta_3(q)$ into its product form:

$$\theta_3(q) = \prod_{i=1}^{\infty}(1+q^{i-\frac{1}{2}})^2(1-q^i). \tag{6.47}$$

This implies that

$$\begin{aligned}
\prod_{i=1}^{\infty}(1+q^{i-\frac{1}{2}}) &= \Big(\frac{\theta_3(q)}{\prod_{i=1}^{\infty}(1-q^i)}\Big)^{\frac{1}{2}} && (6.48\text{a}) \\
&= \Big(\frac{\theta_3(q)}{\eta(q)\,q^{-\frac{1}{24}}}\Big)^{\frac{1}{2}}, && (6.48\text{b})
\end{aligned}$$

where we have used the definition of $\eta(q)$ to rewrite the factor $\prod_{i=1}^{\infty}(1-q^i)$ in eq. (6.48a). Substituting eq. (6.48b) into eq. (6.46b) gives

$$ch[NS](q) = q^{-\frac{1}{48}}\Big(\frac{\theta_3(q)}{\eta(q)\,q^{-\frac{1}{24}}}\Big)^{\frac{1}{2}} = \Big(\frac{\theta_3(q)}{\eta(q)}\Big)^{\frac{1}{2}}. \tag{6.49}$$

The Fock space of a single free fermion on the Ramond sector is spanned by the basis $\{\cdots b_{-3}^{j_3} b_{-2}^{j_2} b_{-1}^{j_1} b_0^{j_0}|0_R\rangle\}$ where j_i is either 0 or 1. The energy of a general basis state is $\frac{1}{16} + \sum_{i=0}^{\infty} i j_i$, where $\frac{1}{16}$ is the vacuum energy of $|0_R\rangle$ as we calculated in eq. (5.36). The character over the entire Ramond Fock space is given by

$$
\begin{aligned}
ch[R](q) &= q^{-\frac{1}{48}}\, q^{\frac{1}{16}} \Big[\sum_{j_0=0}^{1} \sum_{j_1=0}^{1} \sum_{j_2=0}^{1} \sum_{j_3=0}^{1} \cdots \Big] \Big[q^{0 j_0} q^{j_1} q^{2 j_2} q^{3 j_3} \cdots \Big] \\
&= q^{\frac{1}{24}} (1+q^0)(1+q)(1+q^2)(1+q^3)\cdots \\
&= 2\, q^{\frac{1}{24}} \prod_{i=1}^{\infty} (1+q^i), \qquad (6.50)
\end{aligned}
$$

This again can be simplified by making use of the triple product identity [30, 32]:

$$
\begin{aligned}
\theta_2(q) &= q^{\frac{1}{8}} \prod_{i=1}^{\infty} (1+q^i)(1+q^{i-1})(1-q^i) \qquad (6.51a) \\
&= 2 q^{\frac{1}{8}} \prod_{i=1}^{\infty} (1+q^i)^2 (1-q^i), \qquad (6.51b)
\end{aligned}
$$

which implies

$$
\prod_{i=1}^{\infty} (1+q^i) = \Big(\frac{\theta_2(q)}{2 q^{\frac{1}{12}} \eta(q)} \Big)^{\frac{1}{2}}. \qquad (6.52)
$$

Substituting eq. (6.52) into (6.50) gives

$$
ch[R](q) = \sqrt{2} \Big(\frac{\theta_2(q)}{\eta(q)} \Big)^{\frac{1}{2}}. \qquad (6.53)
$$

We are interested in characters of two free fermions defined on each of the four representations: $NS \otimes NS$, $NS \otimes R$, $R \otimes NS$ and $R \otimes R$. These can be obtained by multiplying two single fermionic characters defined on separate sectors:

$$
\begin{aligned}
ch[NS \otimes NS](q) &= ch[NS](q) \times ch[NS](q) \\
&= \Big(\frac{\theta_3(q)}{\eta(q)} \Big)^{\frac{1}{2}} \times \Big(\frac{\theta_3(q)}{\eta(q)} \Big)^{\frac{1}{2}} \\
&= \frac{\theta_3(q)}{\eta(q)}, \qquad (6.54a)
\end{aligned}
$$

$$
ch[R \otimes R](q) = \frac{2\theta_2(q)}{\eta(q)}, \qquad (6.54b)
$$

$$
ch[R \otimes NS](q) = \frac{(2\theta_2(q)\theta_3(q))^{\frac{1}{2}}}{\eta(q)}, \qquad (6.54c)
$$

$$
ch[NS \otimes R](q) = \frac{(2\theta_2(q)\theta_3(q))^{\frac{1}{2}}}{\eta(q)}. \qquad (6.54d)
$$

Summarising the characters of a single boson with that of two free fermions in Table 6.3, we notice that the bosonic character in the NS-representation $\mathbb{F}_0$ is equal to the character of two free fermions both in the NS-sector. The character of a single boson in the R-representation $\mathbb{F}_{1/2}$ is half of that of two fermions both in the R-sector, which is accounted for by the double-degeneracy of the Ramond Fock space vacuum. $R \otimes R$ is a direct sum of two irreducible representations of the two free fermion algebra. If two free fermions are in different sectors, their characters do not contribute to the bosonic character. The

	Single Free Boson	Two Free Fermions
Algebra	$\mathbb{F}_0$	$\mathbb{F}_{1/2}$
Representations	$\mathbb{F}_0$ (NS)	$NS \otimes NS$
	$\mathbb{F}_{1/2}$ (R)	$R \otimes R$
		$NS \otimes R$, $R \otimes NS$
Characters	$ch[\mathbb{F}_0](q) = \theta_3(q)/\eta(q)$	$ch[NS \otimes NS](q) = \theta_3(q)/\eta(q)$
	$ch[\mathbb{F}_{1/2}](q) = \theta_2(q)/\eta(q)$	$ch[R \otimes R](q) = 2\theta_2(q)/\eta(q)$
		$ch[R \otimes NS](q) = ch[NS \otimes R](q) = (2\theta_2(q)\theta_3(q))^{\frac{1}{2}}/\eta(q)$

Table 6.3: A comparison of characters of a free boson and two free fermions in different representations.

agreement in characters of a free boson and two fermions means that their partition functions will also agree. This confirms again the validity of our correspondence.

6.5 The Free Fermion Correlation Functions

Bosonization provides us with an alternative method of computing correlators of free fermions. In this section we will calculate the first few n-point correlation functions of a free fermion using the boson-fermion correspondence and compare the results against those obtained from the fermionic Wick theorem. Recall that free fermions are related to a free boson by eq. (6.13). The two fermions cannot be distinguished through any physical measurements, that is, their correlation functions are exactly the same. We will exemplify the calculation with $\psi_1(z) = \frac{1}{\sqrt{2}}(V_1(z) + V_{-1}(z))$ and keep in mind that $\psi_2(z)$ will give exactly the same result.

We have computed the correlation functions of vertex operators in chapter 4. The results are presented in eq. (4.36), (4.39c), (4.40) and (4.41) from which we are able to calculate the correlators of $\psi_1(z)$:

$$\langle 0_{NS}|\psi_1(z)|0_{NS}\rangle = 0, \tag{6.55a}$$

$$\begin{aligned}
\langle 0_{NS}|\psi_1(z_1)\psi_1(z_2)|0_{NS}\rangle &= \Big(\frac{1}{\sqrt{2}}\Big)^2 \Big\langle \big(V_1(z_1) + V_{-1}(z_1)\big)\big(V_1(z_2) + V_{-1}(z_2)\big)\Big\rangle \\
&= \frac{1}{2}\Big\{\big\langle V_1(z_1)V_1(z_2)\big\rangle + \big\langle V_1(z_1)V_{-1}(z_2)\big\rangle \\
&\qquad + \big\langle V_{-1}(z_1)V_1(z_2)\big\rangle + \big\langle V_{-1}(z_1)V_{-1}(z_2)\big\rangle\Big\} \\
&= \frac{1}{z_1 - z_2},
\end{aligned} \tag{6.55b}$$

$$\begin{aligned}
\langle 0_{NS}|\psi_1(z_1)\psi_1(z_2)\psi_1(z_3)|0_{NS}\rangle &= \Big(\frac{1}{\sqrt{2}}\Big)^3 \Big\langle \big(V_1(z_1) + V_{-1}(z_1)\big)\big(V_1(z_2) + V_{-1}(z_2)\big)\big(V_1(z_3) + V_{-1}(z_3)\big)\Big\rangle \\
&= \Big(\frac{1}{\sqrt{2}}\Big)^3 \Big\{\langle V_1V_1V_1\rangle + \langle V_{-1}V_1V_1\rangle + \langle V_1V_{-1}V_1\rangle \\
&\qquad + \langle V_1V_1V_{-1}\rangle + \langle V_{-1}V_{-1}V_1\rangle + \langle V_1V_{-1}V_{-1}\rangle \\
&\qquad + \langle V_{-1}V_1V_{-1}\rangle + \langle V_{-1}V_{-1}V_{-1}\rangle\Big\} \qquad (6.55c) \\
&= 0. \qquad (6.55d)
\end{aligned}$$

Note that we have omitted the arguments of vertex operators in eq. (6.55c), each term of this expression vanishes because none of them satisfies the neutrality condition discussed after eq. (4.42). In fact, all n-point functions of ψ_1, where n is an odd integer, vanish, because no odd numbers of +1 and -1 can be summed to zero. The 4-point function is computed in exactly the same way but is more complicated:

$$\langle 0|\psi_1(z_1)\psi_1(z_2)\psi_1(z_3)\psi_1(z_4)|0\rangle = \frac{1}{2}\left(\frac{z_{12}\,z_{34}}{z_{13}\,z_{14}\,z_{23}\,z_{24}} + \frac{z_{13}\,z_{24}}{z_{12}\,z_{13}\,z_{23}\,z_{34}} + \frac{z_{14}\,z_{23}}{z_{12}\,z_{13}\,z_{24}\,z_{34}}\right), \tag{6.56}$$

where $z_{ij} = z_i - z_j$.

Let us check the first three correlators using OPEs of free fermions and Wick's theorem. The 1-point function is trivially zero, since we have stated in chapter 4 that the only non-vanishing 1-point function is that of the identity field. The 2-point function can be computed from the OPEs of $\psi_1(z)$ with itself:

$$\begin{aligned}\langle 0_{NS}|\psi_1(z_1)\psi_1(z_2)|0_{NS}\rangle &= \langle 0_{NS}|\frac{1}{z_1 - z_2}|0_{NS}\rangle \\ &= \frac{1}{z_1 - z_2}\langle 0_{NS}|0_{NS}\rangle \\ &= \frac{1}{z_1 - z_2}.\end{aligned} \tag{6.57}$$

To compute the 3-point function, we will apply the fermionic Wick theorem:

$$\begin{aligned}\langle 0_{NS}|\psi_1(z_1)\psi_1(z_2)\psi_1(z_3)|0_{NS}\rangle &= \langle :\psi_1(z_1)\psi_1(z_2)\psi_1(z_3):\rangle + \frac{\langle 0_{NS}|\psi_1(z_3)|0_{NS}\rangle}{z_1 - z_2} \\ &\quad + \frac{\langle 0_{NS}|\psi_1(z_1)|0_{NS}\rangle}{z_2 - z_3} - \frac{\langle 0_{NS}|\psi_1(z_2)|0_{NS}\rangle}{z_1 - z_3} &(6.58a) \\ &= 0. &(6.58b)\end{aligned}$$

There are one zero and three single contractions involved in this calculation, the correlator of a regular term always vanishes. The numerators of the other three terms of eq. (6.58a) also equal to zero because they are the 1-point functions of non-identity fields.

So far in this chapter, we have constructed a boson-fermion correspondence in order to calculate the correlation functions of fermionic fields in Neveu-Schwarz sector. The results were compared with those obtained from Wick's theorem and were identical. In this sense, the correspondence does not bring us any new information about the free fermions. However, we must realise that the procedure of constructing such a correspondence is crucial. In an analogous manner, we shall study how Ramond fields are related to the bosonic theory and use this knowledge to compute their correlators, which are too difficult to calculate through other methods.

6.6 The Ramond Correlation Functions

Since OPEs are not known for the fields in the Ramond sector, we are not able to compute their correlators in a straight forward way. Constructing a correspondence between the Ramond fields and the free boson as we did in section 6.1 and 6.2 solves this problem. First of all, a primary Ramond field $R(z)$ is related to the Ramond vacuum $|0_R\rangle$ through the following state-field correspondence:

$$\lim_{z\to 0} R(z)|0_{NS}\rangle = |0_R\rangle. \tag{6.59}$$

The conformal dimension of $R(z)$ equals the eigenvalue of $|0_R\rangle$ with respect to L_0, which is $\frac{1}{16}$ as we showed in eq. (5.36). We expect $R(z)$ to be related to the free boson theory in a similar way as the free fermion field $\psi(z)$ does, so we can express $R(z)$ in terms of vertex operators $V_p(z)$. In order to match up the conformal dimensions, we expect the bosonic representation of the Ramond field to correspond to the Fock space $\mathcal{F}_{NS}\otimes\mathcal{F}_R$ (or $\mathcal{F}_R\otimes\mathcal{F}_{NS}$, they give identical physical results). There exist two independent Ramond fields $R_1(z)$ and $R_2(z)$ because the vacuum is doubly degenerate in the Ramond sector. Both of them are in the form of a tensor product of two fields: the identity field whose conformal dimension is 0 from $\mathcal{F}_{NS}$, and a field in terms of $V_p(z)$ whose conformal dimension is $\frac{1}{16}$ from $\mathcal{F}_R$, let us call this field $\rho(z)$:

$$R_1(z) = \mathbb{1}(z)\otimes\rho_1(z), \tag{6.60a}$$

$$R_2(z) = \mathbb{1}(z)\otimes\rho_2(z). \tag{6.60b}$$

The total conformal dimension of a field constructed in this manner is $0+\frac{1}{16}$, which matches that of the Ramond field.

Following an analogous procedure as the bosonization of section 6.2, we can readily express the fields $\rho_1(z)$ and $\rho_2(z)$ in terms of the $V_p(z)$. Recall that the conformal dimension of $V_p(z)$ is $\frac{p^2}{2}$, which implies that the only vertex operators with a conformal dimension of $\frac{1}{16}$ are $V_{\frac{1}{\sqrt{8}}}(z)$ and $V_{\frac{-1}{\sqrt{8}}}(z)$. Both $\rho_1(z)$ and $\rho_2(z)$ are therefore linear combinations of $V_{\frac{1}{\sqrt{8}}}(z)$ and $V_{\frac{-1}{\sqrt{8}}}(z)$:

$$\rho_1(z) = \alpha_1 V_{\frac{1}{\sqrt{8}}}(z) + \beta_1 V_{\frac{-1}{\sqrt{8}}}(z), \tag{6.61a}$$

$$\rho_2(z) = \alpha_2 V_{\frac{1}{\sqrt{8}}}(z) + \beta_2 V_{\frac{-1}{\sqrt{8}}}(z), \tag{6.61b}$$

of which the constants α_1, α_2, β_1 and β_2 can be found by constraining the OPEs of $\rho_i(z)$ with itself to be normalised and requiring the orthogonality between $\rho_1(z)$ and $\rho_2(z)$:

$$\rho_1(z)\rho_1(w) = \rho_2(z)\rho_2(w) \sim \frac{1}{(z-w)^{\frac{1}{8}}}, \tag{6.62a}$$

$$\rho_1(z)\rho_2(w) = \rho_2(z)\rho_1(w) \sim 0. \tag{6.62b}$$

The solutions from these constraints are $\alpha_1=\alpha_2=\frac{1}{\sqrt{2}}$, $\beta_1=-\beta_2=\frac{i}{\sqrt{2}}$. The ρ-fields are therefore given by

$$\rho_1(z) = \frac{1}{\sqrt{2}}V_{\frac{1}{\sqrt{8}}}(z) + \frac{1}{\sqrt{2}}V_{\frac{-1}{\sqrt{8}}}(z), \tag{6.63a}$$

$$\rho_2(z) = \frac{i}{\sqrt{2}}V_{\frac{1}{\sqrt{8}}}(z) - \frac{i}{\sqrt{2}}V_{\frac{-1}{\sqrt{8}}}(z). \tag{6.63b}$$

Substituting them back into eq. (6.60) and omitting the identity field, we arrive at the expressions for Ramond fields as follows:

$$\boxed{R_1(z) = \frac{1}{\sqrt{2}} V_{\frac{1}{\sqrt{8}}}(z) + \frac{1}{\sqrt{2}} V_{\frac{-1}{\sqrt{8}}}(z)\,, \qquad R_2(z) = \frac{i}{\sqrt{2}} V_{\frac{1}{\sqrt{8}}}(z) - \frac{i}{\sqrt{2}} V_{\frac{-1}{\sqrt{8}}}(z).} \tag{6.64}$$

This correspondence allows the correlators of Ramond fields to be readily computed in a similar way as we did with the free fermions. Again, we will illustrate the calculation with the first four n-point functions of $R_1(z)$:

$$\langle 0_{NS}|R_1(z)|0_{NS}\rangle = 0, \tag{6.65}$$

$$\begin{aligned}\langle 0_{NS}|R_1(z_1)R_1(z_2)|0_{NS}\rangle &= \big(\frac{1}{\sqrt{2}}\big)^2 \big\langle \big(V_{\frac{1}{\sqrt{8}}}(z_1) + V_{\frac{-1}{\sqrt{8}}}(z_1)\big)\big(V_{\frac{1}{\sqrt{8}}}(z_2) + V_{\frac{-1}{\sqrt{8}}}(z_2)\big)\big\rangle \\ &= \frac{1}{2}\big\{\big\langle V_{\frac{1}{\sqrt{8}}}(z_1)V_{\frac{1}{\sqrt{8}}}(z_2)\big\rangle + \big\langle V_{\frac{1}{\sqrt{8}}}(z_1)V_{\frac{-1}{\sqrt{8}}}(z_2)\big\rangle \\ &\qquad + \big\langle V_{\frac{-1}{\sqrt{8}}}(z_1)V_{\frac{1}{\sqrt{8}}}(z_2)\big\rangle + \big\langle V_{\frac{-1}{\sqrt{8}}}(z_1)V_{\frac{-1}{\sqrt{8}}}(z_2)\big\rangle\big\} \\ &= \frac{1}{(z_1 - z_2)^{\frac{1}{8}}},\end{aligned} \tag{6.66}$$

$$\begin{aligned}\langle 0_{NS}|R_1(z_1)R_1(z_2)R_1(z_3)|0_{NS}\rangle &= \big(\frac{1}{\sqrt{2}}\big)^3 \big\langle \big(V_{\frac{1}{\sqrt{8}}}(z_1) + V_{\frac{-1}{\sqrt{8}}}(z_1)\big)\big(V_{\frac{1}{\sqrt{8}}}(z_2) + V_{\frac{-1}{\sqrt{8}}}(z_2)\big) \\ &\qquad \big(V_{\frac{1}{\sqrt{8}}}(z_3) + V_{\frac{-1}{\sqrt{8}}}(z_3)\big)\big\rangle \\ &= \big(\frac{1}{\sqrt{2}}\big)^3 \big\{\langle V_{\frac{1}{\sqrt{8}}}V_{\frac{1}{\sqrt{8}}}V_{\frac{1}{\sqrt{8}}}\rangle + \langle V_{\frac{-1}{\sqrt{8}}}V_{\frac{1}{\sqrt{8}}}V_{\frac{1}{\sqrt{8}}}\rangle + \langle V_{\frac{1}{\sqrt{8}}}V_{\frac{-1}{\sqrt{8}}}V_{\frac{1}{\sqrt{8}}}\rangle \\ &\qquad + \langle V_{\frac{1}{\sqrt{8}}}V_{\frac{1}{\sqrt{8}}}V_{\frac{-1}{\sqrt{8}}}\rangle + \langle V_{\frac{-1}{\sqrt{8}}}V_{\frac{-1}{\sqrt{8}}}V_{\frac{1}{\sqrt{8}}}\rangle + \langle V_{\frac{1}{\sqrt{8}}}V_{\frac{-1}{\sqrt{8}}}V_{\frac{-1}{\sqrt{8}}}\rangle \\ &\qquad + \langle V_{\frac{-1}{\sqrt{8}}}V_{\frac{1}{\sqrt{8}}}V_{\frac{-1}{\sqrt{8}}}\rangle + \langle V_{\frac{-1}{\sqrt{8}}}V_{\frac{-1}{\sqrt{8}}}V_{\frac{-1}{\sqrt{8}}}\rangle\big\} \\ &= 0.\end{aligned} \tag{6.67}$$

Similarly to the free fermion correlators, all n-point functions, where n is odd, vanish, because they cannot satisfy the neutrality condition. The 4-point function is

$$\begin{aligned}&\langle 0_{NS}|R_1(z_1)R_1(z_2)R_1(z_3)R_1(z_4)|0_{NS}\rangle \\ &\qquad = \frac{1}{2}\left(\left(\frac{z_{12}\,z_{34}}{z_{13}\,z_{14}\,z_{23}\,z_{24}}\right)^{\frac{1}{8}} + \left(\frac{z_{13}\,z_{24}}{z_{12}\,z_{13}\,z_{23}\,z_{34}}\right)^{\frac{1}{8}} + \left(\frac{z_{14}\,z_{23}}{z_{12}\,z_{13}\,z_{24}\,z_{34}}\right)^{\frac{1}{8}}\right).\end{aligned} \tag{6.68}$$

Chapter 7

The Ising Model

The critical Ising model of ferromagnetism is one of the most famous applications of CFT in statistical mechanics. In this chapter, we identify the Ising model with a free fermion model by showing that their primary fields are related by simple current extension. This allows us to construct a correspondence between the Ising model primaries and the bosonic vertex operators as we did for the free fermion in the last chapter. We then calculate the correlation functions of these Ising model primaries which are experimentally verifiable.The calculations in this chapter is from my original research.

7.1 Background

As with many scientific discoveries, the so-called "Ising model" was not named after its inventor, Wilhelm Lenz, but the person who brought attention to it — Ernst Ising — a PhD student of Lenz in 1920. The one-dimensional Ising model was solved by Ising himself in his 1924 thesis [35]. The analytic solution for the more complicated two-dimensional square lattice Ising model was published by Lars Onsager in 1944 [36].

The two-dimensional Ising model is a mathematical description of ferromagnetism in statistical mechanics. It is a square lattice with each lattice point representing a quantum spin. Each spin can be either up or down and is assigned a spin value σ_i of +1 or -1, respectively. As a simple example, Figure 7.1 illustrates the Ising model with a 9×9 square lattice. A spin at position i interacts with its four nearest neighbours at j with an interaction strength J. The total energy of the lattice is given by summing over the interaction energies of all lattice points with their nearest neighbours [14]:

$$E = -J \sum_{i,\, j} \sigma_i \sigma_j, \tag{7.1}$$

where we have chosen J to be positive so that it is energetically favourable to have neighbouring spins aligned in the same direction ($\sigma_i = \sigma_j$).

The Ising model as well as many other statistical models undergo a second-order phase transition at critical temperatures. For ferromagnets, this temperature is referred to as the Curie temperature (T_c), above which the spontaneous magnetisation is lost. When $0 < T < T_c$, Ising model spins flip slowly due to thermal fluctuations. The probability of a flip mainly depends on the energy difference that would occur if flipped, that is, if a flip would cause more spins in its neighbourhood to align in the same direction, then it is more likely to occur than other flips. Above the Curie temperature, the effect of thermal fluctuations dominates the neighbouring effect, spins are flipping so fast that their distribution becomes random. An computer simulated two dimensional Ising model

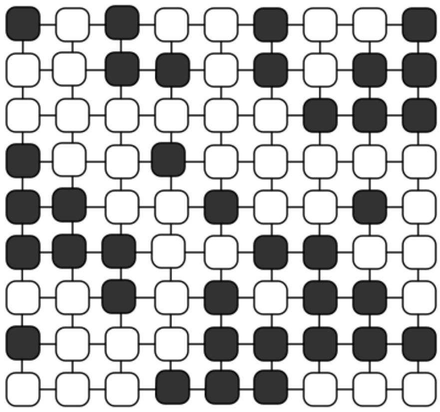

Figure 7.1: A possible configuration of of a 9×9 square lattice Ising model. A black lattice point represents spin up ($\sigma_i = +1$) and a white one represents spin down ($\sigma_i = -1$). This picture is taken from [37].

is illustrated in Figure 7.2.

In statistical mechanics, phase transitions represent the sudden changes in macroscopic properties of thermodynamic systems as a control parameter is varied. For example, at its melting temperature, ice abruptly and discontinuously decreases in volume. A finite amount of heat has been used to change the physical structure of the system, in this case, break the weak molecular bindings. Such phase transitions between solid, liquid and gaseous states of matter are termed *first order phase transitions*.

The two-dimensional Ising model at criticality displays another type of phase transition — a second order phase transition. Such transitions involve sudden changes in the derivatives of macroscopic quantities of the system instead of the quantities themselves as the control parameter varies. No 'latent heat' is required to change the state of the physical system at the second order transition. In the Ising model, the thermodynamical quantity of interest is the magnetisation M, which is defined as the average value of a single spin: $M = \langle \sigma_i \rangle$. As the temperature is varied around T_c, we find M to be continuous, but its first derivative exhibits a divergence at T_c. The Ising model displays a very interesting property at T_c — it is statistically invariant under rescaling, which means that if we zoom in at a typical Ising model configuration at criticality, we will always observe a statistically similar pattern, like a fractal. This is illustrated in Figure 7.3.

The scale invariance property of the two-dimensional Ising model encourages us to describe it (or rather its thermodynamic limit) in terms of conformal field theory. Recall in chapter 2, we have defined a transformation to be conformal if its metric is invariant up to a rescaling factor (eq. (2.2)). The aim of this chapter is to search for a connection between the Ising model and the free fermion theory in order to compute its primary field correlators from those of the free fermion.

7.2 Ising Model-Fermion Correspondence

CFTs with a finite number of primary fields are called *rational conformal field theories (RCFTs)*. In their 1984 paper [11], Belavin, Polyakov and Zamolodchikov studied a special class of RCFTs with central charge $c < 1$, they called them the "*minimal models*" $\mathcal{M}(p, q)$.

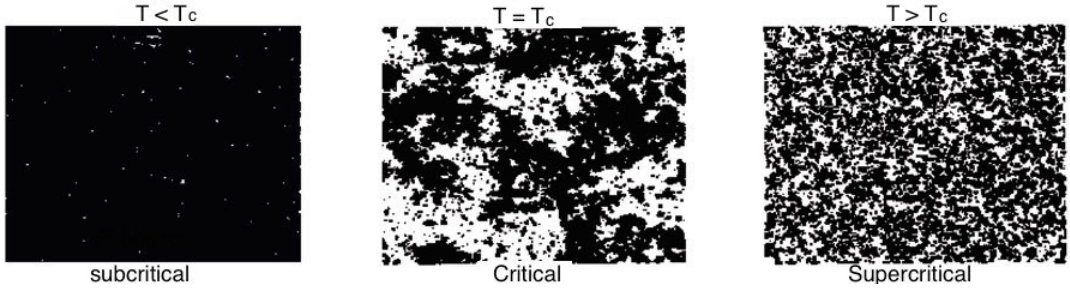

Figure 7.2: A computer simulation of typical two dimensional Ising model configurations at various temperatures, where black and white represent spins of opposite directions. Below T_c, spins tend to align in the same direction due to the neighbouring effect. Above T_c, spins are randomly distributed, no large domains are formed. At the critical temperature, clusters of all sizes can be observed, the pattern is fractal, i.e. scale invariant. These pictures were taken from [38].

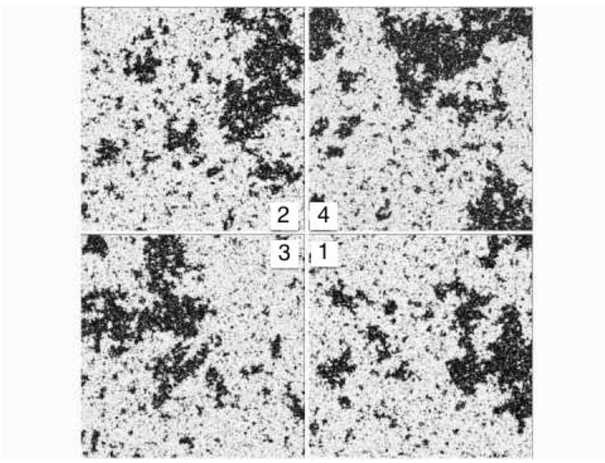

Figure 7.3: A computer simulation of a two dimensional Ising model on 2^{34} lattice points at criticality. The length scale of pictures 1 to 4 are 2^{11}, 2^{13}, 2^{15} and 2^{17} pixels, respectively. All four snapshots contain domains of all sizes, the model is statistically scale invariant. These pictures were taken from [39].

The minimal models are described mathematically by Virasoro algebras characterised by two positive integers p and q ($p, q \geq 2$) with no non-trivial common divisors. The central charge of the minimal model $\mathcal{M}(p,q)$ is then given by [14, 16]

$$c = 1 - \frac{6(p-q)^2}{pq}. \tag{7.2}$$

A canonical example of a minimal model is the two-dimensional Ising model $\mathcal{M}(3,4)$. According to eq. (7.2), the central charge of the Ising model is $\frac{1}{2}$, which is the same as that of a free fermion theory.

Our first task is to find out the primary fields of the Ising model and their conformal dimensions. To do this, we first introduce the idea of "*null states*" or "*unphysical states*". A null or unphysical state $|\chi\rangle$ is a descendant state of a vacuum state which satisfies

$$L_0|\chi\rangle = (h+N)|\chi\rangle, \qquad \text{and} \qquad L_n|\chi\rangle = 0 \text{ for } n > 0. \tag{7.3}$$

Note that $|\chi\rangle$ is then also a vacuum state so it corresponds to a primary field. It is orthogonal to all other quantum states. A null state $|\chi\rangle$ is said to be of "*level-n*" if its energy is $n+h$, where h is the conformal dimension of the vacuum that generates $|\chi\rangle$. For example, let us consider a level-6 null state $|\chi_6\rangle$ generated from the true vacuum $|0\rangle$. It has the general form

$$|\chi_6\rangle = (L_{-2}^3 + c_1 L_{-3}^2 + c_2 L_{-4}L_{-2} + c_3 L_{-6})|0\rangle, \tag{7.4}$$

where c_1, c_2 and c_3 are constants to be determined. Note that L_{-1} is not involved in this general form because $L_{-1}|0\rangle = 0$ by eq. (4.3). The field corresponding to $|\chi_6\rangle$ can be calculated using eq. (4.7) to be

$$\chi_6(z) =: T(z)T(z)T(z) : + c_1 : \partial T(z)\partial T(z) : + \frac{c_2}{2} : \partial^2 T(z)T(z) : + \frac{c_3}{24}\partial^4 T(z). \tag{7.5}$$

Since $\chi_6(z)$ is primary of conformal dimension 6, according to eq. (4.27), its OPE with $T(z)$ must be

$$T(z)\chi_6(w) \sim \frac{6\chi_6(w)}{(z-w)^2} + \frac{\partial\chi_6(w)}{z-w}, \tag{7.6}$$

which imposes constraints on $\chi_6(z)$ and allows us to compute the constants in eq. (7.5). We did this by using a Mathematica package for computing OPEs written by Thielemans [33], it gives us the following expression for $\chi_6(z)$:

$$\chi_6(z) = : T(z)T(z)T(z) : + \frac{93}{64} : \partial T(z)\partial T(z) : - \frac{33}{16} : \partial^2 T(z)T(z) : - \frac{9}{128}\partial^4 T(z). \tag{7.7}$$

Since a null state is orthogonal to any other state, the OPE of the field corresponding to the null state with any other field, including the primary fields, must vanish. Exploiting this fact, we are able to compute the conformal dimensions of the primary fields in the Ising model. Let $\phi_h(z)$ be a primary field of the Ising model, its OPE with the EM-tensor $T(z)$ is again of the following form:

$$T(z)\phi_h(w) \sim \frac{h\phi_h(w)}{(z-w)^2} + \frac{\partial\phi_h(w)}{z-w}, \tag{7.8}$$

where h is the conformal dimension of $\phi_h(z)$. Since $\chi_6(z)$ corresponds to a null state, its

OPE with $\phi_h(z)$ must vanish:

$$\chi_6(z)\phi_h(w) = 0. \tag{7.9}$$

Computing $\chi_6(z)\phi_h(w)$ using the expression for $\chi_6(z)$ in eq. (7.7) and the OPE in eq. (7.8) by Thielemans' Mathematica package, we arrive at the following expression:

$$\begin{aligned}\chi_6(z)\phi_h(w) \sim & \frac{h(2h-1)(16h-1)\phi_h(w)}{32(z-w)^6} + \frac{2(2h-1)(16h-1)\partial\phi_h(w)}{32(z-w)^5} \\ & + \frac{3h(-17+8h) : T(w)\phi_h(w) :}{8(z-w)^4} + \cdots ,\end{aligned} \tag{7.10}$$

which vanishes when the conformal dimension h equals 0, $\frac{1}{2}$ or $\frac{1}{16}$: The first term is identically 0. The other terms vanish when we take the other null fields into account, e.g., the second term vanishes manifestly for $h = \frac{1}{2}$ or $\frac{1}{16}$. For $h = 0$, the primary is $\phi_0 = 1$, which is the only field whose conformal dimension is 0, so $\partial\phi_0 = 0$ and the term vanishes.

We call $\phi_{\frac{1}{2}}(z)$ and $\phi_{\frac{1}{16}}(z)$ the energy field $\epsilon(z)$ and the spin field $\sigma(z)$, respectively. Recall that a free fermion also has a conformal dimension of $\frac{1}{2}$, this motivates us to identify the energy field with the free fermion by comparing the OPE of $\epsilon(z)\epsilon(w)$ with that of $\psi(z)\psi(w)$.

Consider a level-2 null state of the form:

$$|\chi_2\rangle = (L_{-1}^2 + \alpha L_{-2})|\epsilon\rangle, \tag{7.11}$$

which by eq. (4.7) corresponds to the field

$$\chi_2(z) = \partial^2\epsilon(z) + \alpha : T(z)\epsilon(z) : . \tag{7.12}$$

The constant α can be calculated using the orthogonality of $|\chi_2\rangle$ with any other state, in particular, the state $L_{-2}|0\rangle$:

$$\begin{aligned}0 &= \langle\epsilon|L_2(L_{-1}L_{-1} + \alpha L_{-2})|\epsilon\rangle \\ &= \langle\epsilon|[L_2, L_{-1}]\, L_{-1}|\epsilon\rangle + \langle\epsilon|L_{-1}L_2L_{-1}|\epsilon\rangle + \alpha\langle\epsilon|[L_2, L_{-2}]|\epsilon\rangle + \alpha\langle\epsilon|L_{-2}L_2|\epsilon\rangle && (7.13\text{a}) \\ &= \langle\epsilon|3L_1L_{-1}|\epsilon\rangle + \alpha\langle\epsilon|4L_0 + \frac{6}{24}|\epsilon\rangle && (7.13\text{b}) \\ &= 3 + \frac{9\alpha}{4}. && (7.13\text{c})\end{aligned}$$

In eq. (7.13a), since $\epsilon(z)$ is primary, $|\epsilon\rangle$ is annihilated by L_2 in the last term according to eq. (7.3); The L_n adjoint relation in eq. (3.33) implies that $\langle\epsilon|$ is annihilated by L_{-1} in the second term. The commutators in the same line are given by the Virasoro algebra with $c = \frac{1}{2}$ as we stated after eq. (7.2). Recall $\epsilon(z)$ has a conformal dimension of $\frac{1}{2}$, $L_0|\epsilon\rangle = \frac{1}{2}|\epsilon\rangle$ in eq. (7.13b). Eq. (7.13c) implies $\alpha = -\frac{4}{3}$, the field which corresponds to the level-2 null state is therefore given by

$$\chi_2(z) = \partial^2\epsilon(z) - \frac{4}{3} : T(z)\epsilon(z) : . \tag{7.14}$$

Since $|\chi_2\rangle$ is null, we expect the OPE of $\chi_2(z)$ with any other field to be zero, in particular, we consider the following correlator:

$$\langle\chi_2(z)\epsilon(z_1)\epsilon(z_2)\rangle = \partial_z^2\langle\epsilon(z)\epsilon(z_1)\epsilon(z_2)\rangle - \frac{4}{3}\langle : T(z)\epsilon(z) : \epsilon(z_1)\epsilon(z_2)\rangle = 0, \tag{7.15}$$

where we have applied eq. (7.14). To continue with the calculation, we will make use of the following result [4, 16], which is proved in Appendix 7.A:

$$\langle 0|\psi(z)\phi_1(z_1)\cdots\phi_n(z_n)|0\rangle = \sum_{i=1}^{n}\left[\frac{(r-1)h_i}{(z_i-z)^r} - \frac{\partial_{z_i}}{(z_i-z)^{r-1}}\right]\langle 0|\phi(z)\phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n)|0\rangle, \tag{7.16}$$

where $\phi_i(z_i)$ are Virasoro primaries and $|\psi\rangle = L_{-r}|\phi\rangle$, $r > 0$. This result, as well as $: T(z)\epsilon(z) : \longleftrightarrow L_{-2}|\epsilon\rangle$, allows us to compute the second correlator on the right-hand side of eq. (7.15):

$$\langle : T(z)\epsilon(z) : \epsilon(z_1)\epsilon(z_2)\rangle = \left[\frac{1/2}{(z_1-z)^2} - \frac{\partial_{z_1}}{z_1-z} + \frac{1/2}{(z_2-z)^2} - \frac{\partial_{z_2}}{z_2-z}\right]\langle\epsilon(z)\epsilon(z_1)\epsilon(z_2)\rangle, \tag{7.17}$$

where r in eq. (7.42) equals 2. Substituting eq. (7.17) into the vanishing correlator eq. (7.15), we obtain a partial differential equation of variables z, z_1 and z_2:

$$\left\{\partial_z^2 - \frac{4}{3}\left[\frac{1/2}{(z_1-z)^2} - \frac{\partial_{z_1}}{z_1-z} + \frac{1/2}{(z_2-z)^2} - \frac{\partial_{z_2}}{z_2-z}\right]\right\}\langle\epsilon(z)\epsilon(z_1)\epsilon(z_2)\rangle = 0. \tag{7.18}$$

Solving for $\langle\epsilon(z)\epsilon(z_1)\epsilon(z_2)\rangle$ is difficult since this PDE does not have a unique solution. Instead, we make use of the following result on 3-point functions [4, 16], which is generalised from the 2-point function case proved in Appendix 7.B:

$$\langle 0|\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)|0\rangle = \frac{C_{123}}{(z_1-z_2)^{h_1+h_2-h_3}(z_1-z_3)^{h_1-h_2+h_3}(z_2-z_3)^{-h_1+h_2+h_3}}, \tag{7.19}$$

where C_{123} is a constant. This means

$$\langle 0|\epsilon(z)\epsilon(z_1)\epsilon(z_2)|0\rangle = \frac{C_{\epsilon\epsilon\epsilon}}{(z-z_1)^{\frac{1}{2}}(z-z_2)^{\frac{1}{2}}(z_1-z_2)^{\frac{1}{2}}}. \tag{7.20}$$

Substituting into eq. (7.18) tells us if eq. (7.20) is a solution of the PDE; it is only a solution if $C_{\epsilon\epsilon\epsilon} = 0$. Therefore,

$$\langle 0|\epsilon(z)\epsilon(z_1)\epsilon(z_2)|0\rangle = 0. \tag{7.21}$$

Analogously, we find $|\chi_2'\rangle = (L_{-1}^2 - \frac{3}{4}L_{-2})|\sigma\rangle$ to be another null state of the Ising model with the corresponding field $\chi_2'(z) = \partial^2\sigma(z) - \frac{3}{4} : T(z)\sigma(z) :$. Following a similar calculation as those from eq. (7.15) to (7.20), we arrive at the following PDE:

$$\left\{\partial_z^2 - \frac{3}{4}\left[\frac{1/2}{(z_1-z)^2} - \frac{\partial_{z_1}}{z_1-z} + \frac{1/2}{(z_2-z)^2} - \frac{\partial_{z_2}}{z_2-z}\right]\right\}\langle\sigma(z)\epsilon(z_1)\epsilon(z_2)\rangle = 0, \tag{7.22}$$

where $\langle\sigma(z)\epsilon(z_1)\epsilon(z_2)\rangle$, by eq. (7.19), is given by

$$\langle 0|\sigma(z)\epsilon(z_1)\epsilon(z_2)|0\rangle = \frac{C_{\sigma\epsilon\epsilon}}{(z-z_1)^{\frac{1}{16}}(z-z_2)^{\frac{1}{16}}(z_1-z_2)^{\frac{15}{16}}}. \tag{7.23}$$

Again, eq. (7.23) solves eq. (7.22) only when $C_{\sigma\epsilon\epsilon} = 0$, and

$$\langle 0|\sigma(z)\epsilon(z_1)\epsilon(z_2)|0\rangle = 0. \tag{7.24}$$

We now apply results eq. (7.21) and (7.24) to calculate the OPE of two energy fields, whose general form is given by

$$\epsilon(z)\epsilon(w) \sim \frac{\mathbb{1}(w)}{z-w} + \frac{\beta_1\,\sigma(w)}{(z-w)^{15/16}} + \frac{\beta_2\,\epsilon(w)}{(z-w)^{1/2}} \tag{7.25}$$

where β_1 and β_2 are constants to be determined and the power of each pole is given by the conformal dimension of its associated field minus those of fields in the OPE [16]. For example, in the first term of eq. (7.25), the pole associated with the identity field has a power of $h_{\mathbb{1}} - h_\epsilon - h_\epsilon = 0 - \frac{1}{2} - \frac{1}{2} = -1$. We can now compute the correlators $\langle 0|\epsilon(z)\epsilon(z_1)\epsilon(z_2)|0\rangle$ and $\langle 0|\sigma(z)\epsilon(z_1)\epsilon(z_2)|0\rangle$ using this general form of OPE:

$$\begin{aligned}\langle 0|\epsilon(z)\epsilon(z_1)\epsilon(z_2)|0\rangle &\sim \langle 0|\frac{\epsilon(z)\mathbb{1}(z_2)}{z_1-z_2} + \frac{\epsilon(z)\beta_1\,\sigma(z_2)}{(z_1-z_2)^{15/16}} + \frac{\epsilon(z)\beta_2\,\epsilon(z_2)}{(z_1-z_2)^{1/2}}|0\rangle \\ &\sim \frac{\langle\epsilon(z)\rangle}{z_1-z_2} + \frac{\beta_1\langle\epsilon(z)\,\sigma(z_2)\rangle}{(z_1-z_2)^{15/16}} + \frac{\beta_2\langle\epsilon(z)\,\epsilon(z_2)\rangle}{(z_1-z_2)^{1/2}}, &(7.26\text{a}) \\ &\sim \frac{\beta_2\langle\epsilon(z)\,\epsilon(z_2)\rangle}{(z_1-z_2)^{1/2}} &(7.26\text{b})\end{aligned}$$

where the first term of eq. (7.26a) vanishes because $\langle\epsilon(z)\rangle = 0$ as we mentioned after eq. (4.31). The correlator in the second term of this line also vanishes because a 2-point correlation function has the following general form:

$$\langle 0|\phi_1(z_1)\phi_2(z_2)|0\rangle = \frac{C_{12}\delta_{h_1=h_2}}{(z_1-z_2)^{h_1+h_2}}, \tag{7.27}$$

which is proved Appendix 7.B. From eq. (7.21), we know this correlator must be zero, this requires $\beta_2 = 0$ in eq. (7.26b).

Similarly, computing the correlator $\langle 0|\sigma(z)\epsilon(z_1)\epsilon(z_2)|0\rangle$ using the general expansion of $\epsilon(z)\epsilon(w)$ in eq. (7.25) and requiring it to be zero by eq. (7.24), we find $\beta_1 = 0$. Substituting β_1 and β_2 into eq. (7.25), we finally obtain the OPE of two energy fields:

$$\epsilon(z)\epsilon(w) \sim \frac{1}{z-w}, \tag{7.28}$$

which is identical to that of a single free fermion fields as stated in eq. (5.27).

Up to this point, there has been enough information for us to identify the Ising model with a free fermion model: Both theories are described by the Virasoro algebra with a central charge of $\frac{1}{2}$. Both models contain fields of conformal dimensions 0, $\frac{1}{2}$ and $\frac{1}{16}$. The primary fields of conformal dimension 0 are the identity field in both theories. The energy field $\epsilon(z)$ of the Ising model has the same conformal dimension, OPE and therefore correlators as the free fermion field $\psi(z)$. These two fields cannot be distinguished by physical measurements, that is, they are identical. One should now be convinced that the spin field $\sigma(z)$ and the Ramond field $R(z)$, both of conformal dimension $\frac{1}{16}$, must be identified as well.

This identification allows us to write $\epsilon(z)$ and $\sigma(z)$ in terms of vertex operators just as we did with $\psi(z)$ and $R(z)$ in eq. (6.13) and eq. (6.64) :

$$b_{-1/2}|0_{NS}\rangle \longleftrightarrow \psi(z) \equiv \epsilon(z) = \frac{1}{\sqrt{2}}V_1(z) + \frac{1}{\sqrt{2}}V_{-1}(z) \tag{7.29}$$

$$|0_R\rangle \longleftrightarrow R(z) \equiv \sigma(z) = \frac{1}{\sqrt{2}}V_{\frac{1}{\sqrt{8}}}(z) + \frac{1}{\sqrt{2}}V_{\frac{-1}{\sqrt{8}}}(z), \tag{7.30}$$

where we have used $\psi_1(z)$ and $R_1(z)$ instead of the other fields by convention. The two sets of fields are physically equivalent, so they give the same results for OPEs and correlation functions, which we will now compute.

The correspondences in eq. (7.29) and eq. (7.30) allow us to compute OPEs involving $\sigma(z)$ using eq. (4.23):

$$\begin{aligned}
\epsilon(z)\sigma(w) &= \frac{1}{\sqrt{2}}\left(V_1(z) + V_{-1}(z)\right)\frac{1}{\sqrt{2}}\left(V_{\frac{1}{\sqrt{8}}}(w) + V_{-\frac{1}{\sqrt{8}}}(w)\right) \\
&= \frac{1}{2}\left(V_1(z)V_{\frac{1}{\sqrt{8}}}(w) + V_1(z)V_{-\frac{1}{\sqrt{8}}}(w) + V_{-1}(z)V_{\frac{1}{\sqrt{8}}}(w) + V_{-1}(z)V_{-\frac{1}{\sqrt{8}}}(w)\right) \\
&\sim \frac{1}{2(z-w)^{\frac{1}{\sqrt{8}}}}\left(V_{-\frac{1}{\sqrt{8}}+1}(w) + V_{\frac{1}{\sqrt{8}}-1}(w)\right),
\end{aligned} \tag{7.31}$$

$$\sigma(z)\sigma(w) \equiv R_1(z_1)R_1(z_2) \sim \frac{1}{(z_1-z_2)^{\frac{1}{8}}}. \tag{7.32}$$

where we have applied the result from eq. (6.66) for eq. (7.32).

From OPEs eq. (7.28), (7.31) and (7.32), we can compute the correlation functions involving primary fields $\epsilon(z)$ and $\sigma(z)$. (We ignore correlators involving the identity field $\mathbb{1}(z)$, because they are the same as if the $\mathbb{1}(z)$ were absent.) The 1-point function of all fields except the identity field is zero as we have mentioned many times. The 2-point functions are readily computed from the OPEs and eq. (4.42) to be

$$\langle\epsilon(z)\epsilon(w)\rangle = \frac{1}{z-w}, \tag{7.33}$$

$$\langle\epsilon(z)\sigma(w)\rangle = 0, \tag{7.34}$$

$$\langle\sigma(z)\sigma(w)\rangle = \frac{1}{(z_1-z_2)^{\frac{1}{8}}}. \tag{7.35}$$

The 3-point functions are given by

$$\langle\epsilon(z_1)\epsilon(z_2)\epsilon(z_3)\rangle \equiv \langle\psi_1(z_1)\psi_1(z_3)\psi_1(z_2)\rangle = 0, \tag{7.36}$$

$$\langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\rangle \equiv \langle R_1(z_1)R_1(z_2)R_1(z_3)\rangle = 0, \tag{7.37}$$

These two correlators are identified as those we have computed in eq. (6.55d) and (6.67). (Eq. (7.36) was calculated in eq. (7.21) as well.) The mixed correlators are given by

$$\begin{aligned}
\langle\epsilon(z_1)\sigma(z_2)\sigma(z_3)\rangle &= \frac{1}{2\sqrt{2}(z_1-z_2)^{\frac{1}{\sqrt{8}}}}\left\langle\left(V_{-\frac{1}{\sqrt{8}}+1}(z_2) + V_{\frac{1}{\sqrt{8}}-1}(z_2)\right)\left(V_{\frac{1}{\sqrt{8}}}(z_3) + V_{-\frac{1}{\sqrt{8}}}(z_3)\right)\right\rangle \\
&= \frac{1}{2\sqrt{2}(z_1-z_2)^{\frac{1}{\sqrt{8}}}}\left\{\left\langle V_{-\frac{1}{\sqrt{8}}+1}(z_2)V_{\frac{1}{\sqrt{8}}}(z_3)\right\rangle + \left\langle V_{-\frac{1}{\sqrt{8}}+1}(z_2)V_{-\frac{1}{\sqrt{8}}}(z_3)\right\rangle\right.
\end{aligned}$$

$$+\left\langle V_{\frac{1}{\sqrt{8}}-1}(z_2)V_{\frac{1}{\sqrt{8}}}(z_3)\right\rangle+\left\langle V_{\frac{1}{\sqrt{8}}-1}(z_2)V_{-\frac{1}{\sqrt{8}}}(z_3)\right\rangle\Bigg\} \tag{7.38a}$$

$$=0, \tag{7.38b}$$

where each term of eq. (7.38b) vanishes because none of them satisfies the neutrality condition. Similarly, the last 3-point function also vanish:

$$\langle\epsilon(z_1)\epsilon(z_2)\sigma(z_3)\rangle=0, \tag{7.39}$$

which was shown in eq. (7.24).

There are five different 4-point functions in total, we will show results for two of them using the correspondence:

$$\begin{aligned}\langle\epsilon(z_1)\epsilon(z_2)\epsilon(z_3)\epsilon(z_4)\rangle &\equiv \langle\psi_1(z_1)\psi_1(z_2)\psi_1(z_3)\psi_1(z_4)\rangle\\ &=\frac{1}{2}\left(\frac{z_{12}\,z_{34}}{z_{13}\,z_{14}\,z_{23}\,z_{24}}+\frac{z_{13}\,z_{24}}{z_{12}\,z_{13}\,z_{23}\,z_{34}}+\frac{z_{14}\,z_{23}}{z_{12}\,z_{13}\,z_{24}\,z_{34}}\right)\end{aligned} \tag{7.40}$$

$$\begin{aligned}\langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle &\equiv \langle R_1(z_1)R_1(z_2)R_1(z_3)R_1(z_4),\rangle\\ &=\frac{1}{2}\left\{\left(\frac{z_{12}\,z_{34}}{z_{13}\,z_{14}\,z_{23}\,z_{24}}\right)^{\frac{1}{8}}+\left(\frac{z_{13}\,z_{24}}{z_{12}\,z_{13}\,z_{23}\,z_{34}}\right)^{\frac{1}{8}}+\left(\frac{z_{14}\,z_{23}}{z_{12}\,z_{13}\,z_{24}\,z_{34}}\right)^{\frac{1}{8}}\right\},\end{aligned} \tag{7.41}$$

which have been calculated in eq. (6.56) and (6.68), respectively.

7.A A Proof of Eq. (7.42)

In this appendix, we prove the result in eq. (7.42) which has been used in computing the 3-point correlation functions of the Ising model primaries. It tells us how to compute a correlation function which involves a secondary field:

$$\langle 0|\psi(z)\phi_1(z_1)\cdots\phi_n(z_n)|0\rangle = \sum_{i=1}^{n}\left[\frac{(r-1)h_i}{(z_i-z)^r} - \frac{\partial_{z_i}}{(z_i-z)^{r-1}}\right]\langle 0|\phi(z)\phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n)|0\rangle, \tag{7.42}$$

where $\phi_i(z_i)$ are Virasoro primaries and

$$|\psi\rangle = L_{-r}|\phi\rangle, \quad r > 0. \tag{7.43}$$

In chapter 3, we have defined the relations between $T(z)$ and L_n as

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \quad \Longrightarrow \quad L_n = \oint_0 T(z) z^{n+1}\frac{dz}{2\pi i}. \tag{7.44}$$

Applying this result, eq. (7.43) in terms of fields becomes

$$\psi(z) = \oint_z T(w)\psi(z)(w-z)^{-r+1}\frac{dw}{2\pi i}. \tag{7.45}$$

We substitute this into the left-hand side of eq. (7.42), which then becomes

$$\oint_z \frac{\langle 0|T(w)\phi(z)\phi_1(z_1)\cdots\phi_n(z_n)|0\rangle}{(w-z)^{r-1}}\frac{dw}{2\pi i}. \tag{7.46}$$

The w-contour about z can be replaced by a huge contour which encircles not only w but all the other z_i as well. These points are also poles of the correlator in the integrand, so we need to subtract their contributions:

$$\oint_z = \oint_\infty - \sum_{i=1}^{n}\oint_{z_i}, \tag{7.47}$$

where we have denoted the huge contour as a contour around ∞. This huge contour will only contribute if the integrand has a pole at ∞. However, using the Fourier expansion of $\partial\varphi$, one can easily show that the integrand tends to 0 as $w \to \infty$. Therefore, the contour around ∞ does not contribute, i.e.,

$$\oint_z = -\sum_{i=1}^{n}\oint_{z_i}. \tag{7.48}$$

Substituting this into eq. (7.46) and computing the OPE of $T(z)$ with $\phi_i(z_i)$ in the integrand using eq. (4.27) gives

$$\langle 0|\psi(z)\phi_1(z_1)\cdots\phi_n(z_n)|0\rangle = -\sum_{i=1}^{n}\oint_{z_i}\left\{\frac{h_i\langle\phi(z)\phi_1(z_1)\cdots\phi_n(z_n)\rangle}{(w-z)^{r-1}(w-z_i)^2} + \frac{\partial_i\langle\phi(w)\phi_1(z_1)\cdots\phi_n(z_n)\rangle}{(w-z)^{r-1}(w-z_i)}\right\}\frac{dw}{2\pi i} \tag{7.49}$$

Both terms of its right-hand side can be computed using Cauchy's theorem [24], and the equation becomes

$$
\begin{aligned}
\langle 0|\psi(z)\phi_1(z_1)\cdots\phi_n(z_n)|0\rangle &= -\sum_{i=1}^{n} \lim_{z\to z_i} \Big\{ \frac{d}{dz}\Big[\frac{h_i\langle 0|\phi(z)\phi_1(z_1)\cdots\phi_n(z_n)|0\rangle}{(z-w)^{r-1}}\Big] \\
&\qquad + \frac{\partial_{z_i}\langle 0|\phi(z)\phi_1(z_1)\cdots\phi_n(z_n)|0\rangle}{(z-w)^{r-1}}\Big\} \\
&= -\sum_{i=1}^{n} \Big\{ \frac{h_i(1-r)\langle 0|\phi(z)\phi_1(z_1)\cdots\phi_n(z_n)|0\rangle}{(z_i-w)^{r}} \\
&\qquad + \frac{\partial_{z_i}\langle 0|\phi(z)\phi_1(z_1)\cdots\phi_n(z_n)|0\rangle}{(z_i-w)^{r-1}}\Big\} \\
&= \sum_{i=1}^{n} \left[\frac{(r-1)h_i}{(z_i-z)^{r}} - \frac{\partial_{z_i}}{(z_i-z)^{r-1}}\right] \langle 0|\phi(z)\phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n)|0\rangle.
\end{aligned}
\tag{7.50}
$$

Eq. (7.42) is proved.

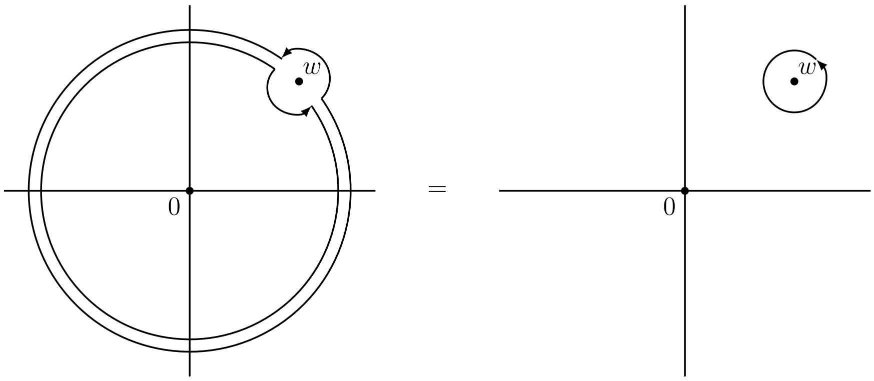

Figure 7.4: The contour subtraction which leads to eq. (7.52b). This picture is taken from [4].

7.B A proof of Eq. (7.27)

This appendix proves eq. (7.27) which we have used to simplify the 3-point functions of the Ising model. It states that

$$\langle 0|\phi_1(z_1)\phi_2(z_2)|0\rangle = \frac{C_{12}\delta_{h_1=h_2}}{(z_1-z_2)^{h_1+h_2}}. \tag{7.51}$$

Let us start with considering the commutation relation between a Virasoro operator L_m and a primary field $\phi(w)$:

$$\begin{aligned}[][L_m, \phi(w)] &= L_m\phi(w) - \phi(w)L_m \\ &= \oint_{0,|z|>|w|} T(z)\phi(w)z^{m+1}\frac{dz}{2\pi i} - \oint_{0,|z|<|w|} \phi(w)T(z)z^{m+1}\frac{dz}{2\pi i} && (7.52\text{a}) \\ &= \oint_w T(z)\phi(w)\frac{dz}{2\pi i}, && (7.52\text{b})\end{aligned}$$

where in line (7.52a), we have applied eq. (7.44) to write L_m in terms of tensor field $T(z)$. Note that the products of the two fields $T(z)$ and $\phi(w)$ are radially ordered, this means that $|z| > |w|$ for the first term and $|z| < |w|$ for the second. Combining these two contours about the origin gives us a single contour about w, this is illustrated in Figure 7.4.

Applying OPE in eq. (4.27) to eq. (7.52b) gives

$$[L_m, \phi(w)] = \oint_w \frac{h\phi(w)}{(z-w)^2}z^{m+1}\frac{dz}{2\pi i} + \oint_w \frac{\partial\phi(w)}{z-w}z^{m+1}\frac{dz}{2\pi i}, \tag{7.53}$$

where h is the conformal dimension of the primary field $\phi(z)$. This equation can then be computed using Cauchy's theorem [24] to be

$$\begin{aligned}[][L_m, \phi(w)] &= \phi(w)\lim_{z\to w}\left\{h\frac{d}{dz}(z^{m+1}) + \partial\phi(w)z^{m+1}\right\} \\ &= h(m+1)w^m\phi(w) + w^{m+1}\partial\phi(w). && (7.54)\end{aligned}$$

This commutation relation imposes constraints on the n-point correlation functions. When

$m = -1$, eq. (7.54) becomes

$$[L_{-1}, \phi(w)] = \partial\phi(w). \tag{7.55}$$

Now consider the commutator $\langle 0|L_{-1}\phi_1(z_1)\cdots\phi_n(z_n)|0\rangle$, which vanishes because $\langle 0|L_{-1} = (L_1|0\rangle)^\dagger = 0$. However, if we try to move the L_{-1} in the correlator to the right repetitively using the commutation relation eq. (7.55), we arrive at

$$\begin{aligned}
0 &= \langle 0|L_{-1}\phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n)|0\rangle \\
&= \langle 0|\phi_1(z_1)L_{-1}\phi_2(z_2)\cdots\phi_n(z_n)|0\rangle + \langle 0|\partial\phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n)|0\rangle \\
&= \langle 0|\phi_1(z_1)\phi_2(z_2)L_{-1}\cdots\phi_n(z_n)|0\rangle + \langle 0|\phi_1(z_1)\partial\phi_2(z_2)\cdots\phi_n(z_n)|0\rangle \\
&\quad + \langle 0|\partial\phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n)|0\rangle.
\end{aligned} \tag{7.56}$$

One can observe that every time we swap L_{-1} with an operator $\phi_i(z_i)$ on its right, we gain an extra term of $\langle 0|\cdots\partial\phi_i(z_i)\cdots|0\rangle$. Keep swapping until L_{-1} reaches the right most position and annihilates $|0\rangle$ (by eq. (4.3)) gives us the following constraint on an n-point function

$$0 = \sum_{j=1}^{n}\partial_j\langle 0|\phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n)|0\rangle. \tag{7.57}$$

Looking back at eq. (7.54), repeating the above calculation with $m = 0$ and $m = 1$ gives us the following constraints

$$\sum_{j=1}^{n}(z_j\partial_j + h_j)\langle 0|\phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n)|0\rangle = 0, \tag{7.58}$$

$$\sum_{j=1}^{n}(z_j^2\partial_j + 2h_jz_j)\langle 0|\phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n)|0\rangle = 0. \tag{7.59}$$

We now study the effect of the above three constraints eq. (7.57), (7.58) and (7.59) on a general 2-point function of primary fields $\langle 0|\phi_1(z_1)\phi_1(z_1)|0\rangle$. First define $z = z_1 + z_2$ and $z_{12} = z_1 - z_2$, eq. (7.57) implies the correlator is a function of z_{12} only, that is, $\langle 0|\phi_1(z_1)\phi_1(z_1)|0\rangle = f(z_{12})$. Eq. (7.58) tells us

$$(z_{12}\partial_{z_{12}} + h_1 + h_2)f(z_{12}) = 0, \tag{7.60}$$

which integrates to

$$f(z_{12}) = \frac{C_{12}}{z_{12}^{h_1+h_2}}, \tag{7.61}$$

where C_{12} is an arbitrary constant. At last we check the last constraint eq. (7.59) on the correlator, which says

$$[zz_{12}\partial_{12} + (h_1 + h_2)z + (h_1 - h_2)z_{12}]f(z_{12}) = 0. \tag{7.62}$$

Substituting the general solution for $f(z_{12})$ in eq. (7.61) into eq. (7.62) gives us

$$(h_1 - h_2)\frac{C_{12}}{z_{12}^{h_1+h_2-1}} = 0, \tag{7.63}$$

which holds for either $h_1 = h_2$ or $C_{12} = 0$. Combining eq. (7.61) and (7.63), we have shown that

$$\langle 0|\phi_1(z_1)\phi_2(z_2)|0\rangle = \frac{C_{12}\delta_{h_1=h_2}}{(z_1-z_2)^{h_1+h_2}}. \tag{7.64}$$

Eq. (7.27) is proved.

Now repeating the same calculation with a general 3-point correlation function $\langle 0|\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)|0\rangle$, we find

$$\langle 0|\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)|0\rangle = \frac{C_{123}}{(z_1-z_2)^{h_1+h_2-h_3}(z_1-z_3)^{h_1-h_2+h_3}(z_2-z_3)^{-h_1+h_2+h_3}}. \tag{7.65}$$

This will be used in computing the 3-point correlator of the Ising primaries in eq. (7.19).

Chapter 8

Conclusion

In this thesis, we have presented a detailed introduction to the basic principles of conformal field theory. This was illustrated by two of the most fundamental models of the CFT – the free boson and the free fermion theories. We have demonstrated the boson-fermion correspondence by associating a pair of free fermions to a single boson and conversely, writing the free fermions in terms of vertex operators, which are functions of the free bosonic field.

We first defined a conformal transformation as one which leaves its metric invariant up to a positive scale factor. Such transformations are angle-preserving which is where the name 'conformal' comes from. We found conformal transformations are special in two-dimensions, where exact solutions can be computed without approximation methods. We broke the conformal algebra into two identical but independent copies — the holomorphic and the anti-holomorphic algebras. In both algebras, we studied the generators for local and global conformal transformations and their commutation relations.

We then proceeded to the study of the free boson, which describes a scalar field defined on a cylinder. To study the field on the complex plane on which the CFT is defined, we applied Wick rotation and changed variables from time and space to z and $\bar{z}$ using a complex exponential. The classical field was then canonically quantised so we could describe it using quantum operators and their commutation relations. We compactified the free boson such that its momentum is discrete as in quantum mechanics. We then introduced the Virasoro operators as a basis which spans the conformal algebra (Virasoro algebra), from which we concluded that the quantum operators must be normally ordered in order to make sense in the quantum picture.

Following the free boson model, we then investigated a special form of algebra in the bosonic theory – the vertex algebra. The vertex algebra consists of fields which are associated with the vacuum quantum states by the state-field correspondence. We then defined radial ordering and the operator product expansions (OPEs) to compute the product of fields with different arguments. The physically measurable quantities of the CFT are defined by its correlation functions. At the end of chapter 4, we presented a formula for calculating the n-point correlation functions of the vertex operators which was used to solve the free fermion and the Ising model in later chapters.

The free fermion model was given a comparable treatment to that of a free boson. A free fermion is described by two spinor fields in the $\mathfrak{su}(2)$ representation. It admits two possible boundary conditions in the space direction, which divides all fermionic states into those of the Neveu-Schwarz sector and those of the Ramond sector. The two sectors were compared and contrasted. We concluded that the conformal algebra for a free fermion is again the Virasoro algebra with a central charge of $\frac{1}{2}$.

By matching up the conformal dimensions and the central charge of the two free theories, we found that a free boson is equivalent to the tensor product of a pair of free fermions. This process is called the fermionization of a free boson, and its reverse relation, the bosonization of free fermions, involves writing each fermion in terms of vertex operators. We confirmed the boson-fermion correspondence by showing the equivalence in their energy-momentum tensors and their partition functions. This correspondence allows us to compute the correlation functions of Ramond fields, which would otherwise be very difficult.

As an application of this boson-fermion correspondence, we finished the thesis with a study of the Ising model at criticality. We first noticed that the Ising model has the same central charge as a free fermion theory. We searched for the null states of the Ising model and used them to compute its primary fields and their conformal dimensions. We found that both theories contain an identity primary field, and we were able to identify the energy primary field of the Ising model with the free fermion field. The spin primary field in the Ising model has the same conformal dimension as that of the Ramond field. We therefore concluded that the Ising model can be identified as the free fermion model.

The boson-fermion correspondence is a remarkably powerful tool that allows us to transfer computations about one theory into another where they are easier. A wide range of experiments [40, 41, 42] have been designed and carried out in order to verify results such as those calculated from the Ising model correlation functions.

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