D1 — PROJECT DESCRIPTION

Project Title

Logarithmic conformal field theory, relaxed highest-weight modules and the 4D/2D correspondence

Investigator/Capability

Research opportunity and performance evidence. I have worked continuously in a research environment since obtaining my PhD at the end of 2005, holding three highly competitive international postdoctoral fellowships, one ARC fellowship and a continuing position at the University of Melbourne. Since returning to Australia in late 2010, I have been the lead CI on successful ARC DP grants worth over $1,000,000 and have built a strong research group of postdoctoral fellows and PhD/master's students. In this time, I have edited one collection of papers, published 27 refereed journal articles and contributed 3 refereed proceedings articles.

My research straddles the boundary between mathematical physics and pure mathematics. Broadly speaking, I specialise in the study of the representation theory of vertex operator algebras (VOAs), these being mathematical structures that axiomatise the symmetries of conformal field theories (CFTs). These days, this is widely regarded as an extremely fertile but technically demanding field. Publications in this area therefore tend to be long (35–40 pages is typical) and citation numbers are closer in line with those of pure mathematics than those of high-energy theoretical physics, say.

Nevertheless, I contend that my publication record is extremely strong. Including arXiv preprints, I have written 44 articles to date, totalling over 1,600 pages, at a rate of 3–4 each year (this is quite reasonable for mathematical physicists and would be considered very productive for a pure mathematician). I have moreover always published in journals of the highest quality, for example Adv. Math., Adv. Theor. Math. Phys., Comm. Math. Phys., Lett. Math. Phys., Nucl. Phys. B and Transform. Groups. Of my articles, 17 are directly relevant to the research being proposed here and several are foundational in the sense that they provided the breakthroughs that have allowed this research to be proposed. The latter are described in detail in F20.

Aside from always publishing in top journals, my research performance may also be judged through citation numbers. Whilst these numbers have remained consistently strong over the last five years, they have increased by around 60% since 2017 due to the increasing relevance of my work to high-energy physicists studying the 4D/2D correspondence. In particular, my breakthrough article [1] on the modular properties of fractional-level $\mathfrak{sl}_2$ models currently has 70 citations whilst my review [2] on logarithmic CFT (LCFT) has 68. Both are catching up fast with my first "logarithmic" article [3] (86 citations) which is known as one of the foundational papers from the mid-2000s that ushered in the resurgence of interest in logarithmic theories. My h-index is currently 20.

My more recent articles are also attracting a significant amount of attention, particularly from high-energy physicists. A three-year old paper [4] on rigorous conclusions that may be drawn for quite general Heisenberg cosets has already attracted 31 citations, whilst one of my papers [5] from last year on rank-1 relaxed modules now has 11. A sequel [6], placed on the arXiv in June this year and currently under review at Adv. Math., develops the technical tools needed to generalise to all ranks and so opens the door to a rigorous complete study of many of the examples that have recently arisen in the 4D/2D correspondence of [7].

Research training, mentoring and supervision. Since returning to Australia, I have been fortunate enough to formally supervise and mentor three outstanding postdoctoral fellows: Simon Wood (PhD ETH Zürich), Kazuya Kawasetsu (PhD Tokyo) and Shashank Kanade (PhD Rutgers). They have gone on to secure continuing positions overseas in Cardiff, Kumamoto and Denver, respectively. Whilst the quality of postdocs in my field tends to be extremely high, I think that it is also fair to say that there is strong demand to join my research group. Here, postdocs are not only immersed in my research methodologies and directions, but they are also initiated into academia by taking on a co-supervisory role through collaborations and regular meetings with (carefully chosen) PhD students. Although their research strengths were obviously the decisive factor in securing their positions, I believe that this mentoring helped my former postdocs to distinguish themselves as mature academics.

Thus far, I have successfully guided three PhD students, as principal supervisor, to completion, one of whom (Tianshu Liu) is presently a postdoc at the Yukawa Institute, Kyoto. Each has produced a high-quality thesis with two substantial publications. This is quite satisfactory given the vast amount of background material that students still need to internalise upon commencing doctoral research in my field. Indeed, the "paper-writing machine" stereotype for PhD students in more experimental disciplines does not apply here. I was also an "unofficial" supervisor of a recently completed PhD student at the ANU and am also currently principal supervisor to one PhD at Melbourne and cosupervisor to one PhD at UQ, both of whom I am currently writing papers with. I have also supervised two MSc students and three

Honours students to completion and have one MSc student presently. Several of my students have gone on to further study at prestigious institutions including Cambridge, Northwestern, Oxford and Texas–Austin.

An additional component of the research training that I provide to students and postdocs (and colleagues for that matter) is the fact that I run regular seminar series, when my teaching load allows, in which they are encouraged to actively participate. Since arriving at Melbourne, I have organised seminars on W-algebras, advanced CFT, VOA-module tensor products (fusion), and string theory for quite diverse groups, all of which were received warmly. I have also had opportunity to formally mentor junior colleagues here in small ways, acting as a grant-writing mentor for Marcy Robertson and Ting Xue, as well as arranging to cosupervise a master's student with Yaping Yang (her first).

Finally, I have also supervised twelve summer students in the last nine years, guiding them through research projects in Lie theory, conformal field theory and string theory. Two more will start this summer. Again, many of these students could be regarded as the *crème de la crème* of mathematics and mathematical physics as evidenced by their subsequent PhD studies as Rhodes Scholars (two!) at Oxford, as well as at Chicago, Harvard and Imperial College London.

Leadership and international research standing. In the LCFT and VOA communities, I think it is fair to say that I am already regarded internationally as one of the leaders. This is especially so for the specialised topic of fractional-level WZW models and their representations, which form the main focus of this proposal.

My national and international standing in these communities is exemplified through consistently high-quality publications, invitations to speak/lecture and a special issue editorship. In 2012, I was commissioned by the editors of the J. Phys. A to write a review on the topic of LCFT and asked if I would consider organising and curating a collection of invited reviews and original research papers on this topic to form a special issue. The result, Vol. 49 No. 40 (approximately 600 pages), is still regarded as the authoritative source for information in this field, though of course things have advanced significantly in the last six years. The preface of this special issue has already been cited 29 times and my own review has 68 citations to date.

I have been an invited speaker at many international conferences devoted to LCFT including the first such workshop which was held at the ETH Zürich in 2009 while I was still a postdoc. Subsequently, it was my pleasure to give the opening lectures of the Institut Henri Poincaré trimestre on CFT in 2011 and speak at an embedded conference. Since then, I have given invited seminars at leading international centres such as RIMS (Kyoto) and the Hausdorff and Erwin Schrödinger Institutes, as well as at the 2018 International Congress on Mathematical Physics in Montréal. At the time, I was also a CRM–Simons Senior Professor at the Centre de Recherches Mathématiques, Montréal.

More recently, I have been developing tools to classify representations of the VOAs that underlie certain classes of LCFTs. This has attracted significant interest from the VOA community with the result being that I am now invited to practically all of their meetings as well. My reputation is such that I am now attracting a steady of stream of top international talent to Australia which benefits both our mathematical physics and representation theory communities. For example, I have hosted Tomoyuki Arakawa, Jürgen Fuchs, Matthias Gaberdiel, Geoff Mason, Dmitri Nikshych, Eric Rowell, Ingo Runkel, Hubert Saleur and Luc Vinet in the last five years and will host Chris Beem, Yi-Zhi Huang, Christoph Schweigert and Katrin Wendland in 2020.

I am also active in organising workshops myself, being responsible for international conferences on CFTs and VOAs at the ANU in 2015 and 2020, at the Centre de Recherches Mathematiques, Montréal, in 2018, and at the new Banff International Research Station, Oaxaca, in 2018. This is complemented by having given invited lectures on conformal field theory for the 2014 AMSI Summer School, at the Chinese Academy of Sciences, Wuhan, (2014 and 2019) and at two MATRIX meetings in 2017. I intend to organise another MATRIX meeting during the term of this fellowship.

Collaboration capability. A casual glance at my publication list in F22 will confirm that I collaborate widely. Almost all of my publications since leaving the postdoc circuit in 2010 are joint, although the vast majority are projects initiated and carried out (or guided in the case of student collaborators) by myself. At an international level, my most frequent collaborators are Thomas Creutzig (Canada), whom I regard as a member of my extended CFT/VOA research group, and my former postdocs Simon Wood (Wales), Shashank Kanade (USA) and Kazuya Kawasetsu (Japan). I would expect to visit and/or invite each of them during the course of the Fellowship.

Outside of the CFT and VOA communities, I have also collaborated with statistical physicists working on loop models, namely Alexi Morin-Duchesne (Belgium), and probabilists working on Schramm-Loewner evolution, namely Kalle Kytölä (Finland). At present, I also have international collaborations approaching fruition with the leading VOA-theorists Dražen Adamović (Croatia) and Cuipo Jiang (China). At a domestic level, I have written two articles already with Jørgen Rasmussen (UQ) at the confluence of statistical physics and VOAs; we will be preparing another with our joint PhD student in the near future. As regards domestic pure mathematics, I have been working with Kevin Coulembier (Sydney) on relaxed module categories and have ongoing collaborative discussions with Dan Murfet and Arun Ram (both at Melbourne).

It is in fact probably true that I have too many collaborations, but there are too many fascinating research directions to explore. In any case, I think it is clear that I am more than capable of building national and international collaborative networks.

Background and Motivation. A conformal field theory (CFT) is a quantum field theory that is invariant under coordinate changes that preserve angles. In two dimensions, the Lie algebra of infinitesimal angle-preserving (conformal) transformations turns out to be (two commuting copies of) the infinite-dimensional *Virasoro algebra*. This remarkable fact means that the physically measurable quantities in many 2D CFTs may be computed exactly, at least in principle, without having to resort to perturbative series expansions or mathematically ill-defined tools (*eg*. path integrals).

The naissance of 2D CFT as an independent discipline is generally regarded as the seminal 1984 paper of Belavin, Polyakov and Zamolodchikov [8]; the field has been extremely active ever since. Historically, interest in 2D CFT was necessitated by applications to statistical mechanics and string theory. These applications, as well as their outgrowths such as the celebrated AdS/CFT correspondence [9] between 10D superstrings on $AdS_5 \times S^5$ and a 4D CFT on the AdS_5 boundary, have cemented CFT's place at the core of modern mathematical physics.

Mathematicians too are fans of 2D CFT. Not only are they rare examples of non-trivial quantum field theories that admit rigorous axiomatisations, their intricate and beautiful mathematical structures have led to the creation of new mathematics, *eg*. modular tensor categories [10, 11], and natural connections between seemingly disparate existing mathematics, *eg*. between modular forms and sporadic groups [12–14]. More broadly, CFT is also behind many of the recent mathematical advances in the study of infinite-dimensional Lie algebras and their representation theories, conformally invariant random processes (Schramm-Loewner evolutions in particular), combinatorial q-series identities, knot theory, and the geometric Langlands program [15].

Recently, physicists have started to explore higher-dimensional CFTs including generalisations of the 4D gauge theories that describe the standard model of particle physics. Being finite-dimensional, the 4D conformal symmetry algebra is not nearly as helpful in analysing these theories as its 2D cousin. However, a lot of interesting progress has been made with supersymmetric gauge theories, in particular when the theory has $N = 2$ supersymmetry, because then it seems that many features of the 4D models have interpretations in terms of certain 2D CFTs!

The most famous of these interpretations is the conjecture of Alday, Gaiotto and Tachikawa [16] in which the Nekrasov partition function [17] of the 4D gauge theory is equated with a correlation function of the 2D Liouville CFT [18]. Fascinating as it is, this is not the 4D/2D correspondence that will be studied here. Instead, we recall the recent discovery of Beem *et al*. [7] that relates a quite general 4D $N = 2$ super-CFT to a 2D CFT in such a way that the correlators of the fields in a certain protected sector of the former are precisely the correlators of the latter. Moreover, there are several invariants of the geometric structure of the 4D theory, *Schur indices* and *Higgs branches* to name two, that can also be elegantly expressed in the terms of 2D CFT data [19].

This is therefore a very general type of 4D/2D correspondence. More interestingly, the types of 2D CFTs that arise are believed [19] to possess a measure of finiteness — technically they are said to be *quasi-lisse* [20] — that gives one hope that they may be completely analysed. This stands in contrast to CFTs like Liouville theory which do not have this finiteness property and are still very far from being completely understood despite decades of intensive attention.

Even more interestingly, in almost all of the many examples of the 4D/2D correspondence of Beem *et al*. in which the 2D CFT has been identified, it has turned out to be a logarithmic CFT (LCFT). The qualifier here comes from the observation that certain correlators in these theories have logarithmic singularities, in contrast to the power law singularities familiar from the *rational* CFTs usually encountered in textbooks. The origin of these logarithms is mathematically interesting: they arise because the chiral symmetry algebra of the CFT, also known as a *vertex operator algebra* (VOA), possesses reducible but indecomposable representations on which the Virasoro zero-mode acts non-diagonalisably [21, 22]. In other words, the relevant category of VOA-modules is *non-semisimple*.

LCFT therefore straddles the border between pure mathematics and mathematical physics, contributing to both. Aside from their appearances in 4D/2D correspondences, they are also known to describe non-local observables in the scaling limit of many critical lattice models, for example percolation and polymers [3, 23, 24], and form an integral part of our understanding of string theories on supermanifolds [25]. Moreover, LCFTs are believed to arise as duals of three-dimensional chiral gravity models [26] and control certain transitions in the quantum Hall effect [27]. On the mathematical side, there has been a huge amount of recent activity devoted to generalising the successes of rational CFT, *eg*. modular tensor categories, to the logarithmic setting [2, 28–35]. Understanding LCFTs is therefore of crucial importance to advancing a wide variety of fields.

Families of logarithmic CFTs. Our main interest here is in certain families of LCFTs that include, or are closely related to, almost all of the examples of 4D/2D correspondences known to date. These are the *fractional-level WZW models* (WZW stands for "Wess–Zumino–Witten") and their associated *W-algebras*. The VOAs $\mathsf{L}_k(\mathfrak{g})$ of the former are parametrised by a choice of simple Lie algebra $\mathfrak{g}$ and a rational number k called the *level* [36, 37]. The W-algebra VOA $\mathsf{W}_k(\mathfrak{g}; f)$ is obtained from $\mathsf{L}_k(\mathfrak{g})$ via a cohomological construction called *quantum hamiltonian reduction* which depends, in addition, on a choice of nilpotent element $f \in \mathfrak{g}$ [38, 39] (taking $f = 0$ recovers $\mathsf{L}_k(\mathfrak{g})$). Fractional-level WZW models (with $k \notin \mathbb{Z}_{\geq 0}$) are always logarithmic, but the story for W-algebraic CFTs is more interesting: they are rational if f is principal [40], but are otherwise believed to be logarithmic for "most" k.

Unfortunately, our understanding of general fractional-level WZW models can only be described as poor. Even those with $\mathfrak{g} = \mathfrak{sl}_2$ were regarded as being "physically sick" [41] until quite recently because of a implicit assumption that they behaved like rational CFTs. More precisely, the assumption that the CFT's space of states could be built from the category of *highest-weight* $\mathsf{L}_k(\mathfrak{sl}_2)$-modules led early researchers to an absurdity: the celebrated Verlinde formula [42] led to *negative* fusion multiplicities [43]. The necessity of including non-highest-weight modules was established somewhat later [44, 45], but a resolution of the Verlinde formula issue for $\mathsf{L}_k(\mathfrak{sl}_2)$ only appeared relatively recently in work of Creutzig and myself [1, 46].

At the centre of this resolution is a type of generalised highest-weight $\mathsf{L}_k(\mathfrak{sl}_2)$-module that has come to be known as a *relaxed* highest-weight module [47]. Classified in [48, 49], they outnumber the highest-weight $\mathsf{L}_k(\mathfrak{sl}_2)$-modules by an uncountably infinite factor and so dominate the spectrum of the CFT. Interestingly, the relaxed highest-weight characters are naturally expressed [5] in terms of characters of its W-algebra, which happens to be a Virasoro minimal model, hence the modular transformations and fusion multiplicities of the fractional-level $\mathfrak{sl}_2$ WZW models are largely determined by those of its W-algebra. Moreover, the characters of the highest-weight $\mathsf{L}_k(\mathfrak{sl}_2)$-modules may be obtained formally in a completion of the space spanned by the relaxed characters, so their modularity and fusion behaviour are also largely determined by the W-algebra.

There are now at least two questions to ask: How does this relaxed highest-weight module paradigm generalise from $\mathfrak{sl}_2$ to higher rank algebras and how do the W-algebras, of which there are generally many, fit into this story? I will explain how I intend to answer these questions below. While answering these questions is obviously crucial to expanding knowledge in LCFT and related mathematical disciplines, they are also essential for deepening our understanding of the 4D/2D correspondence of Beem *et al.* In particular, physicists have recently begun using surface defects to access modified Schur invariants that relate directly to (2D) VOA modules [50]. However, they have thus far only found highest-weight modules in examples. A detailed understanding of their relaxed cousins on the 2D side will therefore provide a much-needed blueprint for physicists struggling to complete their defect categories on the 4D side.

Aims. With this language in place, the general aims of this proposal can now be stated.

Aim 1: **To give the first detailed studies of the higher-rank fractional-level WZW models and W-algebras that arise in the 4D/2D correspondence, including determining characters, modular transformations and fusion multiplicities.**

Aim 2: **To develop general methods for studying fractional-level WZW models and W-algebra CFTs, including super versions.**

Aim 3: **To establish, rigorously and conceptually, the precise relationship between relaxed highest-weight characters and W-algebras.**

I shall outline below how I intend to achieve these aims by describing several projects, each of which forms a significant research direction on its own. As a whole, this proposal is a natural successor to my Discovery Project DP160101520, in which the research foci included the fractional-level WZW models associated to $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{sl}(2|1)$. In line with the intensive nature of a Fellowship, the aims of the present proposal are far more wide-reaching (and challenging).

Project 1. The Deligne exceptional series. In [51], Deligne introduced the following inclusions of simple Lie algebras, now called the *Deligne exceptional series*:

$$\mathfrak{sl}_2 \subset \mathfrak{sl}_3 \subset \mathfrak{g}_2 \subset \mathfrak{so}_8 \subset \mathfrak{f}_4 \subset \mathfrak{e}_6 \subset \mathfrak{e}_7 \subset \mathfrak{e}_8. \tag{1}$$

He observed that the dimensions of certain representations of these Lie algebras could be expressed in an unexpectedly uniform fashion as rational functions of the dual Coxeter number $\mathsf{h}^\vee$. From the perspective of CFT, these Lie algebras recently arose in work of Kawasetsu [52] in which the minimal W-algebras $\mathsf{W}_k(\mathfrak{g}; f_\theta)$ with $\mathfrak{g}$ in (1) and $k = -\frac{\mathsf{h}^\vee}{6}$ were unexpectedly shown to be rational. At about the same time, Beem *et al.* found 4D $N = 2$ super-CFTs whose corresponding 2D CFTs were identified as the fractional-level WZW models $\mathsf{L}_k(\mathfrak{g})$ with $\mathfrak{g}$ in (1) and $k = -\frac{\mathsf{h}^\vee}{6} - 1$ [7].

This project will concentrate on the latter series of 2D CFTs, aiming to analyse as much of their representation theories as possible. For the first member of the series, $\mathsf{L}_{-4/3}(\mathfrak{sl}_2)$, my recent work means that much is already known including the classification of simple relaxed highest-weight modules [48, 49], characters [5], modularity [46], fusion [44, 46] and (conjectural) projective covers [53]. For $\mathsf{L}_{-3/2}(\mathfrak{sl}_3)$, we have only partial knowledge: the relaxed classification appears in [6, 54], whilst I am currently writing two papers [55, 56] that give the characters, modularity and (conjectural) fusion and projectives. For $\mathsf{L}_{-5/3}(\mathfrak{g}_2)$ and $\mathsf{L}_{-2}(\mathfrak{so}_8)$, I have recently completed the relaxed classification [6], but nothing else is known. In all other cases, only highest-weight classifications are known [57, 58].

Aside from the fact that these are surely the most accessible examples to compare with 4D physics results, and more importantly to inspire them to further develop their surface defect technology, the 2D Deligne series CFTs are excellent "practice models" for general fractional-level WZW models. This is because they have the rare property that their minimal W-algebras $\mathsf{W}_k(\mathfrak{g}; f_\theta)$ are the trivial 1-dimensional VOA $\mathbb{C}$ whilst their other W-algebras are 0 [57]. In light of Project 5 below, we therefore expect that the Deligne series CFTs are as simple to analyse as possible, being unmarred by W-algebraic complications. They therefore form a crucial stepping stone along the path to appreciating the full complexity of the fractional-level WZW models.

The Deligne series CFTs therefore make excellent, though challenging, candidates for PhD thesis projects. $\mathsf{L}_{-5/3}(\mathfrak{g}_2)$ in particular is quite accessible, given the trail we have already blazed for $\mathsf{L}_{-3/2}(\mathfrak{sl}_3)$, but still very interesting. Higher-rank examples would, however, require significant computational prowess because of the rapid increase in complexity of the relaxed highest-weight module classification. For example, $\mathsf{L}_{-2}(\mathfrak{so}_8)$ has 24 3-parameter families of relaxed modules which degenerate into 96 2-parameter families, 168 1-parameter families and 49 inequivalent highest-weight modules [6]. Nevertheless, there will be strong interest in these results from 4D physicists, vertex algebraists and representation theorists, so such projects will give the student an excellent start to their academic career.

Project 2. Admissible-level WZW models. The fractional-level WZW models specialise to the original rational WZW models of Witten [59] when the level k is a non-negative integer. These WZW models occupy a central place in rational CFT because almost all of the known rational CFTs can be constructed from them using standard techniques. It is therefore not unreasonable to suppose that something similar will be true for the remaining fractional-level WZW models and LCFT. Indeed, I noted evidence for this explicitly in the case of $\mathfrak{sl}_2$ models in [60]. From the point of view of expanding knowledge and research capacity, as well as for applications of LCFT in physics, it is therefore essential to develop mathematical tools for analysing general fractional-level WZW models.

A particularly attractive place to start this general analysis is by restricting to admissible levels. These were defined by Kac and Wakimoto [37] to be those k satisfying

$$k+\mathsf{h}^\vee = \frac{u}{v}, \quad \text{where } u \text{ and } v \text{ are positive coprime integers with } u \geq \begin{cases} \mathsf{h} & \text{if } \ell \mid v, \\ \mathsf{h}^\vee & \text{if } \ell \nmid v \end{cases}$$

and ℓ is the *lacety* of $\mathfrak{g}$: 1 for types A, D and E; 2 for types B, C and F; 3 for type G. The advantage of these levels is that the highest-weight modules have been classified by Arakawa [61] and their characters are known [37]. My recent results [5, 6] will then yield the relaxed classification and their characters for arbitrary admissible levels, in principle.

Of course, there will still be a huge amount of work to make this classification explicit. First, one has to solve the combinatorial problem of describing the various families of relaxed highest-weight modules and how they degenerate into one another. Next, one needs to understand how these families are related by automorphisms so that characters and modular transforms of the degenerations may be deduced. Finally, the role of the (in general many) W-algebras needs to be precisely determined in order to bring out the (expected) beauty of the final results (*cf.* Project 5). The goal is to describe everything in a Lie-theoretic framework and I expect that certain variants of the Weyl group will end up playing a pivotal role. I am currently working on this for $\mathfrak{sl}_3$ with collaborators Kawasetsu and Wood, along with my PhD student Zac Fehily. Extending this to $\mathfrak{sl}_n$ would already be a major achievement and would be a fine goal for the RA employed alongside this Fellowship.

A secondary goal is to try to "algebraise" the work that this is built on, namely Arakawa's highest-weight classification [61]. This breakthrough work is geometric in flavour, relying heavily on old work of Joseph [62] on primitive spectra and nilpotent orbits. One exciting possibility suggested by my recent work is there should be a relatively elementary proof of Arakawa's result which makes essential use of relaxed highest-weight modules. Such a proof would be extremely exciting because of the chance that it could be generalised to non-admissible levels (*cf.* Project 3), for which the analogue of Joseph's work is currently unknown, and to super-cases (*cf.* Project 4), for which the relevant supergeometric foundations are far less well understood.

Project 3. Non-admissible-level WZW models. Admissible level WZW models have the advantage of having a well developed highest-weight representation theory [37, 61]. In general however, there are many levels, called *fractional levels*, for which the representation theory of $\mathsf{L}_k(\mathfrak{g})$ is interesting. Moreover, the fractional levels coincide with the admissible levels only for $\mathfrak{g} = \mathfrak{sl}_2$. Indeed, the fractional levels are (ignoring the critical level $k = -\mathsf{h}^\vee$ for which there is no CFT) precisely those satisfying [63]

$$\ell(k+\mathsf{h}^\vee) = \frac{u}{v}, \quad \text{where } u \text{ and } v \text{ are positive coprime integers with } u \geq 2.$$

In particular, the levels of the physically important Deligne exceptional series CFTs $\mathsf{L}_k(\mathfrak{g})$, with $\mathfrak{g}$ in (1) and $k = -\frac{\mathsf{h}^\vee}{6} - 1$, are fractional but not admissible when $\operatorname{rank}\mathfrak{g} > 2$, so these examples have already attracted significant interest from physicists. For this reason, several research groups have been investigating non-admissible examples using brute-force methods [52, 64], which don't get very far, or geometric methods [57, 58], which can prove important, but somehow "coarse", properties of the representation theory. Nobody has yet been able to address the detailed structure of the representation theories of these examples.

One reason for this lack of progress is that even brute-force analyses at non-admissible levels require a deep appreciation of quite subtle mathematics, *eg*. singular parabolic affine Kazhdan–Lusztig theory [65] applied to the vacuum module. For the Deligne exceptional series CFTs with rank $\mathfrak{g} > 2$ that arise in the 4D/2D correspondence, this would seem to be the only way forward at present (see Project 1).

However, there are several other series of fractional but non-admissible examples for which there is another way. In particular, the $\mathsf{L}_{-1}(\mathfrak{sl}_n)$ with $n \geq 3$ form one such series because they may be realised [58] as a coset of n bosonic ghost systems by a free boson subalgebra. Given my work on the representation theory of bosonic ghosts [66] and free boson cosets [4], I am confident that a complete analysis of these examples can be achieved. Indeed, this would make an excellent PhD project for a sufficiently strong student. If time permits, other (negative integer) non-admissible levels could then be tackled using the $\mathfrak{sl}_n$ generalisation of the GKO coset construction [67].

Project 4. Generalisation to Lie superalgebras. A natural extension of the previous two projects is to generalise them to fractional-level WZW models with $\mathfrak{g}$ a (basic classical) simple Lie superalgebra. However, there are some highly non-trivial obstacles in place, the most serious of which is the fact that the highest-weight classification is as yet unknown (and will probably stay that way in the near future). For this reason, I propose to study certain low-rank examples using a method called the *inverse coset construction*. This method was introduced in [35] and was used in [53] to give the first complete analysis of the fractional-level WZW models with $\mathfrak{g} = \mathfrak{osp}(1|2)$. With my collaborator Creutzig, I am currently writing a sequel that inverts a new coset realisation to completely analyse, again for the first time, the admissible-level WZW models with $\mathfrak{g} = \mathfrak{sl}(2|1)$ [68].

In this project, Creutzig and I intend to apply the inverse coset construction to investigate the admissible-level WZW models with $\mathfrak{g} = \mathfrak{osp}(3|2)$, $\mathfrak{psl}(2|2)$ and $\mathfrak{d}(2|1;\alpha)$, tackling more general cases if time (and theory) permits. These are the most interesting for physical applications, *eg*. $\mathfrak{psl}(2|2)$ and $\mathfrak{d}(2|1;\alpha)$ play central roles in AdS_3/CFT$_2$ higher spin theory [69] and Mathieu moonshine [70]. Moreover, their minimal quantum hamiltonian reductions are the hugely important $N = 3$, $N = 4$ and big $N = 4$ superconformal algebras, respectively [39], whose non-unitary (and logarithmic) representation theories remain completely unexplored (see [71,72] for some unitary results). The results of this project will therefore be regarded as significant breakthroughs by both physicists and mathematicians.

Project 5. Relaxed highest-weight modules and W-algebras. Recall that there is a family of functors, collectively called quantum hamiltonian reduction and parametrised by a nilpotent element $f \in \mathfrak{g}$, taking representations of a fractional-level WZW model $\mathsf{L}_k(\mathfrak{g})$ to representations of the W-algebra $\mathsf{W}_k(\mathfrak{g};f)$. More precisely, $\mathsf{W}_k(\mathfrak{g};f)$ is determined up to isomorphism by the nilpotent orbit through f of the Lie group G (corresponding to $\mathfrak{g}$) acting by conjugation. These nilpotent orbits are important objects in much of representation theory [73] and form a combinatorially interesting poset under inclusion of their closures. For example, the poset for $\mathfrak{g} = \mathfrak{sl}_n$ is just the set of partitions of n equipped with the dominance ordering.

W-algebras are therefore also ordered according to this poset and there are indeed analogues of the hamiltonian reduction functors that map between W-algebras in the direction dictated by this ordering (in the case of *finite* W-algebras, these functors are referred to as *reduction by stages* [74]). A recent interesting observation of mine is that the relaxed highest-weight modules of $\mathsf{L}_k(\mathfrak{g})$ seem to know about reduction by stages: in several examples, I have shown that the *string functions* of the relaxed characters of a W-algebra are characters of a W-algebra further up the poset. For example, the string functions of the relaxed characters of $\mathsf{L}_k(\mathfrak{sl}_3)$ are characters of its minimal W-algebra, the Bershadsky-Polyakov algebra $W_3^{(2)}$, whilst the string functions of the relaxed $W_3^{(2)}$ characters are characters of the principal W-algebra W_3.

$[3]$	principal: $f = \begin{pmatrix} 0&0&0 \\ 1&0&0 \\ 0&1&0 \end{pmatrix}$	Zamolodchikov algebra W_3
$\downarrow$	$\downarrow$	$\downarrow$
$[2,1]$	minimal: $f = \begin{pmatrix} 0&0&0 \\ 1&0&0 \\ 0&0&0 \end{pmatrix}$	Bershadsky-Polyakov algebra $W_3^{(2)}$
$\downarrow$	$\downarrow$	$\downarrow$
$[1,1,1]$	zero: $f = \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}$	$\mathsf{L}_k(\mathfrak{sl}_3)$

Posets of partitions (left), nilpotent orbits (middle) and W-algebras (right) for $\mathfrak{sl}_3$.

Because the string functions of relaxed characters are constant, it follows that the modular properties and fusion rules of the WZW model will depend explicitly on the modular properties and fusion rules of an appropriate W-algebra. This suggests that a true understanding of the representation theory of $\mathsf{L}_k(\mathfrak{g})$ will involve simultaneously understanding the representation theories of some (possibly all) of its W-algebras! It also indicates that there should be a way of reconstructing the representation theories of $\mathsf{L}_k(\mathfrak{g})$ and its W-algebras from those further up the poset. In particular, we conjecture that it is possible to reconstruct the representation theory of any W-algebra, including $\mathsf{L}_k(\mathfrak{g})$, from that of the principal W-algebra at the top. Note that the principal W-algebra is rational when k is admissible [40].

This project aims to verify and/or refine this beautiful picture by investigating the representation theories of higher-rank W-algebras. In particular, the question of what happens to the string functions when the poset is not linearly ordered, *ie*. the diagram has branches, remains to be answered (branching occurs, *eg*. for $\mathfrak{g} = \mathfrak{sl}_6$, $\mathfrak{sp}_6$, $\mathfrak{so}_8$, *etc*).

I intend to do this by teaming up with my current collaborators Adamović and Kawasetsu to reconstruct admissible-level representation theories using a method we call *inverse hamiltonian reduction*. This was originally introduced by Semikhatov [75] for $\mathfrak{sl}_2$, but was not really utilised until Adamović explained [76] how to use it to construct relaxed highest-weight modules for $\mathfrak{sl}_2$ from representations of the Virasoro minimal models (the principal W-algebras of $\mathfrak{sl}_2$). Recently, the three of us have been generalising this to $\mathfrak{sl}_3$: we have constructed all relaxed $W_3^{(2)}$-modules from W_3-modules and are in the process of lifting this construction to relaxed $\mathsf{L}_k(\mathfrak{sl}_3)$-modules.

I am convinced that inverse reduction is the right conceptual tool to understand the representation theory of WZW models and W-algebras, in particular to explain why they are related through string functions. Mastering this new tool will lead us to beautiful new theorems in this corner of representation theory which I expect to stimulate an enormous amount of exciting new work both in classical representation theory and physical applications.

The above projects serve to indicate the significance of the problems that I am proposing to study during the term of this Fellowship. Some of the these projects are quite specific, though still ambitious, while some are more open-ended. It is, of course, unrealistic to claim that I will solve all of these projects in complete generality over a four year period; in particular Projects 2, 3 and 4 aspire to very general results that would have been unthinkable a few years ago. The point is rather that there is a wide range of substantial and important research projects to be undertaken and that now is the perfect time to be working on them. I hope it is also clear from my dominant position in this field that I am the perfect person to succeed with them.

The projects discussed above are all closely related and it is not easy to disentangle them from one another. Moreover, I have proposed to work on them with different groups of collaborators, hence there is no reason not to work on them in parallel. However, a very rough timeline can be imagined for progress on each, along the following lines:

- Year 1. Commence Projects 1, 2 and 5, with 1 being the initial focus.
- Year 2. Wrap up Project 1, while continuing with 2 and 5. Commence Project 4.
- Year 3. Wrap up Project 5 to some degree, whilst continuing with 2 and 4. Commence Project 3.
- Year 4. Wrap up Projects 2, 3 and 4 as much as possible.

There are of course many more related research directions that I have neglected here due to lack of space including:

- Abelian categories for logarithmic VOAs/CFTs (projective/injective VOA-modules).
- Investigating indecomposable structures of logarithmic VOA-modules and proving fusion rules.
- Generalising modular tensor categories to incorporate non-semisimplicity (logarithmic features).
- Constructing new examples of LCFTs with desirable properties (simple current extensions, orbifolds, cosets).

I am sure that many more will become apparent as the research progresses.

BENEFIT

This proposal addresses fundamental research in two of the enabling sciences, namely mathematical physics and pure mathematics. As such, one cannot (and should not) try to imagine immediate applications to society, the economy or the environment. One may of course speculate about possible long-term benefits, but this is inappropriate at best and damaging to science as a whole at worst. Instead, one must concentrate on the immediate benefit of the proposed research to our overall state of knowledge and our overall research capacity, as stated in the Future Fellowships guideline B1.5a. In particular, two immediate benefits of my continuing presence in Australia is the prestige that it adds to our international reputation for mathematical physics research and the steady stream of top international visitors who disseminate first-hand to our communities the most recent breakthroughs in the field.

Australia has already made significant contributions to LCFT and VOAs. Through its applications to statistical mechanics and string theory, CFT has established itself as a *lingua franca* for modern mathematical physics. Advancing knowledge of LCFTs is crucial for properly understanding universal quantities in condensed matter physics, dualities in superstring theory, and correspondences between theoretical physics and pure mathematics research in general. This project will also bring a bevy of international collaborators to our shores and will provide a supportive environment for training research associates and students. In particular, the creativity and problem solving skills that they will develop through working on these projects will make them valuable assets in academia, defence, the public service, finance and industry. I am therefore certain that funding this project will further improve Australia's standing in fundamental research, consolidate its reputation as a leading international centre for LCFT and VOAs, and contribute to long-term national benefit through training of outstanding personnel.

There are many strong links between the research being proposed here and many current hot topics in mathematics and high-energy physics. As mentioned above, physicists are currently discovering many new and fascinating correspondences relating 2D LCFTs and VOAs to supersymmetric versions of the 4D gauge theories that lie at the heart of the standard model of particle physics. The work proposed here will give us a reasonably complete picture of the 2D side of these correspondences for several infinite families of examples and thereby challenge physicists to extend their results accordingly.

On the pure mathematical side, there are at least three hot areas which are directly affected by this proposal. First, representation theorists studying finite and affine W-algebras are already interested in my work relating the characters of affine VOAs and W-algebras to the "reduction by stages" procedure of finite W-algebras. A second hot area related to this proposal is the question of how to usefully axiomatise a non-semisimple generalisation of a modular tensor category. The new LCFTs that I will construct and analyse will provide crucial data towards this endeavour. A third hot area is the connection with generalisations of modular forms, especially the mock modular forms of Ramanujan. Again, this mock modular behaviour seems to have a natural interpretation in terms of the relaxed modules that arise in LCFT.

I am confident that the research being proposed here will benefit the local and international pure mathematics, mathematical physics and theoretical physics communities. Moreover, the knowledge generated will clearly help to cement Australia's reputation for world-leading research in these fields. The students trained throughout the course of this Fellowship will be in an excellent position to embark on world-leading research in academia or to transition to other fields. Finally, the University of Melbourne has demonstrated its strategic interest in promoting my research field and retaining expertise in this area by supporting it with an exceptionally large Establishment Grant worth $320,000 which will be used to support (and train) a suitably strong RA. All in all, there is clear and significant benefit to Australia's overall state of knowledge and research capacity in the enabling sciences.

Feasibility and Strategic Alignment

The Administering Organisation for this project is the University of Melbourne, where I hold a continuing position. The School of Mathematics and Statistics hosts a very strong mathematical physics group whose interests and expertise have overlap with the research being proposed here. In particular, I expect to benefit tremendously from the expertise of Paul Pearce and Thomas Quella, both of whom have worked in LCFT, and that of Omar Foda and Johanna Knapp, who work in string theory. Moreover, the mathematical physics group as a whole provides a strongly supportive environment for the training and mentoring of postdocs and students of all levels.

The School's pure mathematics group is likewise very interested in the research directions outlined here. As mentioned above, the study of LCFTs and VOAs, and this proposal in particular, makes deep use of representation theory, homological algebra and category theory, drawing from and contributing to progress in these disciplines. Happily, the pure mathematics group has significant expertise in these areas through Nora Ganter, Christian Haesemayer, Peter McNamara, Daniel Murfet, Arun Ram, Marcy Robertson, Kari Vilonen, Ting Xue, Yaping Yang and Gufang Zhao, all of whom also have a keen interest in mathematical physics. Their support will be a key factor in succeeding with the aims of the proposal.

Along with the usual resources given to continuing staff, the Administering Organisation will provide a $320,000 Establishment Grant to support this research. These funds will be used to hire a Postdoctoral Research Assistant who will be intimately involved with many of the proposed projects, a stipend top-up for a PhD student who will work on special cases and/or applications, travel for members of the research group (and for international visitors), and to hold a major international workshop at MATRIX, Creswick, on topics related to CFT and VOAs including representation theory, categories, number theory, and geometry.

With this support and so much relevant local expertise, it is clear that the research environment at the Administering Organisation is ideal for the proposed fellowship. My track record and established international collaboration network likewise indicates that feasibility is not going to be in question. Combined with the clear benefits articulated above, more than appropriate for fundamental research in the enabling sciences, I also think it is clear that this proposal represents excellent value for money. Indeed, successfully completing even some of the projects described above would greatly expand our pool of knowledge about LCFTs and VOAs, especially in regard to the current hot topic of 4D/2D correspondences, whilst simultaneously contributing to the training of a new generation of research scientists.

Communication of Results

As a mathematical physicist, my primary means of disseminating results is through posting articles on the arXiv preprint server (arXiv.org) and then publishing them in the best journals relevant to the topic. It is, of course, a universal truism that fundamental theoretical results, of the type that will be generated here, should be made freely available to the wider academic community. This is reflected in the philosophy behind the arXiv, which proudly stands for never withholding ideas behind paywalls. Effectively, anybody may use the results of my work, with the implicit understanding that my contributions will be acknowledged in their subsequent publications.

Because of the specialised nature of my research, there are relatively few top-quality journals in which I would aim to publish. For this proposal, examples of such journals include Adv. Math., Adv. Theor. Math. Phys., Comm. Contemp. Math., Comm. Math. Phys, Int. Math. Res. Not, J. Phys. A, Lett. Math. Phys, Nucl. Phys. B and Transform. Groups, each of which is universally acknowledged as being among the best in their fields. Based on my past track record, I expect that funding this proposal would result in at least four long papers each year. Moreover, a Fellowship would give me many more of the necessary uninterrupted time periods to concentrate on the deeper problems in the field, thereby leading (one would expect) to even more significant breakthroughs.

My writing is generally praised by the community for its clarity and depth. This is reflected in the fact that my 2014 article [77] was chosen by the editors of the Journal of Physics A as one of their inaugural "Publisher's picks" and my 2015 article [78] was chosen for inclusion in IOPSelect. My philosophy with scientific writing is that the most important part of my job as a researcher is to effectively communicate results, rather than just obtain them. Whilst effective communication can, and often does, take place in seminars and lectures or via direct peer-to-peer discussions, the primary means of information transfer is still through papers. Taking the time to write clearly is therefore crucial for advancing knowledge.

On this topic, I shall of course also take the opportunity to present the work funded by this proposal at national and international conferences, by giving seminars in other centres and departments, and through lecture series to sufficiently advanced audiences. The nature of the field, consisting of abstract fundamental research, means that it is generally difficult to communicate this work directly to non-specialists in mathematics and physics, let along to the general public. Nevertheless, I have experience with giving non-technical lectures on subjects related to the research being proposed here. This includes two "Founders Day" talks at the ANU, given to the entire Research School of Physics and Engineering, and several outreach talks given to local high schools and undergraduate physics and mathematics societies.

Since this is a proposal in mathematical physics which involves no experiments, the main source of generated data is the articles themselves. The source files for these manuscripts are rarely larger than a few hundred kilobytes and will be made available, free of charge and in accordance with the ARC's Open Access policy, on the arXiv preprint server (arXiv.org). When these papers rely on results generated by code written specifically for this fellowship, the code will likewise be made available in the arXiv source files.

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