

RELAXED HIGHEST-WEIGHT MODULES II: CLASSIFICATIONS FOR AFFINE VERTEX ALGEBRAS

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ABSTRACT. This is the second of a series of articles devoted to the study of relaxed highest-weight modules over affine vertex algebras and W-algebras. The first [1] studied the simple “rank-1” affine vertex superalgebras $L_k(\mathfrak{sl}_2)$ and $L_k(\mathfrak{osp}(1|2))$, with the main results including the first complete proofs of certain conjectured character formulae (as well as some entirely new ones). Here, we turn to the question of classifying relaxed highest-weight modules for simple affine vertex algebras of arbitrary rank. The key point is that this can be reduced to the classification of highest-weight modules by generalising Olivier Mathieu’s coherent families [2]. We formulate this algorithmically and illustrate its practical implementation with several detailed examples. We also show how to use coherent family technology to establish the non-semisimplicity of category \mathcal{O} in one of these examples.

1. INTRODUCTION

1.1. **Aims.** The representation theory of the vertex operator superalgebra underlying a given conformal field theory is traditionally assumed to have a highest-weight flavour, especially when the theory in question is rational. However, there is a generalisation that is playing an increasingly important role in studying non-rational examples, namely the *relaxed* highest-weight modules. These were originally named in [3] where such modules over the simple (admissible-level) affine vertex operator algebra $L_k(\mathfrak{sl}_2)$ were used to study the well-known Kazama–Suzuki correspondence [4] with the $N = 2$ superconformal vertex operator superalgebras.

The idea behind the appellation “relaxed” comes from relaxing the definition of a highest-weight vector so that it no longer needs to be annihilated by the positive root vectors of the horizontal subalgebra. A relaxed highest-weight module is then just a module that is generated by a relaxed highest-weight vector. This idea can be applied to quite general classes of vertex operator superalgebras [5] and so relaxed highest-weight modules are potentially important ingredients of a wide variety of conformal field theories.

Interestingly, the simple relaxed highest-weight $L_k(\mathfrak{sl}_2)$ -modules were actually classified in [6], several years before their naming in [3]. They have since been proposed as the main building blocks of the $SL_2(\mathbb{R})$ Wess–Zumino–Witten models [7], found to arise naturally in the fusion rules of $L_{-4/3}(\mathfrak{sl}_2)$ and $L_{-1/2}(\mathfrak{sl}_2)$ [8, 9], and used to analyse the representation theory of the admissible-level \mathfrak{sl}_2 -parafermion theories [10–13]. Moreover, relaxed highest-weight modules have recently been shown to play a central role in conformal field theories based on the vertex operator superalgebras $L_k(\mathfrak{sl}_3)$ [14–16], $L_k(\mathfrak{osp}(1|2))$ [17–19] and $L_k(\mathfrak{sl}(2|1))$ [20].

One of the many reasons to study relaxed highest-weight modules is the belief that such modules are necessary to construct consistent affine conformal field theories at non-rational levels. Indeed, it has been observed in several examples [16, 17, 19, 21–23] that the characters of the representations of a vertex operator superalgebra need not carry a representation of the modular group unless one includes relaxed modules (and their twists by spectral flow automorphisms [24, 25]). Further, this inclusion even allows one, in these cases, to compute the Grothendieck fusion coefficients using a (conjectural) Verlinde formula [26, 27].

From the point of view of this article, however, the most compelling reason to study relaxed highest-weight modules is the fact that they form the largest class of weight modules to which Zhu’s powerful classification methods [28] may be applied. More precisely, the simple relaxed highest-weight modules are the simple objects of a relaxed category \mathcal{R} , see [1, 5] for the definition, that naturally generalises the well-known Bernstein–Gel’fand–Gel’fand category \mathcal{O} . The point is that this is the largest category of weight modules on which Zhu’s functor $Zhu[-]$ (introduced below) has zero kernel.

Our aim here is to provide the means to classify the simple relaxed highest-weight modules, with finite-dimensional weight spaces, over an *arbitrary* affine vertex operator algebra. The restriction to finite-dimensional weight spaces is motivated physically by the need to have well-defined characters, in particular so that the modular invariance of the partition function of the conformal field theory can be verified. Actually, the method works for critical levels as well where one also expects relaxed modules, see [29] for the \mathfrak{sl}_2 case. To the best of our knowledge, relaxed classifications are currently only known for $L_k(\mathfrak{sl}_2)$ [5, 6], $L_k(\mathfrak{osp}(1|2))$ [17, 19, 30] and $L_k(\mathfrak{sl}_3)$ [15]. Our results make it easy to extend these classifications to higher-rank \mathfrak{g} , at least when the level k is admissible [31], thanks to the celebrated highest-weight classification of Arakawa [32].

1.2. Zhu technology. Throughout this work, the underlying field is always implicitly assumed to be the complex numbers \mathbb{C} . Let \mathfrak{g} be a finite-dimensional simple Lie algebra. Recall that the Zhu algebra [28] of a level- k affine vertex algebra $V_k(\mathfrak{g})$ is isomorphic [33] to

$$Z_k = \frac{U(\mathfrak{g})}{I_k}, \quad (1.1)$$

where I_k is some two-sided ideal of $U(\mathfrak{g})$. If $V_k(\mathfrak{g})$ is universal, then $I_k = 0$. If $V_k(\mathfrak{g})$ is not universal, then I_k is non-zero if and only if k is critical, meaning that $k = -h^\vee$, or k satisfies [34]

$$\ell(k + h^\vee) = \frac{u}{v}, \quad \text{for some } u \in \mathbb{Z}_{\geq 2} \text{ and } v \in \mathbb{Z}_{\geq 1} \text{ with } \gcd\{u, v\} = 1. \quad (1.2)$$

Here, ℓ is the lacing number of \mathfrak{g} : $\ell = 1$ for types A, D and E; $\ell = 2$ for types B, C and F; $\ell = 3$ for type G.

The representation theories of a vertex superalgebra and its Zhu algebra are related [28] by a functor $\text{Zhu}[-]$. For an affine vertex algebra $V_k(\mathfrak{g})$, this functor maps the category of relaxed highest-weight $V_k(\mathfrak{g})$ -modules to the category of weight Z_k -modules. If we recall [33] that any $V_k(\mathfrak{g})$ -module is naturally a module over the untwisted affine Kac-Moody algebra $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ (on which the central element K acts as multiplication by k), then this functor has the form $\text{Zhu}[\mathcal{M}] = \mathcal{M}^{t\mathfrak{g}[t]}$ (the elements of \mathcal{M} that are annihilated by $t\mathfrak{g}[t]$).

There is likewise a functor $\text{Ind}[-]$ from the category of weight Z_k -modules to the category of relaxed highest-weight $V_k(\mathfrak{g})$ -modules, obtained by “inducing” and then quotienting by the maximal submodule whose intersection with the original module is zero. We refer to [28, Sec. 2.2] and [35, Sec. 3.2] for a precise definition of what “inducing” means in this context. Using these two functors, Zhu proved the following celebrated result (actually in much greater generality).

Theorem 1.1 (Zhu [28, Thms. 2.2.1 and 2.2.2]).

- (a) *A relaxed highest-weight $V_k(\mathfrak{g})$ -module \mathcal{L} is simple if and only if $\text{Zhu}[\mathcal{L}]$ is a simple weight Z_k -module.*
- (b) *More generally, any Z_k -module \mathcal{M} yields an $\mathbb{Z}_{\geq 0}$ -graded $V_k(\mathfrak{g})$ -module $\text{Ind}[\mathcal{M}]$ such that $\text{Zhu}[\text{Ind}[\mathcal{M}]] \simeq \mathcal{M}$ and 0 is the only submodule of $\text{Ind}[\mathcal{M}]$ whose intersection with the “top space” \mathcal{M} is zero.*

To classify the simple relaxed highest-weight modules of the affine vertex algebra $V_k(\mathfrak{g})$, it therefore suffices to classify the simple weight modules of \mathfrak{g} that are annihilated by the Zhu ideal I_k .

If $I_k = 0$, which occurs when $V_k(\mathfrak{g})$ is universal, our task is then to classify all the weight modules of \mathfrak{g} . This is quite ambitious and has in fact only been completed for $\mathfrak{g} = \mathfrak{sl}_2$ (see [36] for a textbook treatment). However, as noted above, we actually want to restrict to weight modules with finite-dimensional weight spaces. Then, we are in better shape because this class of \mathfrak{g} -modules was classified, for all finite-dimensional simple Lie algebras \mathfrak{g} , by Mathieu [2] (building on work of Fernando [37]). For this purpose, Mathieu introduced highly reducible \mathfrak{g} -modules called coherent families whose properties reduced the classification problem to the classification of highest-weight \mathfrak{g} -modules satisfying certain easily analysed conditions.

In this paper, we are interested in the case in which $I_k \neq 0$. We will therefore extend Mathieu’s result to a classification of all simple weight Z -modules with finite-dimensional weight spaces, where Z is the quotient of $U(\mathfrak{g})$ by an arbitrary two-sided ideal I . More precisely, we use Mathieu’s theory of coherent families to reduce this

classification to a classification problem involving highest-weight Z -modules. In particular, if the classification of simple highest-weight Z -modules is already known, then our results allow one to algorithmically classify all the simple weight Z -modules with finite-dimensional weight spaces (see Section 8 and the examples detailed in Section 9). Specialising Z to the Zhu algebra Z_k of $V_k(\mathfrak{g})$ and applying Theorem 1.1a, we then recover the relaxed highest-weight classification that we are interested in here.

1.3. Results. In this section, we present our results in the context of classifying certain types of relaxed highest-weight $V_k(\mathfrak{g})$ -modules with finite-dimensional weight spaces. As mentioned above, these results actually hold for ideals more general than the Zhu ideals I_k and are stated as such in the rest of the paper.

Before stating our main theorems, we shall need to introduce some definitions. First, we generalise Mathieu's notion [2] of a coherent family of \mathfrak{g} -modules to families of l -modules, where l is an arbitrary finite-dimensional reductive Lie algebra. Fixing a Cartan subalgebra \mathfrak{h} of l , we let $\mathfrak{s} = [l, l]$ and $\mathfrak{h}_{\mathfrak{s}} = \mathfrak{h} \cap \mathfrak{s}$. Then, a *coherent family* of l -modules is a weight l -module satisfying the following three properties: its (weight) support is a single coset $\zeta + \mathfrak{h}_{\mathfrak{s}}^*$ (for some $\zeta \in \mathfrak{h}^*$); its non-zero weight spaces all have the same dimension; any element of the centraliser of \mathfrak{h} in $U(l)$ defines a polynomial function on the support given by the trace of the element's action on each weight space. We refer to Definition 2.2 below for further discussion.

The reason why we need this minor generalisation of coherent families is that we require a further generalisation that also accounts for Fernando's work [37]. For this, we consider parabolic subalgebras $\mathfrak{p} \subseteq \mathfrak{g}$ and take l to be the corresponding Levi factor. Parabolic induction then defines a functor that maps a weight l -module to a weight \mathfrak{g} -module, canonically embedding the former in the latter. We define the *almost-simple quotient* of a parabolically induced module to be the quotient by the sum of all the submodules that have zero intersection with the image of this embedding.

With this, we can finally define the promised generalisation of coherent families: a *parabolic family* of \mathfrak{g} -modules is the almost-simple quotient of the parabolic induction of some coherent family of l -modules, see Definition 3.3. One useful property of a coherent family \mathcal{C} of l -modules is that it always contains an *infinite-dimensional highest-weight* submodule \mathcal{H} [2]. The almost simple quotient of the parabolic induction of \mathcal{H} is then an infinite-dimensional highest-weight \mathfrak{g} -submodule of the parabolic family induced from \mathcal{C} . We shall refer to the highest-weight \mathfrak{g} -modules obtained in this fashion as being *l-bounded*, referring to Definition 4.3 below for further details. Note that not every infinite-dimensional highest-weight submodule of a parabolic family is automatically l -bounded.

We can now present our first main theorem. Recall that \mathfrak{g} denotes a finite-dimensional simple Lie algebra, $\widehat{\mathfrak{g}}$ its untwisted affinisation and $V_k(\mathfrak{g})$ one of the corresponding affine vertex algebras of level $k \in \mathbb{C}$.

Main Theorem 1. *Suppose that \mathcal{L} is a simple level- k relaxed highest-weight $\widehat{\mathfrak{g}}$ -module, with finite-dimensional weight spaces, that is not highest-weight with respect to any Borel subalgebra. Then, \mathcal{L} is a $V_k(\mathfrak{g})$ -module if and only if $\text{Zhu}[\mathcal{L}]$ is a submodule of an irreducible semisimple parabolic family \mathcal{P} of \mathfrak{g} -modules that has a simple l -bounded highest-weight submodule \mathcal{H} whose Zhu-induction $\text{Ind}[\mathcal{H}]$ is a $V_k(\mathfrak{g})$ -module. Here, l denotes the Levi factor of the parabolic subalgebra associated with \mathcal{P} .*

The notion of irreducibility and semisimplicity for parabolic families is defined in Section 3, see Equation (3.2).

This result follows immediately by combining Theorem 1.1a with Theorem 4.5 below. What it means is that if one is able to classify the simple highest-weight $V_k(\mathfrak{g})$ -modules and understand the highest-weight submodules of every parabolic family of $\widehat{\mathfrak{g}}$ -modules, then one can deduce the classification of the simple *relaxed* highest-weight $V_k(\mathfrak{g})$ -modules. We shall see how this works with a series of examples in Section 9.

Our second main theorem extends the first to cover certain types of non-simple, but indecomposable, relaxed highest-weight $V_k(\mathfrak{g})$ -modules. Given a root α of \mathfrak{g} , we say that a \mathfrak{g} -module \mathcal{M} is α -*bijective* if the corresponding root vector acts bijectively.

Main Theorem 2. *Let $\mathfrak{p} \subseteq \mathfrak{g}$ be a parabolic subalgebra of \mathfrak{g} with a non-abelian Levi factor \mathfrak{l} and let \mathcal{C} be an irreducible α -bijective coherent family of \mathfrak{l} -modules, for some root α of \mathfrak{l} . Let \mathcal{P} denote the parabolic family of \mathfrak{g} -modules induced from \mathcal{C} and let \mathcal{H} be a simple \mathfrak{l} -bounded highest-weight submodule of \mathcal{P} . Then, if $\text{Ind}[\mathcal{H}]$ is a $V_k(\mathfrak{g})$ -module, then so is every subquotient of $\text{Ind}[\mathcal{P}]$.*

This result follows from Theorem 1.1b and Theorem 5.3. The condition of α -bijectivity ensures that \mathcal{P} is not semisimple, hence that it has reducible but indecomposable subquotients from which we obtain reducible but indecomposable $V_k(\mathfrak{g})$ -modules by Zhu-induction. Of course, identifying these indecomposable $V_k(\mathfrak{g})$ -modules may be quite difficult in practice. In Section 10, we consider an illustrative application that features a non-semisimple parabolic family of \mathfrak{so}_8 -modules.

We mention that the motivation for wanting to construct such non-simple indecomposable relaxed highest-weight $V_k(\mathfrak{g})$ -modules stems from the observation [9, 38] that such modules seem to be building blocks for constructing projective covers (in a category that naturally extends the relaxed category \mathcal{R} by spectral flow). These projective covers are, in turn, believed to be the natural building blocks of the state space of the conformal field theory [26, 38]. Unfortunately, these covers are currently not even known to exist for any non-rational affine theory, though conjectural structures for $L_k(\mathfrak{sl}_2)$ and $L_k(\mathfrak{osp}(1|2))$ may be found in [19, 39, 40].

1.4. Outline. We start by recalling Mathieu's definition of a coherent family of \mathfrak{g} -modules in Section 2 and by immediately generalising it to coherent families of modules over a reductive Lie algebra \mathfrak{l} . This section also introduces some convenient definitions and summarises some of the important results of Fernando and Mathieu that are needed in what follows. Section 3 then introduces a new notion, which we call a parabolic family of \mathfrak{g} -modules, and formalises the relationship between parabolic and coherent families in terms of restriction- and induction-type functors.

The classification work begins in Section 4. For a quotient Z of $U(\mathfrak{g})$ by an arbitrary ideal, we identify the simple weight Z -modules, with finite-dimensional weight spaces, as simple submodules of certain semisimple parabolic families of \mathfrak{g} -modules (Theorem 4.5). This proves Main Theorem 1. The extension to α -bijective indecomposable modules, needed for Main Theorem 2, is then proven in Section 5 (Theorem 5.3), now using non-semisimple parabolic families.

Having proven these classification theorems, we next turn to the question of how to efficiently analyse the combinatorics of parabolic families so as to be able to exploit existing highest-weight classification results. For this, we first summarise Mathieu's explicit classification of coherent families in Section 6. Interestingly, it turns out that coherent families are usually, but not always, completely distinguished by their central characters. Section 7 then describes when two highest-weight modules appear as submodules of the same coherent/parabolic family and discusses how the Weyl group acts on parabolic families.

This material is combined with Theorem 4.5 in Section 8 and the result is summarised in terms of an algorithm for classifying simple weight Z -modules with finite-dimensional weight spaces. In Section 9, we use this algorithm to classify the simple relaxed highest-weight modules of some interesting examples, taking Z to be the Zhu algebra Z_k of a simple affine vertex operator algebra $L_k(\mathfrak{g})$. Specifically, we address the admissible-level cases $L_{-3/2}(\mathfrak{sl}_3)$, $L_{-1/2}(\mathfrak{sp}_4)$ and $L_{-5/3}(\mathfrak{g}_2)$ as well as the non-admissible-level case $L_{-2}(\mathfrak{so}_8)$. We hope that these illustrations will provide the reader with a taste of the utility of our results.

Finally, we give an application of the utility of Theorem 5.3 in Section 10. Specifically, we use it to show that the simple affine vertex operator algebra $L_{-2}(\mathfrak{so}_8)$ not only admits non-semisimple relaxed highest-weight modules, but it in fact also admits non-semisimple highest-weight modules. We believe that this is the first demonstration of non-semisimplicity in category \mathcal{O} for a *quasilisse* [41] affine vertex operator algebra.

In the future, we intend to explore more families of higher-rank classifications in order to better understand the general features of relaxed highest-weight modules. We also intend to generalise the methodology developed here to affine vertex superalgebras and the associated W-algebras and superalgebras. Note that there are many

interesting cases [13, 39, 42–53] in which a vertex algebra possesses continuously parametrised “coherent” families consisting of *highest-weight* modules. We also hope to generalise our treatment of weight modules so as to study these cases. The next instalment of this series [54] will address the important problem of computing the character of more general relaxed highest-weight modules, thus generalising the rank-1 results of [1].

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2. COHERENT FAMILIES

In [2], Olivier Mathieu introduced the notion of a coherent family as a fundamental tool for completing the classification of simple weight modules with finite-dimensional weight spaces over a finite-dimensional simple Lie algebra \mathfrak{g} . Fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. We let $\text{supp}(\mathcal{M}) \subseteq \mathfrak{h}^*$ denote the *support* (the set of weights) of a module \mathcal{M} and write $\mathcal{M}(\mu)$ for the weight space of \mathcal{M} corresponding to the weight $\mu \in \mathfrak{h}^*$. Let $U(\mathfrak{g})^{\mathfrak{h}}$ denote the centraliser of \mathfrak{h} in the universal enveloping algebra $U(\mathfrak{g})$. Mathieu’s definition is then as follows.

Definition 2.1. *Let \mathfrak{g} be a finite-dimensional simple Lie algebra. A coherent family of \mathfrak{g} -modules is a weight \mathfrak{g} -module \mathcal{C} for which:*

- *There exists $d \in \mathbb{Z}_{>0}$, called the degree of \mathcal{C} , such that $\dim \mathcal{C}(\mu) = d$ for all $\mu \in \mathfrak{h}^*$.*
- *Given any $U \in U(\mathfrak{g})^{\mathfrak{h}}$, the function taking $\mu \in \mathfrak{h}^*$ to $\text{tr}_{\mathcal{C}(\mu)} U$ is polynomial in μ .*

In particular, the support of a coherent family is all of \mathfrak{h}^* .

We shall need analogues of these families for certain finite-dimensional *reductive* Lie algebras \mathfrak{l} , each also coming with a fixed Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{l}$. We let $\mathfrak{s} = [\mathfrak{l}, \mathfrak{l}]$ denote the derived subalgebra of \mathfrak{l} and choose a Cartan subalgebra of \mathfrak{s} to be $\mathfrak{h}_{\mathfrak{s}} = \mathfrak{h} \cap \mathfrak{s}$. We then have $\mathfrak{l} = \mathfrak{s} \oplus \mathfrak{z}$ and $\mathfrak{h} = \mathfrak{h}_{\mathfrak{s}} \oplus \mathfrak{z}$, where \mathfrak{z} is the centre of \mathfrak{l} .

Definition 2.2. *Let \mathfrak{l} be a finite-dimensional reductive Lie algebra. A coherent family of \mathfrak{l} -modules is a weight \mathfrak{l} -module \mathcal{C} for which:*

- *$\text{supp}(\mathcal{C}) = \zeta + \mathfrak{h}_{\mathfrak{s}}^*$, for some $\zeta \in \mathfrak{h}^*$.*
- *There exists $d \in \mathbb{Z}_{>0}$, called the degree of \mathcal{C} , such that $\dim \mathcal{C}(\mu) = d$ for all $\mu \in \text{supp}(\mathcal{C})$.*
- *Given any $U \in U(\mathfrak{l})^{\mathfrak{h}}$, the function taking $\mu \in \text{supp}(\mathcal{C})$ to $\text{tr}_{\mathcal{C}(\mu)} U$ is polynomial in μ .*

This reduces to Mathieu’s definition when \mathfrak{l} is simple.

This reduction of the support from \mathfrak{h}^* to $\zeta + \mathfrak{h}_{\mathfrak{s}}^*$ is motivated by the idea that a given polynomial action on a suitable *infinite-dimensional* submodule automatically determines the action on the entire coherent family. As we shall see, the simple ideals of \mathfrak{l} may have infinite-dimensional submodules that can be used for such purposes, while the abelian ideal \mathfrak{z} of course does not.

A coherent family \mathcal{C} of \mathfrak{l} -modules is therefore highly reducible in general, decomposing as

$$\mathcal{C} \simeq \bigoplus_{\lambda \in \text{supp}(\mathcal{C})/\mathcal{Q}_\mathfrak{l}} \mathcal{C}_\lambda, \quad (2.1)$$

where Q_l denotes the root lattice of l (which coincides with that of \mathfrak{s}). If at least one of the \mathcal{C}_λ is simple, then the coherent family \mathcal{C} is said to be *irreducible*. Likewise, \mathcal{C} is called a *semisimple* coherent family if all of the \mathcal{C}_λ are semisimple l -modules.

We note the special case in which l is abelian, hence $\mathfrak{z} = \mathfrak{h} = l$, $\mathfrak{s} = \mathfrak{h}_\mathfrak{s} = 0$ and $U(l)^\mathfrak{h} = U(\mathfrak{z})$. Then, the support of a coherent family \mathcal{C} of l -modules is a singleton $\{\zeta\}$. It follows that $\mathcal{C} \simeq \mathcal{C}_\zeta$ is a (possibly non-semisimple) extension of d copies of the simple \mathfrak{z} -module of weight ζ . It is clear that the trace of the action of each $U \in U(\mathfrak{z})$ amounts to multiplication by $d\zeta(U)$, where d is the degree of \mathcal{C} . This is clearly polynomial in ζ . If \mathcal{C} is irreducible, then it is automatically semisimple with degree $d = 1$. Indeed, in this case, \mathcal{C} is actually simple as a \mathfrak{z} -module.

A somewhat less trivial example is $l = \mathfrak{sl}_2$ for which $U(l)^\mathfrak{h}$ is the polynomial ring generated by \mathfrak{h} and the centre $U(l)^\mathfrak{l}$, the latter being polynomials in the quadratic Casimir Ω . The classification of simple weight modules (with finite-dimensional weight spaces) is therefore elementary, see [36, Thm. 3.32] for example. Indeed, a simple weight module is either highest-weight, lowest-weight, or dense, where we recall that a weight l -module \mathcal{N} is said to be *dense* if $\text{supp}(\mathcal{N}) = \lambda + Q_l$, for some $\lambda \in \mathfrak{h}^*$. The summands \mathcal{C}_λ of an irreducible semisimple coherent family \mathcal{C} over \mathfrak{sl}_2 are thus either direct sums of simple highest- and lowest-weight modules or are simple and dense. Moreover, the latter case is generic, occurring whenever there are no $\mu \in \lambda + Q_l$ satisfying the highest-weight condition relating μ to the eigenvalue of Ω . Note that Ω acts as a constant on each simple summand of \mathcal{C} , by Schur's lemma, hence it must act as a constant on all of \mathcal{C} in order to act polynomially.

We consider one last example: $l = \mathfrak{gl}_2$, for which we have $\mathfrak{z} \simeq \mathfrak{gl}_1$ and $\mathfrak{s} \simeq \mathfrak{sl}_2$. A simple weight l -module is therefore a highest-weight, lowest-weight or dense \mathfrak{s} -module tensored by a one-dimensional \mathfrak{z} -module. Our definition for an irreducible degree- d coherent family \mathcal{C} of \mathfrak{gl}_2 -modules is now seen to reduce to the tensor product of an irreducible degree- d coherent family of \mathfrak{sl}_2 -modules with a fixed simple \mathfrak{gl}_1 -module. Indeed, if $\text{supp}(\mathcal{C}) = \zeta + \mathfrak{h}_\mathfrak{s}^*$, then one may choose $\zeta \in \mathfrak{z}^* \subset \mathfrak{h}^*$ to be the unique weight of the fixed \mathfrak{gl}_1 -module.

The picture for irreducible \mathfrak{sl}_2 (and \mathfrak{gl}_2) coherent families \mathcal{C} is then that they decompose into direct summands \mathcal{C}_λ that are simple and dense for all but a finite number of $\lambda \in \text{supp}(\mathcal{C})/Q_l$. The non-simple summands have highest- and lowest-weight composition factors that share their central character (Ω -eigenvalue) with the simple summands. Unfortunately, this picture only generalises partially to higher ranks. We prepare some convenient terminology.

Definition 2.3. *A finite-dimensional reductive Lie algebra is said to be of AC-type if its simple ideals are all of types A and C.*

We recall that the *type* of a finite-dimensional simple Lie algebra refers to the name given to its Dynkin diagram. Thus, \mathfrak{sl}_n is of type A while \mathfrak{sp}_{2n} is of type C, for all $n \in \mathbb{Z}_{\geq 2}$. For our purposes, it is convenient to regard $\mathfrak{sl}_2 \simeq \mathfrak{sp}_2$ as being of type A only (see Section 6).

Proposition 2.4 (Fernando [37, Thm. 5.2 and Rem. 5.4]). *A finite-dimensional reductive Lie algebra l admits a simple dense module if and only if it is of AC-type.*

Despite this, coherent families provide the means to construct and understand simple weight modules with finite-dimensional weight spaces, as we shall discuss below (see Theorem 3.2). First, we collect some useful definitions.

Definition 2.5.

- A bounded l -module is an infinite-dimensional weight module for which there is a (finite) upper bound on the multiplicities (the dimensions of the weight spaces). The maximal multiplicity is called the degree of the l -module.

- The essential support $\text{ess-supp}(\mathcal{N})$ of a bounded \mathfrak{l} -module \mathcal{N} is the set of weights whose multiplicities are maximal.

We remark that Mathieu calls a weight module with uniformly bounded multiplicities *admissible*. We prefer not to use this terminology as it clashes, in our intended application, with a similar widely used terminology for certain affine vertex algebras and their modules [31].

We note that the simple weight \mathfrak{sl}_2 -modules (with finite-dimensional weight spaces) are all bounded. However, this situation is not typical: for example, a Verma module of a finite-dimensional simple Lie algebra \mathfrak{g} is bounded if and only if $\mathfrak{g} = \mathfrak{sl}_2$. On the other hand, a simple dense \mathfrak{l} -module is *torsion-free* [37], meaning that the root vectors of \mathfrak{l} act injectively, hence its (non-zero) multiplicities are constant. Simple dense modules for $\mathfrak{l} \neq \mathfrak{h}$ are thus always bounded.

We conclude this section by quoting some fundamental results for coherent families, proofs for all of which may be found in Mathieu's article [2]. In fact, we present adaptations of Mathieu's results which apply to coherent families for finite-dimensional reductive Lie algebras. These adaptations are quite straightforward and follow immediately from the standard decomposition of a reductive Lie algebra into its simple and abelian ideals, as noted in [2, Sec. 1]. The case where the Lie algebra is abelian is excluded for simplicity.

Proposition 2.6 (Mathieu [2]). *Let \mathfrak{l} be a finite-dimensional non-abelian reductive Lie algebra. Then:*

- [Prop. 3.5ii] *The essential spectrum of a simple bounded \mathfrak{l} -module is Zariski-dense in $\zeta + \mathfrak{h}_s^*$, for some $\zeta \in \mathfrak{h}^*$.*
- [Prop. 4.8i] *Every simple bounded \mathfrak{l} -module embeds into a unique irreducible semisimple coherent family.*
- [Prop. 4.8ii] *Every infinite-dimensional submodule of an irreducible coherent family of degree d is bounded and its degree is also d .*
- [Lem. 5.3ii] *Coherent families exist if and only if \mathfrak{l} is of AC-type (compare Proposition 2.4).*
- [Prop. 5.7] *Given an irreducible semisimple coherent family, there is a choice of Borel subalgebra for \mathfrak{l} such that the family contains a simple bounded highest-weight module.*

3. PARABOLIC FAMILIES

Let \mathfrak{g} be a finite-dimensional simple Lie algebra and $\mathfrak{p} \subseteq \mathfrak{g}$ be a parabolic subalgebra. We choose, once and for all, a Cartan subalgebra \mathfrak{h} for \mathfrak{g} and restrict the parabolics we consider to always contain \mathfrak{h} . Let \mathfrak{u} denote the nilradical of \mathfrak{p} , $\mathfrak{l} = \mathfrak{p}/\mathfrak{u}$ its Levi factor and \mathfrak{u}^- the nilradical opposite to \mathfrak{u} , so that $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}$ (as vector spaces). We denote the derived subalgebra of \mathfrak{l} by \mathfrak{s} and let $\mathfrak{h}_s = \mathfrak{h} \cap \mathfrak{s}$. Finally, let \mathfrak{z} be the centre of \mathfrak{l} so that $\mathfrak{l} = \mathfrak{s} \oplus \mathfrak{z}$ and $\mathfrak{h} = \mathfrak{h}_s \oplus \mathfrak{z}$.

Given a choice of Borel, hence a set of simple/positive roots for \mathfrak{g} , the parabolics containing the Borel are in bijection with the subsets of the set of simple roots. In particular, such a subset S defines \mathfrak{l} and \mathfrak{u} as follows: \mathfrak{l} is spanned by \mathfrak{h} and the root vectors whose roots are integer linear combinations of the elements of S , while \mathfrak{u} is spanned by all the remaining positive root vectors. A useful consequence that we shall use several times is that the root lattice of \mathfrak{l} (and \mathfrak{s}) has zero intersection with the monoid $\Delta_{\mathfrak{u}}^{\geq}$ generated by the roots whose root vectors span \mathfrak{u} .

If \mathfrak{p} is a Borel subalgebra of \mathfrak{g} , then $\mathfrak{l} = \mathfrak{z} = \mathfrak{h}$, $\mathfrak{s} = 0$ and \mathfrak{u} is the nilradical of the Borel. This corresponds to taking $S = \emptyset$. At the other extreme, taking S to be the set of all simple roots corresponds to $\mathfrak{p} = \mathfrak{g}$, whence $\mathfrak{l} = \mathfrak{s} = \mathfrak{g}$ and $\mathfrak{u} = 0$. A useful motivating example is that of $\mathfrak{g} = \mathfrak{sl}_3$ and $|S| = 1$. This leads to 6-dimensional parabolics \mathfrak{p} with $\mathfrak{l} \simeq \mathfrak{gl}_2$, hence $\mathfrak{s} \simeq \mathfrak{sl}_2$ and $\mathfrak{z} \simeq \mathfrak{gl}_1$, while \mathfrak{u} is spanned by two commuting root vectors.

Given a parabolic $\mathfrak{p} \subseteq \mathfrak{g}$ as above, there are two important functors that relate weight modules over \mathfrak{g} and \mathfrak{l} . First, there is the restriction $\mathcal{R}_{\mathfrak{p}}$ that maps a weight \mathfrak{g} -module \mathcal{M} to its subspace $\mathcal{M}^{\mathfrak{u}}$ of vectors annihilated by \mathfrak{u} . As $[\mathfrak{u}, \mathfrak{l}] \subseteq \mathfrak{u}$, $\mathcal{R}_{\mathfrak{p}}\mathcal{M} = \mathcal{M}^{\mathfrak{u}}$ is naturally an \mathfrak{l} -module. To introduce the second functor, recall that the

parabolic induction of a weight \mathfrak{l} -module \mathcal{N} is defined to be the \mathfrak{g} -module that results from letting \mathfrak{u} act as 0 and then inducing as $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathcal{N}$. If \mathcal{N} is simple, then its parabolic induction will have a unique simple quotient. In general, we define the *almost-simple quotient* of the parabolic induction of a weight \mathfrak{l} -module to be the \mathfrak{g} -module obtained by quotienting by the sum of all modules that have zero intersection with the subspace $\mathbb{1} \otimes \mathcal{N}$. We denote by $\mathcal{I}_{\mathfrak{p}}$ the functor on a weight \mathfrak{l} -module that first parabolically induces to a \mathfrak{g} -module and then replaces the result by its almost-simple quotient.

Proposition 3.1. *The functors $\mathcal{I}_{\mathfrak{p}}$ and $\mathcal{R}_{\mathfrak{p}}$ satisfy the following properties:*

- (a) $\mathcal{R}_{\mathfrak{p}}$ is inclusion-preserving and maps simple weight \mathfrak{g} -modules to simple weight \mathfrak{l} -modules (or 0).
- (b) $\mathcal{I}_{\mathfrak{p}}$ maps simple weight \mathfrak{l} -modules to simple weight \mathfrak{g} -modules.
- (c) $\mathcal{R}_{\mathfrak{p}} \mathcal{I}_{\mathfrak{p}} \mathcal{N} \simeq \mathcal{N}$, for all weight \mathfrak{l} -modules \mathcal{N} .
- (d) If \mathcal{N} is a simple weight \mathfrak{l} -module that embeds in a weight \mathfrak{l} -module \mathcal{N}' , then $\mathcal{I}_{\mathfrak{p}} \mathcal{N}$ embeds in $\mathcal{I}_{\mathfrak{p}} \mathcal{N}'$.

Proof. We first prove part **a**. The fact that $\mathcal{R}_{\mathfrak{p}}$ preserves inclusions is clear. Suppose then that \mathcal{M} is a simple weight \mathfrak{g} -module with $\mathcal{R}_{\mathfrak{p}} \mathcal{M} \neq 0$ and that v_1 and v_2 are (non-zero) weight vectors in $\mathcal{R}_{\mathfrak{p}} \mathcal{M} \subseteq \mathcal{M}$. Since v_1 and v_2 are annihilated by \mathfrak{u} , Poincaré–Birkhoff–Witt and simplicity imply that $U(\mathfrak{u}^- \oplus \mathfrak{l}) \cdot v_i = U(\mathfrak{g}) \cdot v_i = \mathcal{M}$, for $i = 1, 2$. In particular, $v_1 = U_1 v_2$ and $v_2 = U_2 v_1$ for some $U_1, U_2 \in U(\mathfrak{u}^- \oplus \mathfrak{l})$. But, $v_1 = U_1 U_2 v_1$ requires that $U_1, U_2 \in U(\mathfrak{l})$ because the $\mathbb{Z}_{\geq 0}$ -span of the roots of \mathfrak{u}^- , which are all negative with respect to an appropriate Borel, have zero intersection with the root lattice of \mathfrak{l} . Thus, v_1 and v_2 lie in the same \mathfrak{l} -submodule of $\mathcal{R}_{\mathfrak{p}} \mathcal{M}$, proving that the latter is simple.

For part **b**, suppose that \mathcal{N} is a simple weight \mathfrak{l} -module. Since any non-zero $w \in \mathcal{N}$ is cyclic, $\mathbb{1} \otimes w$ (or rather its image in the almost-simple quotient) must generate $\mathcal{I}_{\mathfrak{p}} \mathcal{N}$. However, the submodule generated by any non-zero $v \in \mathcal{I}_{\mathfrak{p}} \mathcal{N}$ must contain an element of the form $\mathbb{1} \otimes w$, for some $w \in \mathcal{N}$, because otherwise its intersection with $\mathbb{1} \otimes \mathcal{N}$ would be zero. This submodule is thus $\mathcal{I}_{\mathfrak{p}} \mathcal{N}$, proving that the latter is simple.

To prove part **c**, first note that $\mathcal{N} \simeq \mathbb{1} \otimes \mathcal{N} \subseteq \mathcal{R}_{\mathfrak{p}} \mathcal{I}_{\mathfrak{p}} \mathcal{N}$ because $\mathfrak{u} \cdot (\mathbb{1} \otimes \mathcal{N}) = 0$. If this inclusion were strict, then Poincaré–Birkhoff–Witt would imply that there exists a non-zero $v \in \mathfrak{u}^- U(\mathfrak{u}^-) \otimes \mathcal{N}$ with $\mathfrak{u} \cdot v = 0$. Since $[\mathfrak{l}, \mathfrak{u}^-] \subseteq \mathfrak{u}^-$, we would have

$$U(\mathfrak{g}) \cdot v = U(\mathfrak{u}^- \oplus \mathfrak{l}) \cdot v \subseteq \mathfrak{u}^- U(\mathfrak{u}^-) \otimes \mathcal{N}. \quad (3.1)$$

But then, v would generate a non-zero submodule of $\mathcal{I}_{\mathfrak{p}} \mathcal{N}$ whose intersection with $\mathbb{1} \otimes \mathcal{N}$ is zero, a contradiction. We therefore conclude that the inclusion is an equality.

Finally, inducing from weight \mathfrak{p} -modules to \mathfrak{g} -modules is exact, by Poincaré–Birkhoff–Witt. The sum of the submodules of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathcal{N}$ whose intersection with $\mathbb{1} \otimes \mathcal{N}$ is zero therefore embeds into the sum of the submodules of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathcal{N}'$ whose intersection with $\mathbb{1} \otimes \mathcal{N}'$ is zero, so it follows that we have a morphism from $\mathcal{I}_{\mathfrak{p}} \mathcal{N}$ to $\mathcal{I}_{\mathfrak{p}} \mathcal{N}'$. This morphism is non-zero since it is non-zero on $\mathbb{1} \otimes \mathcal{N}$, hence it is injective by the simplicity of $\mathcal{I}_{\mathfrak{p}} \mathcal{N}$ (part **b**). This proves part **d**. ■

Unsurprisingly, parabolic subalgebras turn out to be important when classifying simple weight modules. For this, the following result is germane.

Theorem 3.2 (Fernando [37, Thm. 4.18]). *Every simple weight \mathfrak{g} -module with finite-dimensional weight spaces is isomorphic to $\mathcal{I}_{\mathfrak{p}} \mathcal{N}$, for some parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$, with Levi factor \mathfrak{l} of AC-type, and some simple dense \mathfrak{l} -module \mathcal{N} .*

We note that if the parabolic is a Borel (so $\mathfrak{l} = \mathfrak{h}$), then all simple \mathfrak{l} -modules are dense and parabolic induction results in highest-weight \mathfrak{g} -modules. Of course, parabolic induction does nothing if $\mathfrak{l} = \mathfrak{g}$.

Suppose now that the reductive subalgebra $\mathfrak{l} \subseteq \mathfrak{g}$ is of AC-type. Then, there exists a semisimple coherent family \mathcal{C} for \mathfrak{l} , by Proposition 2.6d. As a natural generalisation of coherent families, we offer the following definition.

Definition 3.3. A parabolic family of \mathfrak{g} -modules is a module \mathcal{P} isomorphic to $\mathcal{I}_{\mathfrak{p}}\mathcal{C}$, for some parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$, whose Levi factor l is of AC-type, and some coherent family \mathcal{C} of l -modules.

Obviously, a coherent family is just a parabolic family corresponding to the parabolic subalgebra \mathfrak{g} . Note that since $\mathcal{P} \simeq \mathcal{I}_{\mathfrak{p}}\mathcal{C}$ by definition, we always have $\mathcal{R}_{\mathfrak{p}}\mathcal{P} \simeq \mathcal{C}$, by Proposition 3.1c.

We remark that we were tempted to instead coin the term “parabolic coherent family” for the \mathfrak{g} -modules of Definition 3.3. However, these families are not necessarily “coherent” in the sense of Mathieu because their weight multiplicities need not be constant, even if we restrict to weights that differ by elements of the weight space $\mathfrak{h}_{\mathfrak{g}}^*$ of l . Nevertheless, Fernando’s theorem suggests that we will be able to use parabolic families to classify weight modules.

As for coherent families, a parabolic family

$$\mathcal{P} = \bigoplus_{\lambda \in \mathfrak{h}^*/\mathbb{Q}_{\mathfrak{g}}} \mathcal{P}_{\lambda} \quad (3.2)$$

is said to be *irreducible*, if at least one of the \mathcal{P}_{λ} is a simple \mathfrak{g} -module, and *semisimple*, if all of the \mathcal{P}_{λ} are semisimple. By Proposition 3.1, these notions are equivalent to \mathcal{C} being irreducible and semisimple, respectively.

It is convenient at this point to note two useful facts.

Proposition 3.4.

- (a) [2, Lem. 3.3] *The direct summands \mathcal{C}_{λ} of a coherent family \mathcal{C} of l -modules have finite length.*
- (b) [37, Thm. 4.21] *Every finitely generated weight \mathfrak{g} -module with finite-dimensional weight spaces has finite length.*

From part a, we learn that the \mathcal{C}_{λ} are finitely generated, hence so are the $\mathcal{P}_{\lambda} \simeq \mathcal{I}_{\mathfrak{p}}\mathcal{C}_{\lambda}$. Consequently, the direct summands \mathcal{P}_{λ} of a parabolic family \mathcal{P} also have finite length, by part b. It follows that any given parabolic family of \mathfrak{g} -modules has a *semisimplification*, this being the semisimple parabolic family of \mathfrak{g} -modules obtained by replacing each of its direct summands by the direct sum of the summand’s composition factors. Clearly, the semisimplification of an irreducible parabolic family will also be irreducible.

If \mathfrak{p} is a Borel and \mathcal{C} is an irreducible semisimple coherent family for $l = \mathfrak{h}$, then \mathcal{C} is just a one-dimensional \mathfrak{h} -module. The parabolic family $\mathcal{P} = \mathcal{I}_{\mathfrak{p}}\mathcal{C}$ is thus a simple highest-weight module (with respect to the Borel \mathfrak{p}). When $\mathfrak{p} = \mathfrak{g}$, we instead get $\mathcal{P} = \mathcal{C}$. This construction therefore interpolates between simple highest-weight modules and coherent families for \mathfrak{g} .

We conclude this section with a few simple observations about the relationship between coherent and parabolic families.

Proposition 3.5. *Let \mathcal{C} be a coherent family of l -modules and let $\mathcal{P} = \mathcal{I}_{\mathfrak{p}}\mathcal{C}$ be the associated parabolic family of \mathfrak{g} -modules. Then:*

- (a) *The l -module embedding $\mathcal{C} \simeq \mathbb{1} \otimes \mathcal{C} \hookrightarrow \mathcal{P}$ has the property that the weight spaces satisfy $\mathcal{C}(\mu) = \mathcal{P}(\mu)$, for all $\mu \in \text{supp}(\mathcal{C})$.*
- (b) *The function $\text{tr}_{\mathcal{P}(\mu)} U$ is polynomial in $\mu \in \text{supp}(\mathcal{C})$ for any $U \in \mathcal{U}(\mathfrak{g})^{\mathfrak{h}}$.*
- (c) *If \mathcal{M} is a simple quotient of \mathcal{P} , then $\mathcal{R}_{\mathfrak{p}}\mathcal{M}$ is a simple quotient of \mathcal{C} .*

Proof. For a, first note that the Poincaré–Birkhoff–Witt theorem gives $\mathcal{P}(\mu) = \mathcal{C}(\mu) + (u^{-}\mathcal{U}(u^{-}) \otimes \mathcal{C})(\mu)$. Since $\mu \in \text{supp}(\mathcal{C}) = \zeta + \mathfrak{h}_{\mathfrak{g}}^*$ and the weights of $u^{-}\mathcal{U}(u^{-})$ have empty intersection with $\mathfrak{h}_{\mathfrak{g}}^*$, it follows that $(u^{-}\mathcal{U}(u^{-}) \otimes \mathcal{C})(\mu) = 0$. This proves the first assertion. The same intersection argument also shows that $\mathcal{U}(\mathfrak{g})^{\mathfrak{h}}$ may be decomposed, again *à la* Poincaré–Birkhoff–Witt, as $\mathcal{U}(l)^{\mathfrak{h}} \oplus (\mathcal{U}(\mathfrak{g})u)^{\mathfrak{h}}$. Since \mathcal{C} is a coherent family

for l , the elements of $U(l)^{\mathfrak{b}}$ act polynomially on the $\mathcal{C}(\mu)$ with $\mu \in \text{supp}(\mathcal{C})$, while those of $(U(\mathfrak{g})u)^{\mathfrak{b}}$ act as 0. Assertion **b** now follows from **a**.

To prove **c**, suppose that we have a simple quotient $\pi: \mathcal{P} \twoheadrightarrow \mathcal{M}$. Composing this with the inclusion $\mathcal{C} \hookrightarrow \mathcal{P}$ from part **a**, we get an l -module homomorphism $v \in \mathcal{C} \mapsto \pi(1 \otimes v) \in \mathcal{M}$ whose image is easily checked to lie in $\mathcal{R}_p \mathcal{M} = \mathcal{M}^u$. If we assume that the image $\pi(1 \otimes \mathcal{C})$ is 0, then we get

$$\mathcal{M} = \pi(\mathcal{P}) = \pi(U(\mathfrak{g}) \cdot 1 \otimes \mathcal{C}) = U(\mathfrak{g}) \cdot \pi(1 \otimes \mathcal{C}) = 0, \quad (3.3)$$

a contradiction. We conclude that the composition $\mathcal{C} \rightarrow \mathcal{R}_p \mathcal{M}$ is surjective as the target is a simple l -module, by Proposition 3.1a. \blacksquare

4. SIMPLE MODULE CLASSIFICATION

As before, let \mathfrak{g} be a finite-dimensional simple Lie algebra with fixed Cartan subalgebra \mathfrak{h} . Our aim in this section is to classify the simple weight \mathfrak{g} -modules, with finite-dimensional weight spaces, that are annihilated by some two-sided ideal l of $U(\mathfrak{g})$. To this end, we introduce

$$Z = \frac{U(\mathfrak{g})}{l} \quad (4.1)$$

and study the classification of simple weight Z -modules. The motivating example corresponds to taking l to be the Zhu ideal l_k and Z to be the Zhu algebra Z_k of the simple level- k affine vertex algebra associated to \mathfrak{g} , as in Section 1. Another important example corresponds to taking l to be the annihilating ideal of a given simple \mathfrak{g} -module.

We shall first determine when a given coherent family of \mathfrak{g} -modules is a Z -module before upgrading the result to parabolic families of \mathfrak{g} -modules. This case serves to illustrate the strategy of the general proof with a minimum of complications. Recall that $U(\mathfrak{g})^{\mathfrak{b}}$ denotes the centraliser of \mathfrak{h} in $U(\mathfrak{g})$. Let

$$A = l \cap U(\mathfrak{g})^{\mathfrak{b}} \quad (4.2)$$

and note that A is a two-sided ideal of $U(\mathfrak{g})^{\mathfrak{b}}$. We commence with the following very useful lemma, whose underlying idea is surely well known (see [55] for example).

Lemma 4.1. *A simple weight \mathfrak{g} -module \mathcal{M} is a Z -module if and only if $A \cdot v = 0$ for some non-zero $v \in \mathcal{M}$.*

Proof. If \mathcal{M} is a Z -module, then $l \cdot \mathcal{M} = 0$, hence $A \cdot \mathcal{M} = 0$ as required. Suppose therefore that $A \cdot v = 0$ for some non-zero weight vector $v \in \mathcal{M}$. We may decompose l as $A \oplus B$, where the elements of B have non-zero weights. As v is cyclic and l is a right-ideal of $U(\mathfrak{g})$, we have

$$l \cdot \mathcal{M} = l \cdot U(\mathfrak{g}) \cdot v = l \cdot v = A \cdot v \oplus B \cdot v = B \cdot v. \quad (4.3)$$

However, the (non-zero) elements of B have non-zero weights, so it follows that $v \notin B \cdot v$. This proves that $l \cdot \mathcal{M}$ is a proper submodule of \mathcal{M} , hence it is 0 because \mathcal{M} is simple. \blacksquare

Consider now a semisimple coherent family \mathcal{C} for \mathfrak{g} . We choose a simple bounded submodule $\mathcal{H} \subset \mathcal{C}$ (this exists by Proposition 2.6e). We shall suppose that \mathcal{H} is a Z -module, so that $l \cdot \mathcal{H} = 0$. Then, $\text{tr}_{\mathcal{H}(\mu)} a = 0$ for all $a \in l$ and $\mu \in \text{supp}(\mathcal{H})$. But, $\mathcal{H}(\mu) = \mathcal{C}(\mu)$ for all $\mu \in \text{ess-supp}(\mathcal{H})$ (Proposition 2.6c), a set that is Zariski dense in \mathfrak{h}^* (Proposition 2.6a). We therefore have

$$\text{tr}_{\mathcal{C}(\mu)} a = 0, \quad \text{for all } a \in A \text{ and } \mu \in \text{supp}(\mathcal{C}) = \mathfrak{h}^*, \quad (4.4)$$

since the trace of the action of $a \in A \subset l$ is polynomial in μ . Now, $\dim \mathcal{C}(\mu) < \infty$, so replacing a in (4.4) by a^n , $n \in \mathbb{Z}_{>0}$, shows that 0 is the only eigenvalue of a . We conclude that every $a \in A$ acts nilpotently on every $\mathcal{C}(\mu)$, $\mu \in \mathfrak{h}^*$.

Choose a simple direct summand $\mathcal{M} \subset \mathcal{C}$. Then, each non-zero weight space $\mathcal{M}(\mu)$ is a simple $U(\mathfrak{g})^{\flat}$ -module. It follows that $A \cdot \mathcal{M}(\mu)$ is either 0 or $\mathcal{M}(\mu)$ because A is an ideal in $U(\mathfrak{g})^{\flat}$. But, $A \cdot \mathcal{M}(\mu) = \mathcal{M}(\mu) \neq 0$ would imply that any non-zero v generates $\mathcal{M}(\mu)$, as an A -module, and so satisfies $v \in A \cdot v$. However, having $v = av$, for some $a \in A$, contradicts our earlier conclusion that a must act nilpotently on $\mathcal{M}(\mu) \subseteq \mathcal{C}(\mu)$. We therefore conclude that $A \cdot \mathcal{M}(\mu) = 0$ for all $\mu \in \mathfrak{h}^*$, hence that A annihilates \mathcal{M} . By Lemma 4.1, every simple direct summand $\mathcal{M} \subset \mathcal{C}$ is a Z -module, hence so is \mathcal{C} , as desired.

Note that every simple bounded highest-weight Z -module \mathcal{H} embeds into some irreducible semisimple coherent family \mathcal{C} (Proposition 2.6b). By the above argument, \mathcal{C} and all its direct summands are Z -modules. By choosing \mathcal{H} above to be highest-weight, we see that the classification of semisimple coherent families that are Z -modules is therefore essentially equivalent to that of simple bounded highest-weight Z -modules.

Proposition 4.2. *An irreducible semisimple coherent family for \mathfrak{g} is a Z -module if and only if any (and thus all) of its bounded highest-weight submodules are.*

Clearly, every infinite-dimensional highest-weight submodule of a coherent family is bounded.

We now extend this to a classification of *all* simple weight Z -modules, with finite-dimensional weight spaces, in terms of the classification of highest-weight Z -modules. Recall the restriction- and induction-type functors $\mathcal{R}_{\mathfrak{p}}$ and $\mathcal{I}_{\mathfrak{p}}$ from Section 3.

Definition 4.3. *Given a parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ with Levi factor \mathfrak{l} , we say that a \mathfrak{g} -module \mathcal{M} is \mathfrak{l} -bounded if $\mathcal{R}_{\mathfrak{p}}\mathcal{M}$ is a bounded \mathfrak{l} -module.*

Proposition 4.4. *Given a choice of parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$, with non-abelian Levi factor \mathfrak{l} of AC-type, an irreducible semisimple parabolic family for \mathfrak{g} will be a Z -module if and only if any (and thus all) of its \mathfrak{l} -bounded highest-weight submodules are.*

Proof. Let \mathcal{P} be such an irreducible semisimple parabolic family and let $\mathcal{C} = \mathcal{R}_{\mathfrak{p}}\mathcal{P}$, so that \mathcal{C} is a coherent family of \mathfrak{l} -modules with $\mathcal{P} \simeq \mathcal{I}_{\mathfrak{p}}\mathcal{C}$. Suppose that $\mathcal{H} \subset \mathcal{P}$ is a simple \mathfrak{l} -bounded submodule that happens to be a Z -module: $a \cdot \mathcal{H}(\mu) = 0$ for all $a \in \mathfrak{l}$ and $\mu \in \text{supp}(\mathcal{H})$. We now focus on the \mathfrak{l} -submodule $\mathcal{R}_{\mathfrak{p}}\mathcal{H}$ of \mathcal{C} , restricting a to A and μ to $\text{ess-supp}(\mathcal{R}_{\mathfrak{p}}\mathcal{H})$. As $\mathcal{R}_{\mathfrak{p}}\mathcal{H}$ is a simple bounded \mathfrak{l} -submodule of \mathcal{C} , by Proposition 3.1a, its essential support is Zariski-dense in $\text{supp}(\mathcal{C})$. Moreover, $\mu \mapsto \text{tr}_{\mathcal{P}(\mu)} a$ is polynomial in $\mu \in \text{supp}(\mathcal{C})$, for each $a \in A$, by Proposition 3.5b. We therefore conclude, as in the coherent family argument above, that A acts nilpotently on each $\mathcal{P}(\mu) = \mathcal{C}(\mu)$ with $\mu \in \text{supp}(\mathcal{C})$.

Any simple \mathfrak{g} -submodule $\mathcal{M} \subset \mathcal{P}$ has a non-zero image under $\mathcal{R}_{\mathfrak{p}}$ because a zero image would mean that \mathcal{M} has zero intersection with $\mathcal{R}_{\mathfrak{p}}\mathcal{P} \simeq \mathcal{C}$ and hence be zero in $\mathcal{P} \simeq \mathcal{I}_{\mathfrak{p}}\mathcal{C}$. We may therefore choose a (non-zero) weight vector $v \in \mathcal{R}_{\mathfrak{p}}\mathcal{M}$ and let μ denote its weight. Since $\mathcal{R}_{\mathfrak{p}}\mathcal{M} \subset \mathcal{C}$, by Proposition 3.1a, it follows that $\mu \in \text{supp}(\mathcal{C})$ and so A acts nilpotently on $v \in (\mathcal{R}_{\mathfrak{p}}\mathcal{M})(\mu)$. As above, v generating the simple $U(\mathfrak{g})^{\flat}$ -module $(\mathcal{R}_{\mathfrak{p}}\mathcal{M})(\mu)$ under the action of A contradicts the nilpotence of this action. A must therefore annihilate $v \in \mathcal{M}$, whence \mathcal{M} must be a Z -module, by Lemma 4.1, and the proof is complete. \blacksquare

We are now ready to prove our classification result.

Theorem 4.5. *Let \mathfrak{g} be a finite-dimensional simple Lie algebra and let Z be a quotient of $U(\mathfrak{g})$ by a two-sided ideal. Then, a simple weight \mathfrak{g} -module \mathcal{M} , with finite-dimensional weight spaces, is a Z -module if and only if either of the following statements hold:*

- \mathcal{M} is a highest-weight Z -module, with respect to some Borel subalgebra of \mathfrak{g} .

- *There is a parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$, with non-abelian Levi factor \mathfrak{l} of AC-type, and a corresponding irreducible semisimple parabolic family \mathcal{P} of \mathfrak{g} -modules such that \mathcal{M} is isomorphic to a submodule of \mathcal{P} and some submodule of \mathcal{P} is an \mathfrak{l} -bounded highest-weight \mathbb{Z} -module.*

Proof. Proposition 4.4 shows that every submodule \mathcal{M} of such a parabolic family \mathcal{P} is a \mathbb{Z} -module. Conversely, let \mathcal{M} be a simple weight \mathbb{Z} -module, with finite-dimensional weight spaces. We assume that \mathcal{M} is not highest-weight, with respect to any Borel. Then, Theorem 3.2 says that $\mathcal{M} \simeq \mathcal{S}_{\mathfrak{p}}\mathcal{N}$, for some parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$, with non-abelian Levi factor \mathfrak{l} of AC-type, and some simple dense \mathfrak{l} -module \mathcal{N} . As simple dense modules over a non-abelian \mathfrak{l} are bounded (Section 2), \mathcal{N} embeds in an irreducible semisimple coherent family \mathcal{C} of \mathfrak{l} -modules, by Proposition 2.6b. But, Proposition 2.6e ensures that \mathcal{C} contains a simple bounded highest-weight submodule \mathcal{H} . It thus follows from Proposition 3.1d that the irreducible semisimple parabolic family $\mathcal{P} = \mathcal{S}_{\mathfrak{p}}\mathcal{C}$ contains the simple \mathfrak{l} -bounded highest-weight \mathfrak{g} -module $\mathcal{S}_{\mathfrak{p}}\mathcal{H}$. It only remains to show that $\mathcal{S}_{\mathfrak{p}}\mathcal{H}$ is a \mathbb{Z} -module. However, this follows from Proposition 4.4 and the fact that \mathcal{M} is a simple \mathfrak{l} -bounded \mathbb{Z} -module. ■

This theorem reduces the classification of simple weight \mathbb{Z} -modules to that of simple highest-weight \mathbb{Z} -modules and parabolic subalgebras with non-abelian Levi factors of AC-type. The former is a difficult problem in general, through tractable in many important cases, while the latter is essentially combinatorial. Note that Main Theorem 1 is a straightforward corollary of Theorem 4.5 with \mathfrak{l} taken to be the Zhu ideal of the simple level- k affine vertex algebra $L_k(\mathfrak{g})$.

5. INDECOMPOSABLE MODULES

In this section, we study irreducible, but non-semisimple, parabolic families in order to determine when certain indecomposable \mathfrak{g} -modules are \mathbb{Z} -modules. Let $\Delta_{\mathfrak{l}}$ denote the root system of \mathfrak{l} and let e^{α} denote the root vector corresponding to the root $\alpha \in \Delta_{\mathfrak{l}}$.

Definition 5.1. *A weight module \mathcal{N} over \mathfrak{l} is α -bijective, for some given $\alpha \in \Delta_{\mathfrak{l}}$, if e^{α} acts bijectively on \mathcal{N} .*

Many examples of such modules were constructed by Mathieu [2, Lem. 4.5] using a powerful tool called twisted localisation. In particular, for any irreducible semisimple coherent family \mathcal{C}' , there are α -bijective coherent families \mathcal{C} such that \mathcal{C}' is the semisimplification of \mathcal{C} . For $\mathfrak{g} = \mathfrak{sl}_2$, the dense direct summands \mathcal{C}_{λ} of an α -bijective coherent family may also be constructed explicitly, see [56, Sec. 7.8.16] or [36, Sec. 3.3], or as modules induced from one-dimensional modules of the centraliser $U(\mathfrak{sl}_2)^{\mathfrak{b}}$, see [36, Ex. 3.99] or [1, Sec. 3.2]. We note that this induction procedure can also result in indecomposable dense modules that are not α -bijective for any root α .

Given a simple bounded \mathfrak{l} -module \mathcal{N} , let $\Delta^{\text{inj}}(\mathcal{N})$ denote the additive monoid generated by the roots $\alpha \in \Delta$ whose root vectors e^{α} act injectively on \mathcal{N} . We need two straightforward results about these monoids.

Proposition 5.2 (Mathieu [2]). *Let \mathcal{N} be a simple bounded \mathfrak{l} -module. Then:*

- [Lem. 3.1] *The group-completion of the monoid $\Delta^{\text{inj}}(\mathcal{N})$ is $Q_{\mathfrak{l}}$ (the root lattice of \mathfrak{l}).*
- [Prop. 3.5i] *For any $\lambda \in \text{ess-supp}(\mathcal{N})$, we have $\lambda + \Delta^{\text{inj}}(\mathcal{N}) \subseteq \text{ess-supp}(\mathcal{N})$.*

Our goal is to prove the following theorem.

Theorem 5.3. *Let \mathfrak{p} be a parabolic subalgebra of \mathfrak{g} with non-abelian Levi factor \mathfrak{l} of AC-type. Let \mathcal{C} be an irreducible α -bijective coherent family of \mathfrak{l} -modules, for some $\alpha \in \Delta_{\mathfrak{l}}$. Let \mathcal{P} denote the irreducible parabolic family of \mathfrak{g} -modules induced from \mathcal{C} . Suppose that an \mathfrak{l} -bounded highest-weight submodule \mathcal{H} of \mathcal{P} is a \mathbb{Z} -module. Then \mathcal{P} , and hence all its subquotients, are also \mathbb{Z} -modules.*

The idea behind the proof is that if such a parabolic family has an l -bounded highest-weight Z -module as a submodule, then all its simple quotients are also Z -modules, by Theorem 4.5, hence the ideal l must map each direct summand of the parabolic family into its radical. We will show that the weight multiplicities of each simple quotient are frequently equal to those of the corresponding direct summand so that those of the radical are frequently zero. This happens sufficiently often to prove that l in fact maps each direct summand of the parabolic family to zero.

There are technical details required to make this idea precise. For these, we have the following four lemmas. The first follows immediately from the well-known fact, see [56, Cor. 2.3.8] for example, that $U(\mathfrak{g})$ is noetherian.

Lemma 5.4. *The ideal l is finitely generated as a left-ideal of $U(\mathfrak{g})$.*

Lemma 5.5. *Let \mathcal{C} be an α -bijective coherent family of l -modules and \mathcal{P} be the induced irreducible parabolic family of \mathfrak{g} -modules, as in Theorem 5.3. Then every simple quotient of \mathcal{P} is l -bounded.*

Proof. Let \mathcal{M} be a simple quotient of \mathcal{P} . By Proposition 3.5c, $\mathcal{R}_p\mathcal{M}$ is a simple quotient of \mathcal{C} , so its multiplicities are uniformly bounded. Let $\eta: \mathcal{C} \rightarrow \mathcal{R}_p\mathcal{M}$ denote the quotient map. The lemma will therefore follow if we can show that $\mathcal{R}_p\mathcal{M}$ is infinite-dimensional. Assume the contrary: that $\dim \mathcal{R}_p\mathcal{M} < \infty$. Then, there exists $\mu \in \text{supp}(\mathcal{R}_p\mathcal{M})$ such that $\mu - \alpha \notin \text{supp}(\mathcal{R}_p\mathcal{M})$. As e^α acts bijectively on \mathcal{C} , we obtain

$$(\mathcal{R}_p\mathcal{M})(\mu) = \eta(\mathcal{C}(\mu)) = \eta(e^\alpha \cdot \mathcal{C}(\mu - \alpha)) = e^\alpha \cdot \eta(\mathcal{C}(\mu - \alpha)) = e^\alpha \cdot (\mathcal{R}_p\mathcal{M})(\mu - \alpha) = 0, \quad (5.1)$$

a contradiction. It follows that $\mathcal{R}_p\mathcal{M}$ must be infinite-dimensional, completing the proof. \blacksquare

Lemma 5.6. *Let \mathcal{N} be a simple bounded l -module. Then, for any finite subset S of Q_l , there is a weight $\mu \in \text{ess-supp}(\mathcal{N})$ such that $\mu + S \subset \text{ess-supp}(\mathcal{N})$.*

Proof. Note first that $\text{ess-supp}(\mathcal{N})$ is not empty (Proposition 2.6a). So, choose $\lambda \in \text{ess-supp}(\mathcal{N})$ and let $S = \{\beta_1, \dots, \beta_n\} \subset Q_l$. Since the monoid $\Delta^{\text{inj}}(\mathcal{N})$ generates Q_l (Proposition 5.2a), the elements of S have the form $\beta_i = \mu_i - \nu_i$, with $\mu_i, \nu_i \in \Delta^{\text{inj}}(\mathcal{N})$. Set $\nu = \nu_1 + \dots + \nu_n$ and $\mu = \lambda + \nu$. Then, $\nu \in \Delta^{\text{inj}}(\mathcal{N})$ so $\mu \in \text{ess-supp}(\mathcal{N})$ (Proposition 5.2b). Moreover, $\nu + S \subset \Delta^{\text{inj}}(\mathcal{N})$ and so $\mu + S \subset \text{ess-supp}(\mathcal{N})$ (Proposition 5.2b again). This proves the assertion. \blacksquare

Recall that $\Delta_{\mathfrak{u}}^{\geq}$ denotes the monoid generated by the roots of \mathfrak{u} ; it satisfies $\Delta_{\mathfrak{u}}^{\geq} \cap Q_l = 0$. The monoid generated by the roots of \mathfrak{u}^- is therefore $-\Delta_{\mathfrak{u}}^{\geq}$.

Lemma 5.7. *If v is a non-zero weight vector in \mathcal{P} , then $(U(\mathfrak{u}) \cdot v) \cap (1 \otimes \mathcal{C})$ is non-zero.*

Proof. Suppose that $U(\mathfrak{u}) \cdot v$ has zero intersection with $1 \otimes \mathcal{C}$. Acting with $U(l)$ will not change this because it just adds elements of Q_l to the weights and the weights of \mathcal{C} are already a shift of Q_l . Moreover, acting with $U(\mathfrak{u}^-)$ will not change this intersection either because $-\Delta_{\mathfrak{u}}^{\geq} \cap Q_l = 0$. It now follows from Poincaré–Birkhoff–Witt that the $U(\mathfrak{g})$ -submodule generated by v has zero intersection with $1 \otimes \mathcal{C}$ and is therefore 0, by definition of $\mathcal{P} = \mathcal{I}_p\mathcal{C}$. \blacksquare

We now prove Theorem 5.3.

Proof of Theorem 5.3. Recall the decomposition (3.2) of \mathcal{P} into \mathfrak{g} -submodules \mathcal{P}_λ , $\lambda \in \mathfrak{h}^*/Q_{\mathfrak{g}}$, which need not be simple. We will show that each of the \mathcal{P}_λ are annihilated by l . To do this, fix $\lambda \in \mathfrak{h}^*/Q_{\mathfrak{g}}$ and let \mathcal{M}_i , $i \in I$, denote the simple quotients of \mathcal{P}_λ . We recall that the \mathcal{P}_λ are finite-length (Proposition 3.4), hence I is non-empty. Since each \mathcal{M}_i is isomorphic to a submodule of the semisimplification of \mathcal{P} , which contains an l -bounded highest-weight Z -module by hypothesis, it follows from Theorem 4.5 that \mathcal{M}_i is likewise a Z -module.

Thus, $l \cdot \mathcal{M}_i = 0$ for all $i \in I$, hence $l \cdot \mathcal{P}_\lambda \subseteq \text{rad } \mathcal{P}_\lambda$, the radical of \mathcal{P}_λ (the intersection of its maximal proper submodules).

Suppose now that $\mu \in \text{ess-supp}(\mathcal{R}_p \mathcal{M}_i)$. Then, we have

$$\dim \mathcal{M}_i(\mu) \geq \dim(\mathcal{R}_p \mathcal{M}_i)(\mu) = \dim \mathcal{C}_\lambda(\mu) = \dim \mathcal{P}_\lambda(\mu), \quad (5.2)$$

by Proposition 2.6c and Proposition 3.5a, which establishes the equality $\dim \mathcal{M}_i(\mu) = \dim \mathcal{P}_\lambda(\mu)$. We conclude that $(\text{rad } \mathcal{P}_\lambda)(\mu) = 0$ whenever $\mu \in \text{ess-supp}(\mathcal{R}_p \mathcal{M}_i)$ for some $i \in I$.

Lemma 5.4 ensures that there exist a finite number of elements U_1, \dots, U_n that generate l as a left-ideal of $U(\mathfrak{g})$. Without loss of generality, we may take these elements to be weight vectors of $U(\mathfrak{g})$, denoting their weights by $\nu_j, j = 1, \dots, n$. For each j , define a set $S_j \subset \mathfrak{h}^*$ by

$$S_j = (\nu_j + \Delta_{\mathfrak{u}}^{\geq}) \cap \mathcal{Q}_l \quad (5.3)$$

and note that each S_j is finite, a fact that is easily established by expanding ν_j in a basis of \mathfrak{h}^* consisting of roots of l and \mathfrak{u} . Let S denote the union of the S_j . As each \mathcal{M}_i is simple and l -bounded, by Lemma 5.5, it now follows from Lemma 5.6 (with $\mathcal{N} = \mathcal{R}_p \mathcal{M}_i$) that there exists a weight $\mu_i \in \text{ess-supp}(\mathcal{R}_p \mathcal{M}_i)$, for each $i \in I$, such that $\mu_i + S \subset \text{ess-supp}(\mathcal{R}_p \mathcal{M}_i)$.

Recall that $U_j \cdot \mathcal{P}_\lambda(\mu_i) \subseteq (\text{rad } \mathcal{P}_\lambda)(\mu_i + \nu_j)$. We want to show that the right-hand side, and thus the left-hand side, of this inclusion is zero. To do so, act with $U(\mathfrak{u})$ in order to bring the weight $\mu_i + \nu_j$ back to an element of $\mu_i + \mathcal{Q}_l$. In other words, consider the subspace $(U(\mathfrak{u}) \cdot U_j \cdot \mathcal{P}_\lambda(\mu_i)) \cap (\mathbb{1} \otimes \mathcal{C}_\lambda)$ corresponding to weights lying in $(\mu_i + \nu_j + \Delta_{\mathfrak{u}}^{\geq}) \cap (\mu_i + \mathcal{Q}_l) = \mu_i + S_j \subset \text{ess-supp}(\mathcal{R}_p \mathcal{M}_i)$. We conclude that

$$(U(\mathfrak{u}) \cdot U_j \cdot \mathcal{P}_\lambda(\mu_i)) \cap (\mathbb{1} \otimes \mathcal{C}_\lambda) \subseteq (\text{rad } \mathcal{P}_\lambda)(\mu_i + S_j) = 0, \quad (5.4)$$

because the radical vanishes for weights in the essential support of some $\mathcal{R}_p \mathcal{M}_i$. It now follows from Lemma 5.7 that $U_j \cdot \mathcal{P}_\lambda(\mu_i) = 0$, for each $j = 1, \dots, n$, as desired.

Finally, the U_j generate l as a left-ideal, so $l \cdot \mathcal{P}_\lambda(\mu_i) = 0$ for all $i \in I$. As l is also a right-ideal of $U(\mathfrak{g})$, it therefore annihilates the submodule of \mathcal{P}_λ generated by each $\mathcal{P}_\lambda(\mu_i), i \in I$. It therefore annihilates the sum of these submodules, which is clearly \mathcal{P}_λ . \mathcal{P}_λ is thus a \mathbb{Z} -module, for all $\lambda \in \mathfrak{h}^*/\mathcal{Q}_{\mathfrak{g}}$, hence so is \mathcal{P} . \blacksquare

Our Main Theorem 2 is obtained by applying the induction functor of Zhu and Li to the parabolic family of \mathbb{Z} -modules guaranteed by Theorem 5.3 (see Theorem 1.1b). In this application, \mathbb{Z} is taken to be the Zhu algebra of an affine vertex operator algebra $V_{\mathfrak{k}}(\mathfrak{g})$ (as in Section 1.3).

It is natural to consider a direct proof of Theorem 4.5 using the twisted localisation functors introduced by Mathieu in [2] and we hope to come back to this point in the future. This leads us to ask if Theorem 5.3 can likewise be proved using localisation. This is unclear to us at present because it is not obvious that every α -bijective parabolic family can be constructed in this fashion.

6. COHERENT FAMILIES OF SIMPLE LIE ALGEBRAS OF AC-TYPE

In this section, we recall Mathieu's explicit classification [2] of irreducible semisimple coherent families over a finite-dimensional simple Lie algebra \mathfrak{g} . As mentioned above, there are no coherent families if \mathfrak{g} is not of type A or C. For this classification, it will be convenient to introduce some terminology for weights $\lambda \in \mathfrak{h}^*$. In particular, we say that λ is *integral*, *shifted-singular* or *shifted-regular* if $\lambda + \rho$ belongs to the weight lattice $\mathcal{P}_{\mathfrak{g}}$, lies on a Weyl chamber wall, or lies in the interior of a Weyl chamber, respectively.

Choose a Borel subalgebra of \mathfrak{g} , hence a notion of being highest-weight. Let $\mathcal{B}_{\mathfrak{g}}$ denote the set of weights $\lambda \in \mathfrak{h}^*$ such that the simple highest-weight \mathfrak{g} -module of highest weight λ is bounded. As semisimple coherent families are invariant under the action of the Weyl group [2, Prop. 6.2], it does not matter which choice of Borel we make. Note that the set $\mathcal{B}_{\mathfrak{g}}$ is empty if \mathfrak{g} is not of type A or C (Proposition 2.6b and d).

6.1. **Type A.** Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ with $n \geq 1$. With respect to our chosen Borel, we have simple roots $\alpha_1, \dots, \alpha_n$, highest root θ , Weyl vector ρ and dominant integral weights $P_{\mathfrak{g}}^{\geq}$. For $\lambda \in \mathfrak{h}^*$, set

$$A(\lambda) = \{i \in \{1, \dots, n\} : (\lambda + \rho, \alpha_i^{\vee}) \notin \mathbb{Z}_{>0}\}, \quad (6.1)$$

where the Killing form is normalised so that $(\theta, \theta) = 2$. Note that $A(\lambda) = \emptyset$ if and only if $\lambda \in P_{\mathfrak{g}}^{\geq}$.

Proposition 6.1 (Mathieu [2, Lem. 8.1 and Prop. 8.5]). *For $\mathfrak{g} = \mathfrak{sl}_{n+1}$, the set $\mathcal{B} = \mathcal{B}_{\mathfrak{g}}$ consists of the elements $\lambda \in \mathfrak{h}^*$ that satisfy at least one of the following conditions:*

- (a) $A(\lambda) = \{1\}$ or $\{n\}$.
- (b) $A(\lambda) = \{i\}$ with $1 < i < n$ and either $(\lambda + \rho, \alpha_i^{\vee} + \alpha_{i-1}^{\vee}) \in \mathbb{Z}_{>0}$ or $(\lambda + \rho, \alpha_i^{\vee} + \alpha_{i+1}^{\vee}) \in \mathbb{Z}_{>0}$.
- (c) $A(\lambda) = \{i, i+1\}$ with $1 \leq i < n$ and $(\lambda + \rho, \alpha_i^{\vee} + \alpha_{i+1}^{\vee}) \in \mathbb{Z}_{>0}$.

For example, only (a) applies when $\mathfrak{g} = \mathfrak{sl}_2$, hence $\mathcal{B}_{\mathfrak{sl}_2} = \mathfrak{h}^* \setminus P_{\mathfrak{sl}_2}^{\geq}$ is the set of weights whose Dynkin label is not a non-negative integer. For $\mathfrak{g} = \mathfrak{sl}_3$, (b) does not apply and $\mathcal{B}_{\mathfrak{sl}_3}$ is the union of two sets: one consisting of the weights that have precisely one non-negative integer Dynkin label and the other consisting of the weights with no non-negative integer Dynkin labels but for which the sum of the Dynkin labels lies in $\mathbb{Z}_{\geq -1}$.

For $\lambda, \mu \in \mathcal{B}$, we write $\lambda \rightarrow \mu$ if there exists $i \in A(\lambda)$ with $\mu = s_i \cdot \lambda$. Here, s_i is the simple reflection w_{α_i} of the Weyl group $W \simeq S_{n+1}$ of \mathfrak{g} and \cdot denotes the shifted action of W on \mathfrak{h}^* : $w \cdot \lambda = w(\lambda + \rho) - \rho$. For $\mathfrak{g} = \mathfrak{sl}_2$, we thus have $\lambda \rightarrow s_1 \cdot \lambda$ whenever the Dynkin label λ_1 of $\lambda \in \mathcal{B}_{\mathfrak{sl}_2}$ is not in $\mathbb{Z}_{\leq -2}$; however when $\lambda_1 \in \mathbb{Z}_{\leq -2}$, there is no $\mu \in \mathcal{B}_{\mathfrak{sl}_2}$ satisfying $\lambda \rightarrow \mu$. The relation \rightarrow is obviously not an equivalence relation, but it defines on \mathcal{B} the structure of a directed graph: the vertices are the weights of \mathcal{B} and we have an edge from λ to μ if and only if $\lambda \rightarrow \mu$. We shall denote the set of connected components of this graph by $\mathcal{B}/(\rightarrow)$.

We have the following classification result.

Proposition 6.2 (Mathieu [2, Thm. 8.6]). *There is a bijective correspondence between the set of (equivalence classes of) irreducible semisimple coherent families of \mathfrak{sl}_{n+1} -modules and the set $\mathcal{B}/(\rightarrow)$ of connected components of \mathcal{B} . This correspondence sends an irreducible semisimple coherent family \mathcal{C} to the set*

$$\{\lambda \in \mathfrak{h}^* : \lambda \notin P_{\mathfrak{g}}^{\geq} \text{ and } \mathcal{L}_{\lambda} \subset \mathcal{C}\} \in \mathcal{B}/(\rightarrow) \quad (6.2)$$

of highest weights of infinite-dimensional highest-weight submodules of \mathcal{C} .

This shows that irreducible semisimple coherent families of \mathfrak{sl}_{n+1} -modules are completely characterised by their bounded highest-weight submodules (and in fact, a single representative will do). Because all elements of $U(\mathfrak{g})^{\mathfrak{h}}$ act polynomially on a given coherent family \mathcal{C} , it follows that each element of the centre $U(\mathfrak{g})^{\mathfrak{h}}$, that is each Casimir operator, acts as a constant on \mathcal{C} . In other words, \mathcal{C} has a definite central character. It is therefore natural to ask whether the central character also completely characterises an irreducible semisimple coherent family. The answer is interesting: “usually, but not always”.

We recall that two highest-weight modules have the same central character if and only if their highest weights are related by the shifted action of W . Given $\lambda \in \mathcal{B}$, the question asked above amounts to deciding whether $(W \cdot \lambda) \cap \mathcal{B}$ is a single connected component in \mathcal{B} or not.

Proposition 6.3 (Mathieu [2, Lem. 8.3]).

- (a) *If $\lambda \in \mathcal{B}$ is integral, then the connected component $[\lambda] \in \mathcal{B}/(\rightarrow)$ has n elements. Otherwise, $[\lambda]$ has $n+1$ elements.*
- (b) *The intersection $(W \cdot \lambda) \cap \mathcal{B}$ is a single connected component in \mathcal{B} unless λ is shifted-regular and integral, in which case it is the union of n connected components.*

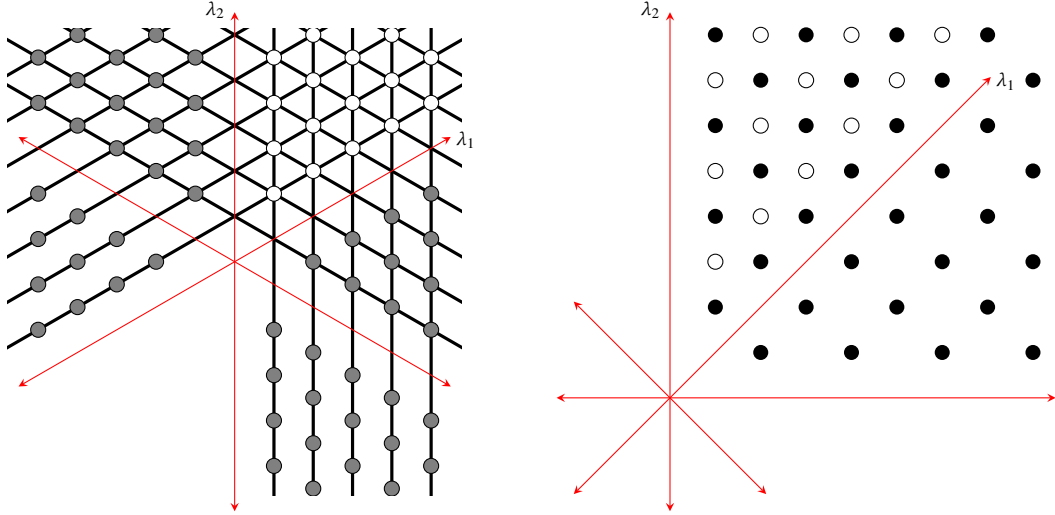


FIGURE 1. At left, the set $\mathcal{B}_{\mathfrak{sl}_3}$, depicted as black lines, in (the real slice of) the weight space. The white circles correspond to the dominant integral weights, which are not in $\mathcal{B}_{\mathfrak{sl}_3}$, while the grey circles indicate the shifted-regular integral weights in $\mathcal{B}_{\mathfrak{sl}_3}$. The red lines correspond to the shifted Weyl chamber walls. At right, the set $\mathcal{B}_{\mathfrak{sp}_4}$, with the same conventions except that $\mathcal{B}_{\mathfrak{sp}_4}$ is discrete and is therefore represented by the black circles.

We conclude that an irreducible semisimple coherent family of \mathfrak{sl}_{n+1} -modules is completely characterised by its central character unless its highest-weight submodules have shifted-regular integral highest weights (and if one does, then they all do).

We illustrate these ideas for $\mathfrak{g} = \mathfrak{sl}_2$. In this case, the connected components of $\mathcal{B}_{\mathfrak{sl}_2} = \mathfrak{h}^* \setminus \mathcal{P}_{\mathfrak{sl}_2}^{\geq}$ have the form $[\lambda] = \{\lambda\}$, if λ is integral, and $[\lambda] = \{\lambda, s_1 \cdot \lambda\}$ otherwise. The set of connected components of $\mathcal{B}_{\mathfrak{sl}_2}$ thus decomposes into shifted-regular integral, shifted-singular integral, and non-integral weights as follows:

$$\mathcal{B}_{\mathfrak{sl}_2}/(\rightarrow) = \bigcup_{\lambda \in \mathbb{Z}_{\leq -2}} \{\lambda\omega_1\} \cup \{-\omega_1\} \cup \bigcup_{\lambda \in \mathbb{C} \setminus \mathbb{Z}} \{\lambda\omega_1, -(\lambda+2)\omega_1\}. \quad (6.3)$$

Moreover, the central character always completely characterises the coherent families. While there exist shifted-regular integral weights in $\mathcal{B}_{\mathfrak{sl}_2}$ (those with $\lambda \in \mathbb{Z}_{\leq -2}$), the (partial) W -orbits $(W \cdot \lambda) \cap \mathcal{B}_{\mathfrak{sl}_2} = \{\lambda\}$ coincide with the connected components in this case, consistent with Proposition 6.3 (because $n = 1$).

The case $\mathfrak{g} = \mathfrak{sl}_3$ is more typical and we illustrate the set $\mathcal{B}_{\mathfrak{sl}_3}$ in Figure 1 for convenience. As before, $\mathcal{B}_{\mathfrak{sl}_3}$ is partitioned into shifted-regular integral, shifted-singular integral, and non-integral weights. It is easy to see that each non-integral weight $\lambda \in \mathcal{B}_{\mathfrak{sl}_3}$ gives rise to a length-6 (shifted) W -orbit whose intersection with $\mathcal{B}_{\mathfrak{sl}_3}$ consists of three weights and represents one connected component. The shifted-singular integral weights correspond to the intersections of the red and black lines. This singularity means that the W -orbit's length is only 3, but one element necessarily lies outside $\mathcal{B}_{\mathfrak{sl}_3}$. The remaining two weights again form a single connected component. Finally, each shifted-regular integral weight yields a length-6 W -orbit whose intersection with $\mathcal{B}_{\mathfrak{sl}_3}$ has four elements. Because weights linked by \rightarrow must be related by a simple Weyl reflection, the intersection splits into two connected components of two elements each.

6.2. Type C. The situation is somewhat more straightforward for $\mathfrak{g} = \mathfrak{sp}_{2n}$ (with $n \geq 2$). We fix an ordering of the simple roots in which consecutive roots are connected in the Dynkin diagram, $\alpha_1, \dots, \alpha_{n-1}$ are short and α_n is long.

Proposition 6.4 (Mathieu [2, Lems. 9.1 and 9.2]). *For $\mathfrak{g} = \mathfrak{sp}_{2n}$, the set $\mathcal{B} = \mathcal{B}_{\mathfrak{g}}$ consists of the elements $\lambda \in \mathfrak{h}^*$ which satisfy all of the following conditions:*

- (a) $(\lambda, \alpha_i^\vee) \in \mathbb{Z}_{\geq 0}$ for any $i \neq n$.
- (b) $(\lambda, \alpha_n^\vee) \in \mathbb{Z} + \frac{1}{2}$.
- (c) $(\lambda, \alpha_{n-1}^\vee + 2\alpha_n^\vee) \in \mathbb{Z}_{\geq -2}$.

Note that \mathcal{B} is clearly discrete in this case. We illustrate $\mathcal{B}_{\mathfrak{sp}_4}$ in Figure 1 (right).

For $\lambda, \mu \in \mathcal{B}$, we write $\lambda \rightarrow \mu$ if $\mu = s_n \cdot \lambda$, where we recall that the Weyl group of \mathfrak{sp}_{2n} is $W \simeq S_n \times \mathbb{Z}_2^n$.

Proposition 6.5 (Mathieu [2, Thm. 9.3]).

- (a) *There is a bijective correspondence between the set of (equivalence classes of) irreducible semisimple coherent families of \mathfrak{sp}_{2n} -modules and the set $\mathcal{B}/(\rightarrow)$ of connected components in \mathcal{B} . This correspondence sends an irreducible semisimple coherent family \mathcal{C} to the set*

$$\{\lambda \in \mathfrak{h}^* : \lambda \notin P_{\mathfrak{g}}^{\geq} \text{ and } \mathcal{L}_\lambda \subset \mathcal{C}\} \in \mathcal{B}/(\rightarrow) \quad (6.4)$$

of highest weights of infinite-dimensional highest-weight submodules of \mathcal{C} .

- (b) *Every connected component $[\lambda] \in \mathcal{B}/(\rightarrow)$ has 2 elements and the intersection $(W \cdot \lambda) \cap \mathcal{B}$ is always a single connected component in \mathcal{B} .*

We conclude that an irreducible semisimple coherent family of \mathfrak{sp}_{2n} -modules is always completely characterised by its central character.

7. THE COMBINATORICS OF CLASSIFYING WEIGHT MODULES

To apply our classification in concrete examples, we first need to clarify which (infinite-dimensional) highest-weight \mathfrak{g} -modules appear in any given parabolic family. Recall that the Weyl group W of \mathfrak{g} is generated by the reflections w_α , $\alpha \in \Delta_{\mathfrak{g}}$. We recall the definition of the small Weyl group, following [57].

Definition 7.1. *Given a simple weight \mathfrak{g} -module \mathcal{M} , let $\Delta_{\mathcal{M}}^{\text{nil}}$ denote the set of roots $\alpha \in \Delta_{\mathfrak{g}}$ whose positive and negative root vectors act locally nilpotently on \mathcal{M} . The small Weyl group $W_{\mathcal{M}}$ of \mathcal{M} is then the subgroup of W generated by the w_α with $\alpha \in \Delta_{\mathcal{M}}^{\text{nil}}$.*

The small Weyl group of a simple dense \mathfrak{g} -module is therefore trivial because all root vectors act injectively. It is easy to see that the action of a root vector on any simple \mathfrak{g} -module is either injective or locally nilpotent (the set of vectors on which the action is locally nilpotent is a submodule).

An easy way to appreciate the small Weyl group is to look at the case in which \mathcal{M} is a simple highest-weight \mathfrak{sl}_2 -module, with respect to some Borel subalgebra \mathfrak{b} . Then, there are two possibilities:

- (a) \mathcal{M} is finite-dimensional, so $\Delta_{\mathcal{M}}^{\text{nil}} = \Delta_{\mathfrak{sl}_2}$ and $W_{\mathcal{M}} = W \simeq \mathbb{Z}_2$.
- (b) \mathcal{M} is infinite-dimensional, so $\Delta_{\mathcal{M}}^{\text{nil}} = \emptyset$ and $W_{\mathcal{M}} = 1$.

There are of course $|W| = 2$ choices of Borel (containing our fixed Cartan subalgebra \mathfrak{h}). In the first case, \mathcal{M} is highest-weight with respect to either choice of Borel; in the second, only one choice makes \mathcal{M} highest-weight. Now consider this from the perspective of the parabolics (in this case, Borels). A simple highest-weight module \mathcal{M} has the form $\mathcal{I}_{\mathfrak{b}}\mathbb{C}_\lambda$, for some simple \mathfrak{h} -module \mathbb{C}_λ (λ is the highest weight of \mathcal{M}), and some Borel \mathfrak{b} . If \mathcal{M} is finite-dimensional, then it is also highest-weight with respect to the other Borel $w(\mathfrak{b})$, though its highest weight is no longer λ but $w(\lambda) = -\lambda$. Thus, $\mathcal{I}_{\mathfrak{b}}\mathbb{C}_\lambda \simeq \mathcal{M} \simeq \mathcal{I}_{w(\mathfrak{b})}\mathbb{C}_{w(\lambda)}$ when $w \in W_{\mathcal{M}}$.

In general, the small Weyl group describes exactly this lack of uniqueness in representing a simple module through parabolic induction. Recall from Theorem 3.2 that every simple weight \mathfrak{g} -module \mathcal{M} , with finite-dimensional weight spaces, has the form $\mathcal{M} \simeq \mathcal{I}_{\mathfrak{p}}\mathcal{N}$, for some parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ and some simple dense module \mathcal{N} over the Levi factor \mathfrak{l} of \mathfrak{p} .

Lemma 7.2 (Dimitrov–Mathieu–Penkov [57, Thm. 6.1]). *Given a simple weight \mathfrak{g} -module \mathcal{M} , with finite-dimensional weight spaces, the choice of $(\mathfrak{p}, \mathcal{N})$ is unique up to the action of the small Weyl group $W_{\mathcal{M}}$.*

In other words, if we also have $\mathcal{M} \simeq \mathcal{I}_{\mathfrak{p}'}\mathcal{N}'$ for some parabolic $\mathfrak{p}' \subseteq \mathfrak{g}$ and some simple dense module \mathcal{N}' over the Levi factor of \mathfrak{p}' , then there exists $w \in W_{\mathcal{M}}$ such that $\mathfrak{p}' = w(\mathfrak{p})$ and $\mathcal{N}' \simeq w(\mathcal{N})$.

At this point, it is convenient to describe an often more practical means of computing the small Weyl group of a simple weight \mathfrak{g} -module \mathcal{M} . Let $\mathfrak{p} \subseteq \mathfrak{g}$ be a parabolic subalgebra with Levi factor \mathfrak{l} . We choose a set of simple roots $\Pi_{\mathfrak{g}}$ of \mathfrak{g} such that the corresponding root vectors all belong to \mathfrak{p} . This ensures, in particular, that $\Pi_{\mathfrak{g}}$ includes a set $\Pi_{\mathfrak{l}}$ of simple roots of \mathfrak{l} . Define $\Pi_{\mathfrak{l}}^{\perp}$ to be the subset of $\Pi_{\mathfrak{g}}$ consisting of the simple roots that are orthogonal to those of \mathfrak{l} . The simple coroot α_i^{\vee} corresponding to any $\alpha_i \in \Pi_{\mathfrak{l}}^{\perp}$ therefore acts on $\mathcal{R}_{\mathfrak{p}}\mathcal{M}$ as multiplication by $\lambda_i = (\lambda, \alpha_i^{\vee})$, where λ is any weight of $\text{supp}(\mathcal{R}_{\mathfrak{p}}\mathcal{M})$, by Proposition 3.1a and Schur's lemma.

Proposition 7.3 (Mathieu [2, Prop. 1.3(ii)]). *Suppose that $\mathcal{M} \simeq \mathcal{I}_{\mathfrak{p}}\mathcal{N}$, for some simple dense \mathfrak{l} -module \mathcal{N} . Then, the small Weyl group $W_{\mathcal{M}}$ is the subgroup of W generated by the simple Weyl reflections s_i with $\alpha_i \in \Pi_{\mathfrak{l}}^{\perp}$ and $\lambda_i \in \mathbb{Z}_{\geq 0}$.*

Proof. As we were not able to find a detailed proof of this useful result in the literature, we provide one for the reader's convenience. Let $\Pi(\mathcal{M})$ denote the set of $\alpha_i \in \Pi_{\mathfrak{l}}^{\perp}$ for which $\lambda_i \in \mathbb{Z}_{\geq 0}$. This set of simple roots corresponds to a Lie subalgebra $\mathfrak{g}(\mathcal{M})$ of \mathfrak{g} with root system $\Delta(\mathcal{M})$. We shall prove the proposition by showing that $\Delta(\mathcal{M}) = \Delta_{\mathcal{M}}^{\text{nil}}$.

Suppose first that α is a positive root in $\Delta_{\mathcal{M}}^{\text{nil}}$. As the root vectors of \mathfrak{l} act injectively on the simple dense \mathfrak{l} -module $\mathcal{R}_{\mathfrak{p}}\mathcal{M} \subset \mathcal{M}$, we must have $e^{\alpha} \in \mathfrak{u}$. Thus, e^{α} annihilates any weight vector $v \in \mathcal{R}_{\mathfrak{p}}\mathcal{M}$. If λ is the weight of v , then the action of f^{α} on v is only locally nilpotent when $(\lambda, \alpha^{\vee}) \in \mathbb{Z}_{\geq 0}$.

If there exists $\beta \in \Delta_{\mathfrak{l}}$ with $(\beta, \alpha^{\vee}) \neq 0$, then without loss of generality we may assume that this quantity is positive. Since f^{β} acts injectively, it follows that $(f^{\beta})^n v$ is a non-zero element of the \mathfrak{l} -module $\mathcal{R}_{\mathfrak{p}}\mathcal{M}$, for any $n \in \mathbb{Z}_{\geq 0}$. The actions of e^{α} and f^{α} on $(f^{\beta})^n v = 0$ are therefore zero and locally nilpotent, respectively, for all $n \in \mathbb{Z}_{\geq 0}$, hence we must have $(\lambda - n\beta, \alpha^{\vee}) \in \mathbb{Z}_{\geq 0}$ for all $n \in \mathbb{Z}_{\geq 0}$. Since $(\beta, \alpha^{\vee}) > 0$, this is a contradiction, proving that $(\beta, \alpha^{\vee}) = 0$ for all $\beta \in \Delta_{\mathfrak{l}}$.

It remains to show that α is a linear combination of simple roots in $\Delta(\mathcal{M})$. For this, we induct over the height of α . If the height is 1, then α is simple and the conditions established above prove that it is in $\Delta(\mathcal{M})$. We may therefore assume that the height of α is greater than 1, so $\alpha = \beta + \gamma$ for some $\beta, \gamma \in \Delta_{\mathfrak{g}}$, and that the statement has been proven for all roots in $\Delta_{\mathcal{M}}^{\text{nil}}$ of lower height than α . There are three cases to consider:

- 1) f^{β} and f^{γ} act injectively on \mathcal{M} . Then, $-\alpha = -\beta - \gamma \in C_{\mathcal{M}}$, the cone (monoid) generated by the roots whose root vectors act injectively on \mathcal{M} . But, this contradicts [57, Lem. 5.1(iii)] which says that $C_{\mathcal{M}}$ has trivial intersection with the root lattice $\mathbb{Q}_{\mathcal{M}}^{\text{nil}}$ generated by $\Delta_{\mathcal{M}}^{\text{nil}}$. This case is therefore impossible.
- 2) f^{β} acts injectively on \mathcal{M} whilst f^{γ} acts locally nilpotently. But then, $-\beta = \gamma - \alpha \in \mathbb{Q}_{\mathcal{M}}^{\text{nil}}$ which again contradicts [57, Lem. 5.1(iii)] and is thus impossible.
- 3) f^{β} and f^{γ} act locally nilpotently on \mathcal{M} . Then, $\beta, \gamma \notin \Delta_{\mathfrak{l}}$ so $e^{\beta}, e^{\gamma} \in \mathfrak{u}$ and so $\beta, \gamma \in \Delta_{\mathcal{M}}^{\text{nil}}$. By induction, β and γ are linear combinations of simple roots in $\Delta(\mathcal{M})$ and hence so is α .

This establishes that $\Delta_{\mathcal{M}}^{\text{nil}} \subseteq \Delta(\mathcal{M})$.

For the opposite inclusion, note that it is enough to prove that e^{α} and f^{α} , for $\alpha \in \Delta(\mathcal{M})$ positive, act locally nilpotently on a single non-zero vector $v \in \mathcal{M}$, since \mathcal{M} is simple (Proposition 3.1b). Noting that α being orthogonal to $\Delta(\mathcal{M})$ implies that $e^{\alpha} \in \mathfrak{u}$, we may take $v \in \mathcal{R}_{\mathfrak{p}}\mathcal{M}$ so that $e^{\alpha}v = 0$.

To show that f^{α} acts locally nilpotently on v requires more work. We will actually show something a little stronger, namely that the $\mathfrak{g}(\mathcal{M})$ -module \mathcal{D} generated from v is finite-dimensional. Let λ be the weight of v and recall that $\lambda_i \in \mathbb{Z}_{\geq 0}$ for all $\alpha_i \in \Pi(\mathcal{M})$. We will prove that the only $w \in \mathcal{D}$ which are annihilated by the e^{β} , with $\beta \in \Delta(\mathcal{M})$ positive, are multiples of v .

Let us therefore assume that such a w exists, but is not a multiple of v . It is therefore not in $\mathcal{R}_p\mathcal{M}$. However, as \mathcal{M} is simple (as a \mathfrak{g} -module), there exists $U \in \mathcal{U}(\mathfrak{g})$ such that $Uw = v$. We can write U as a linear combination of monomials which are ordered as follows: root vectors of \mathfrak{u}^- appear to the left of negative root vectors of \mathfrak{l} , which appear to the left of positive root vectors of \mathfrak{l} ; finally, root vectors of \mathfrak{u} appear to the right. As $v \in \mathcal{R}_p\mathcal{M}$, we may in fact assume that no root vectors of \mathfrak{u}^- appear. Similarly, as $w \notin \mathcal{R}_p\mathcal{M}$, we may assume that at least one root vector of \mathfrak{u} appears in each monomial. This may be sharpened by partitioning the positive roots of \mathfrak{g} into those of \mathfrak{l} , those of $\mathfrak{g}(\mathcal{M})$ and the remainder $\tilde{\Delta}^+$. As the positive root vectors of $\mathfrak{g}(\mathcal{M})$ annihilate w by hypothesis, we may thus assume that at least one e^β , with $\beta \in \tilde{\Delta}^+$, appears in each monomial.

It follows from the orthogonality of $\Pi_{\mathfrak{l}}$ and $\Pi(\mathcal{M})$ that any given $\beta \in \tilde{\Delta}^+$ may be decomposed as $\alpha_i + \beta'$, where α_i is in the complement $\tilde{\Pi}$ of $\Pi_{\mathfrak{l}} \sqcup \Pi(\mathcal{M})$ in $\Pi_{\mathfrak{g}}$ and β' is a non-negative-integer linear combination of simple roots. The weight of U is therefore a linear combination of simple roots in which the coefficient of α_i is positive. However, this contradicts $w \in \mathcal{D} = \mathcal{U}(\mathfrak{g}(\mathcal{M}))v$ because the latter implies that only the coefficients of the simple roots of $\mathfrak{g}(\mathcal{M})$ can be non-zero. This contradiction establishes that the vector w does not exist, hence that f^α acts locally nilpotently on v as required. \blacksquare

We note that computing $W_{\mathcal{M}}$ can be done directly at the level of the Dynkin diagrams $\Gamma_{\mathfrak{g}}$ and $\Gamma_{\mathfrak{s}}$ of \mathfrak{g} and the semisimple subalgebra $\mathfrak{s} = [\mathfrak{l}, \mathfrak{l}]$ of \mathfrak{l} , respectively. The latter is of course the subdiagram of $\Gamma_{\mathfrak{g}}$ consisting of the nodes corresponding to the simple roots $\Pi_{\mathfrak{l}} \subseteq \Pi_{\mathfrak{g}}$ and the edges connecting them. The simple roots $\Pi_{\mathfrak{l}}^{\perp}$ thus correspond to the nodes of $\Gamma_{\mathfrak{g}}$ that are neither in $\Gamma_{\mathfrak{s}}$ nor are directly connected to any node in $\Gamma_{\mathfrak{s}}$. Moreover, for each such node, the scalar λ_i is just the (necessarily common) Dynkin label of the weights of $\mathcal{R}_p\mathcal{M}$.

We illustrate this with a simple example: $\mathfrak{g} = \mathfrak{sl}_4$ and $\mathcal{M} = \mathcal{S}_p\mathcal{N}$, where \mathfrak{p} is the parabolic subalgebra of \mathfrak{g} corresponding to α_1 and \mathcal{N} is a simple dense module over the Levi factor $\mathfrak{l} \simeq \mathfrak{sl}_2 \oplus \mathfrak{gl}_1^{\oplus 2}$. Since the Dynkin diagram of $\mathfrak{s} \simeq \mathfrak{sl}_2$ is realised as the first node of that of \mathfrak{sl}_4 , we see that $\Pi_{\mathfrak{l}}^{\perp} = \{\alpha_3\}$ as the second node is directly connected to the first. If the third Dynkin label of any (and thus every) weight of \mathcal{N} is a non-negative integer, then the small Weyl group of \mathcal{M} is $W_{\mathcal{M}} = \langle s_3 \rangle \simeq \mathbb{Z}_2$; otherwise, it is $\mathbb{1}$.

Given a semisimple weight module $\mathcal{M} = \bigoplus_i \mathcal{M}_i$, where the \mathcal{M}_i are simple and weight with finite-dimensional weight spaces, we define the small Weyl group of \mathcal{M} to be $W_{\mathcal{M}} = \bigcap_i W_{\mathcal{M}_i}$. In particular, consider the small Weyl group of an irreducible semisimple parabolic family $\mathcal{P} = \mathcal{S}_p\mathcal{C}$ of \mathfrak{g} -modules, where \mathfrak{p} has non-abelian Levi factor \mathfrak{l} and \mathcal{C} is a coherent family of \mathfrak{l} -modules. The following proposition now follows from Theorem 3.2 and Lemma 7.2.

Proposition 7.4. *Given a parabolic family \mathcal{P} of \mathfrak{g} -modules, the choice of $(\mathfrak{p}, \mathcal{C})$ is unique up to the action of the small Weyl group $W_{\mathcal{P}}$.*

As one might hope, the simple dense submodules of a parabolic family of \mathfrak{g} -modules all have the same small Weyl group. We may therefore compute $W_{\mathcal{P}}$ using the method discussed above. More importantly, we do not have to perform uncountably many computations in order to deduce the small Weyl groups of all its simple submodules.

Proposition 7.5. *Let \mathcal{N} be a simple dense \mathfrak{l} -submodule of \mathcal{C} and $\mathcal{M} = \mathcal{S}_p\mathcal{N}$ the corresponding simple submodule of $\mathcal{P} = \mathcal{S}_p\mathcal{C}$. Then, the small Weyl group of \mathcal{M} is contained in the small Weyl group of every submodule of \mathcal{P} . In particular, $W_{\mathcal{P}} = W_{\mathcal{M}}$.*

Proof. Choose a positive root $\alpha \in \Delta_{\mathcal{M}}^{\text{nil}}$. Then, $e^\alpha \in \mathfrak{u}$ so $e^\alpha v = 0$ for all $v \in \mathcal{R}_p\mathcal{M} = \mathcal{N}$. Moreover, because the action of f^α on v is locally nilpotent, $\alpha^\vee = [e^\alpha, f^\alpha]$ acts on v as multiplication by some non-negative integer λ_α . This integer is v -independent because α^\vee is orthogonal to $\Delta_{\mathfrak{l}}$ (Proposition 7.3). As \mathcal{N} is dense, it is bounded and so its weights are Zariski-dense in $\text{supp}(\mathcal{C})$ (Proposition 2.6a). Because $\alpha^\vee \in \mathfrak{h} \subset \mathcal{U}(\mathfrak{l})^{\mathfrak{b}}$, it acts polynomially on \mathcal{C} . We therefore conclude that α^\vee acts as multiplication by λ_α on all of \mathcal{C} .

In particular, α^\vee acts as multiplication by $\lambda_\alpha \in \mathbb{Z}_{\geq 0}$ on any simple submodule $\mathcal{N}' \subset \mathcal{C} = \mathcal{R}_\mathfrak{p}\mathcal{P}$. Thus, e^α acts on \mathcal{N}' as 0 and so f^α acts locally nilpotently on the subspace \mathcal{N}' of the \mathfrak{g} -module $\mathcal{M}' = \mathcal{I}_\mathfrak{p}\mathcal{N}' \subset \mathcal{P}$. As the action of a root vector on a simple module is either injective or locally nilpotent, this proves that e^α and f^α both act locally nilpotently on \mathcal{M}' . In other words, $\alpha \in \Delta_{\mathcal{M}'}^{\text{nil}}$ and so $\Delta_{\mathcal{M}}^{\text{nil}} \subseteq \Delta_{\mathcal{M}'}^{\text{nil}}$. The small Weyl group of every simple submodule of \mathcal{P} thus contains that of \mathcal{M} , completing the proof. \blacksquare

8. A CLASSIFICATION ALGORITHM

We shall now combine Theorem 4.5 with the theory developed in Sections 6 and 7 to present an algorithm whose input is the classification of simple highest-weight Z -modules and whose output is the classification of all simple weight Z -modules with finite-dimensional weight spaces. In Section 9 below, we shall illustrate this algorithm with several examples in which Z is the Zhu algebra Z_k of a simple affine vertex operator algebra $L_k(\mathfrak{g})$. In this case, the algorithm then implies the classification of the simple relaxed highest-weight $L_k(\mathfrak{g})$ -modules, by Theorem 1.1a, again assuming finite-dimensional weight spaces.

Fix a set of simple roots $\Pi_\mathfrak{g} = \{\alpha_1, \dots, \alpha_r\}$ of \mathfrak{g} , where r is the rank of \mathfrak{g} . We shall refer to a parabolic subalgebra of \mathfrak{g} as being *standard* (with respect to $\Pi_\mathfrak{g}$) if it contains all of the simple root vectors e^{α_i} , $i = 1, \dots, r$. We shall similarly call a parabolic family $\mathcal{P} = \mathcal{I}_\mathfrak{p}\mathcal{C}$ *standard* when it is induced from a coherent family \mathcal{C} over the Levi factor \mathfrak{l} of a standard parabolic \mathfrak{p} . We recall that a standard parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ is completely determined by the set $S \subseteq \{1, \dots, r\}$ of indices i (or Dynkin nodes) for which the negative simple root vectors f^{α_i} also belong to \mathfrak{p} .

Just as every parabolic subalgebra of \mathfrak{g} may be obtained from a standard parabolic subalgebra by acting with the Weyl group W , every parabolic family may similarly be obtained from a standard parabolic family using W . Since coherent families of \mathfrak{l} -modules are invariant under the action of the Weyl group $W_\mathfrak{l} \subseteq W$ of \mathfrak{l} , as is \mathfrak{p} , it follows that the parabolic family \mathcal{P} is also preserved by $W_\mathfrak{l}$. Moreover, the small Weyl group $W_\mathcal{P}$ preserves \mathcal{P} but not necessarily \mathcal{C} or \mathfrak{p} . Thus, the W -orbit of each standard parabolic family \mathcal{P} gives $|W|/(|W_\mathcal{P}||W_\mathfrak{l}|)$ different parabolic families. Indeed, the classification of parabolic families of \mathfrak{g} -modules reduces to that of standard parabolic families and the computation of their small Weyl groups (see Propositions 7.4 and 7.5).

The basic idea of the classification algorithm is to choose a standard parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ and determine which, if any, of the simple highest-weight Z -modules are \mathfrak{l} -bounded. By Theorem 4.5, each such module is contained in an irreducible semisimple standard parabolic family of Z -modules and every simple weight Z -module, with finite-dimensional weight spaces, is contained in a W -twist of such a parabolic family. To assist with determining when a highest-weight module is \mathfrak{l} -bounded, write

$$\mathfrak{l} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_m \oplus \mathfrak{z}, \quad (8.1)$$

where the \mathfrak{s}_i are simple ideals and \mathfrak{z} is the centre of \mathfrak{l} . We let $\pi_{\mathfrak{s}_i}$ denote the orthogonal projection onto \mathfrak{s}_i and let \mathbb{C}_μ denote the one-dimensional \mathfrak{z} -module whose sole weight is μ .

The classification algorithm is then as follows.

Algorithm. *Let \mathfrak{g} be a finite-dimensional simple Lie algebra and let Z be a quotient of $U(\mathfrak{g})$ by a two-sided ideal. Assume that the simple highest-weight Z -modules have been classified.*

- *Consider each non-empty subset $S \subseteq \{1, \dots, r\}$ and determine if the corresponding standard parabolic subalgebra \mathfrak{p} is of AC-type. This is easy to check by looking at the connected components of the Dynkin diagram of \mathfrak{s} , where $\mathfrak{s} = [\mathfrak{l}, \mathfrak{l}]$ and \mathfrak{l} is the Levi factor of \mathfrak{p} .*
- *If \mathfrak{p} is of AC-type, consider the highest weight λ of each simple highest-weight Z -module \mathcal{H} and compute the projections $\pi_{\mathfrak{s}_i}(\lambda)$, $i = 1, \dots, m$, onto the weight spaces of the simple ideals \mathfrak{s}_i of \mathfrak{l} .*

- For each $i = 1, \dots, m$, use Propositions 6.1 and 6.4 to determine whether $\pi_{\mathfrak{s}_i}(\lambda) \in \mathcal{B}_{\mathfrak{s}_i}$. If so, then there is an irreducible semisimple coherent family \mathcal{C}_i of \mathfrak{s}_i -modules containing the simple highest-weight \mathfrak{s}_i -module of highest weight $\pi_{\mathfrak{s}_i}(\lambda)$.
- If \mathcal{C}_i exists for all $i = 1, \dots, m$, then there is an irreducible semisimple standard parabolic family

$$\mathcal{P} = \mathcal{I}_{\mathfrak{p}}(\mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_m \otimes \mathbb{C}_{\mu}), \quad \mu = \lambda - \sum_{i=1}^m \pi_{\mathfrak{s}_i}(\lambda), \quad (8.2)$$

that contains \mathcal{H} . \mathcal{P} is thus a \mathbb{Z} -module, by Theorem 4.5, hence so are all its direct summands.

- Determine which λ give the same parabolic family by using Propositions 6.2, 6.3 and 6.5 to compute the connected components $[\pi_{\mathfrak{s}_i}(\lambda)] \in \mathcal{B}_{\mathfrak{s}_i} / (\rightarrow)$.
- For each irreducible semisimple standard parabolic family \mathcal{P} of \mathbb{Z} -modules found, act with representatives of $\mathbb{W} / (\mathbb{W}_{\mathcal{P}} \times \mathbb{W}_1)$ to obtain a complete set of irreducible semisimple parabolic families of \mathbb{Z} -modules.

Along with the simple highest-weight \mathbb{Z} -modules, the direct summands of the irreducible semisimple parabolic families of \mathbb{Z} -modules found with this algorithm form a complete set, up to isomorphism, of simple weight \mathbb{Z} -modules with finite-dimensional weight spaces.

9. EXAMPLES

In this section, we apply the classification algorithm to some concrete examples of simple vertex operator algebras $L_{\mathfrak{k}}(\mathfrak{g})$ in order to classify the irreducible semisimple standard parabolic (and coherent) families of the corresponding Zhu algebras $Z_{\mathfrak{k}} = \text{Zhu}[L_{\mathfrak{k}}(\mathfrak{g})]$. By Theorem 1.1a, this yields a classification of all the simple relaxed highest-weight modules (with finite-dimensional weight spaces) of the vertex operator algebra. We recall that non-standard parabolic families are obtained by twisting standard ones by elements of the Weyl group of \mathfrak{g} , as described in Propositions 7.4 and 7.5. We use the same notations as in Section 7 and will assume that all the parabolic families considered in this section are both semisimple and irreducible.

9.1. Example: $L_{\mathfrak{k}}(\mathfrak{sl}_2)$ for \mathfrak{k} admissible. We warm up with the familiar case of $\mathfrak{g} = \mathfrak{sl}_2$ with \mathfrak{k} admissible and non-integral:

$$k + 2 = \frac{u}{v}, \quad \text{where } u, v \in \mathbb{Z}_{\geq 2} \text{ and } \gcd\{u, v\} = 1. \quad (9.1)$$

Let α_1 denote the simple root of \mathfrak{sl}_2 , so that $\omega_1 = \frac{1}{2}\alpha_1$ is the fundamental weight.

The simple highest-weight $L_{\mathfrak{k}}(\mathfrak{sl}_2)$ -modules were originally classified in [6, 58], see also [5]. Their Zhu images are the highest-weight \mathfrak{sl}_2 -modules $\mathcal{L}_{r,s}$ of highest weights $\lambda_{r,s} = (r - 1 - \frac{u}{v}s)\omega_1$, where $r = 1, 2, \dots, u-1$ and $s = 0, 1, \dots, v-1$. Note that the $\mathcal{L}_{r,s}$ with $s > 0$ are bounded, so $\lambda_{r,s} \in \mathcal{B}_{\mathfrak{sl}_2}$ for all $s > 0$. As these $\lambda_{r,s}$ are never integral, the connected component $[\lambda_{r,s}]$ has two elements (Proposition 6.3): $\lambda_{r,s}$ and $s_1 \cdot \lambda_{r,s} = \lambda_{u-r, v-s}$. (Here, s_1 denotes the simple Weyl reflection of \mathfrak{sl}_2 .)

Each distinct connected component $[\lambda_{r,s}]$, for $r = 1, 2, \dots, u-1$ and $s = 1, 2, \dots, v-1$, therefore gives rise to a distinct (standard) coherent family $\mathcal{C}_{[r,s]}$ of $Z_{\mathfrak{k}}$ -modules, making $\frac{1}{2}(u-1)(v-1)$ families in all. As coherent families are invariant under the action of the Weyl group $\mathbb{W} \simeq \mathbb{Z}_2$, there is no need to consider non-standard families. Moreover, the $\mathcal{C}_{[r,s]}$ are distinguished by their central characters (Proposition 6.3 again), meaning the eigenvalues of the quadratic Casimir. Along with the $\mathcal{L}_{r,0}$, $r = 1, 2, \dots, u-1$, the simple direct summands of the $\mathcal{C}_{[r,s]}$ exhaust the simple weight $Z_{\mathfrak{k}}$ -modules with finite-dimensional weight spaces. Note that because coherent families are \mathbb{W} -invariant, the lowest-weight modules $s_1(\mathcal{L}_{r,s})$ and $s_1(\mathcal{L}_{u-r, v-s})$, with $s > 0$, are also contained in $\mathcal{C}_{[r,s]}$ and are thus also $Z_{\mathfrak{k}}$ -modules.

The corresponding $L_{\mathfrak{k}}(\mathfrak{sl}_2)$ -modules therefore provide a classification of the simple relaxed highest-weight modules (with finite-dimensional weight spaces). Each of these coherent families of $L_{\mathfrak{k}}(\mathfrak{sl}_2)$ -modules is determined by its conformal weight $\Delta_{r,s}$, which is proportional to the eigenvalue of the quadratic Casimir of

\mathfrak{sl}_2 on the Zhu image $\mathcal{C}_{[r,s]}$:

$$\Delta_{r,s} = \frac{1}{2(k+2)}(\lambda_{r,s}, \lambda_{r,s} + 2\rho) = \frac{v}{2u} \frac{(r-1 - \frac{u}{v}s)(r+1 - \frac{u}{v}s)}{2} = \frac{(vr-us)^2 - v^2}{4uv}. \quad (9.2)$$

These results reproduce exactly the known classification of relaxed highest-weight $L_k(\mathfrak{sl}_2)$ -modules that was obtained in [5, 6] using more arduous methods.

9.2. Example: $L_{-3/2}(\mathfrak{sl}_3)$. We next consider $\mathfrak{g} = \mathfrak{sl}_3$ with simple roots α_1 and α_2 , giving fundamental weights $\omega_1 = \frac{1}{3}(2\alpha_1 + \alpha_2)$ and $\omega_2 = \frac{1}{3}(\alpha_1 + 2\alpha_2)$. The level in this example is $k = -\frac{3}{2}$ which is admissible. Moreover, $L_{-3/2}(\mathfrak{sl}_3)$ is the second member of a family of vertex operator algebras related to the Deligne exceptional series — the first being $L_{-4/3}(\mathfrak{sl}_2)$ — that have recently attracted much attention in mathematics and physics, see [41, 59] for example.

The simple highest-weight $L_{-3/2}(\mathfrak{sl}_3)$ -modules were originally classified in [60]. The corresponding simple modules over the Zhu algebra $Z_{-3/2}$ are as follows: One finite-dimensional highest-weight module \mathcal{L}_0 and three infinite-dimensional highest-weight modules \mathcal{L}_{Λ_1} , \mathcal{L}_{Λ_2} , \mathcal{L}_{Λ_3} , where $\Lambda_1 = -\frac{3}{2}\omega_1$, $\Lambda_2 = -\frac{3}{2}\omega_2$ and $\Lambda_3 = -\frac{1}{2}(\omega_1 + \omega_2)$. Here, the subscripts indicate the highest weight.

We will now extend this to a classification of standard parabolic families of $Z_{-3/2}$ -modules. Recall that a standard parabolic subalgebra is determined by the subset S of $\{1, 2\}$ corresponding to which negative simple root vectors it contains. When S is empty, the parabolic is the standard Borel and so the corresponding parabolic families are just the highest-weight $Z_{-3/2}$ -modules given above.

At the other extreme, $S = \{1, 2\}$ corresponds to $\mathfrak{p} = \mathfrak{l} = \mathfrak{g}$ and so parabolic families reduce to coherent families. It is easy to check from Proposition 6.1 that the highest weights Λ_i , $i = 1, 2, 3$, of the infinite-dimensional highest-weight $L_{-3/2}(\mathfrak{sl}_3)$ -modules listed above all belong to $\mathcal{B}_{\mathfrak{sl}_3}$. Indeed, Λ_1 and Λ_2 satisfy condition (a) while Λ_3 satisfies condition (c). None of these weights are integral, so each belongs to a connected component of $\mathcal{B}_{\mathfrak{sl}_3}/(\rightarrow)$ with three elements (Proposition 6.3). There is therefore only one connected component, hence only one coherent family \mathcal{C} of $Z_{-3/2}$ -modules. It is characterised by its central character which coincides with the common central character of the Λ_i . Moreover, $\mathcal{L}_{\Lambda_i} \subset \mathcal{C}$, for each $i = 1, 2, 3$.

Next, set $S = \{1\}$, which corresponds to $\mathfrak{l} \simeq \mathfrak{sl}_2 \oplus \mathfrak{gl}_1$. We orthogonally project each of the Λ_i onto the weight space of $\mathfrak{s} \simeq \mathfrak{sl}_2$, here realised as $\mathbb{C}\alpha_1$, obtaining

$$\Lambda_1 = -\frac{3}{4}\alpha_1 - \frac{3}{4}\omega_2, \quad \text{hence} \quad \pi_{\mathfrak{s}}(\Lambda_1) = -\frac{3}{2}\omega_1^{\mathfrak{s}}, \quad (9.3)$$

and similarly $\pi_{\mathfrak{s}}(\Lambda_2) = 0$ and $\pi_{\mathfrak{s}}(\Lambda_3) = -\frac{1}{2}\omega_1^{\mathfrak{s}}$ (see Figure 2). Here, $\pi_{\mathfrak{s}}$ denotes the orthogonal projection and $\omega_1^{\mathfrak{s}}$ denotes the fundamental weight of \mathfrak{s} . We find that $\pi_{\mathfrak{s}}(\Lambda_1), \pi_{\mathfrak{s}}(\Lambda_3) \in \mathcal{B}_{\mathfrak{sl}_2}$, while $\pi_{\mathfrak{s}}(\Lambda_2) \notin \mathcal{B}_{\mathfrak{sl}_2}$. In other words, the $L_{-3/2}(\mathfrak{sl}_3)$ -modules \mathcal{L}_{Λ_1} and \mathcal{L}_{Λ_3} are \mathfrak{l} -bounded, hence they correspond to parabolic families. In fact, they correspond to the same parabolic family because $\pi_{\mathfrak{s}}(\Lambda_1)$ and $\pi_{\mathfrak{s}}(\Lambda_3)$ belong to the same connected component in $\mathcal{B}_{\mathfrak{sl}_2}/(\rightarrow)$.

Thus, there is just one standard parabolic family \mathcal{P}^1 of $Z_{-3/2}$ -modules corresponding to $S = \{1\}$ and it contains both \mathcal{L}_{Λ_1} and \mathcal{L}_{Λ_3} . It is induced from the coherent family $\mathcal{C}_q \otimes \mathcal{C}_\mu$ of \mathfrak{l} -modules, where

$$q = (\pi_{\mathfrak{s}}(\Lambda_1), \pi_{\mathfrak{s}}(\Lambda_1) + 2\rho^{\mathfrak{s}}) = (\pi_{\mathfrak{s}}(\Lambda_3), \pi_{\mathfrak{s}}(\Lambda_3) + 2\rho^{\mathfrak{s}}) = -\frac{3}{8} \quad (9.4)$$

and $\mu = \Lambda_1 - \pi_{\mathfrak{s}}(\Lambda_1) = \Lambda_3 - \pi_{\mathfrak{s}}(\Lambda_3) = -\frac{3}{4}\omega_2$

are the eigenvalue of the quadratic Casimir (central character) of \mathfrak{sl}_2 and the \mathfrak{gl}_1 -weight, respectively.

Similarly, $S = \{2\}$ also yields precisely one standard parabolic family $\mathcal{P}^2 = \mathcal{S}_{\mathfrak{p}}(\mathbb{C}_{-3/8} \otimes \mathbb{C}_{-3\omega_1/4})$. It contains both \mathcal{L}_{Λ_2} and \mathcal{L}_{Λ_3} , hence $\mathcal{P}^2 \subset \mathcal{C}$ as well. Clearly, this parabolic family may be obtained from that found when $S = \{1\}$ by twisting by the conjugation automorphism (the outer automorphism of \mathfrak{sl}_3 that acts as -1 on \mathfrak{h}).

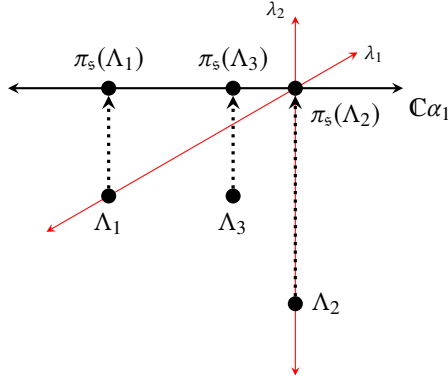


FIGURE 2. Projecting the \mathfrak{sl}_3 -weights Λ_i , $i = 1, 2, 3$, onto the weight space $\mathbb{C}\alpha_1$ corresponding to $S = \{1\}$. λ_1 and λ_2 indicate the directions of increasing Dynkin labels.

Thus, for a given Borel subalgebra, there is 1 finite-dimensional $Z_{-3/2}$ -module \mathcal{L}_0 ; 3 infinite-dimensional highest-weight $Z_{-3/2}$ -modules \mathcal{L}_{Λ_i} , $i = 1, 2, 3$; 2 standard parabolic families \mathcal{P}^1 and \mathcal{P}^2 of $Z_{-3/2}$ -modules corresponding to $\mathfrak{l} \simeq \mathfrak{gl}_2$; and 1 coherent family \mathcal{C} of $Z_{-3/2}$ -modules. The small Weyl groups are as follows.

\mathcal{M}	\mathcal{L}_0	\mathcal{L}_{Λ_1}	\mathcal{L}_{Λ_2}	\mathcal{L}_{Λ_3}	$\mathcal{P}^1, \mathcal{P}^2$	\mathcal{C}
$W_{\mathcal{M}}$	W	$\langle s_2 \rangle$	$\langle s_1 \rangle$	1	1	1

Twisting \mathcal{L}_0 by W , which amounts to changing the Borel, thus leads to $|W/W_{\mathcal{L}_0}| = 1$ finite-dimensional simple highest-weight module. Similarly, we get 3 twists each for \mathcal{L}_{Λ_1} and \mathcal{L}_{Λ_2} , while \mathcal{L}_{Λ_3} gets 6. The action of W on the parabolic corresponding to $S = \{1\}$ results in $|W/W_{\mathfrak{sl}_2}| = 3$ parabolics, because the parabolic families are invariant under acting with the Weyl group of \mathfrak{l} (which coincides with that of \mathfrak{sl}_2). There are another 3 coming from $S = \{2\}$, hence we have 6 parabolic families in total. Finally, there is only a single coherent family.

The resulting classification of simple relaxed highest-weight $L_{-3/2}(\mathfrak{sl}_3)$ -modules (with finite-dimensional weight spaces) appears to be consistent with the Gelfand-Tsetlin classification reported in [15]. Some relaxed highest-weight modules for this vertex operator algebra were also constructed in [14], but with no claim of completeness. An analysis of the characters, modular properties and Grothendieck fusion rules of all these modules will appear in [16].

9.3. Example: $L_{-1/2}(\mathfrak{sp}_4)$. Consider now a non-simply-laced admissible-level example: $\mathfrak{g} = \mathfrak{sp}_4$ and $k = -\frac{1}{2}$. Recall from Section 6.2 that we take α_1 to be short and α_2 long, so that the fundamental weights are $\omega_1 = \alpha_1 + \frac{1}{2}\alpha_2$ and $\omega_2 = \alpha_1 + \alpha_2$.

The simple highest-weight $L_{-1/2}(\mathfrak{sp}_4)$ -modules were first classified in [55]. There turn out to be four simple highest-weight $Z_{-1/2}$ -modules, of which two are finite-dimensional (\mathcal{L}_0 and \mathcal{L}_{ω_1}) and two are not. The highest weights of the infinite-dimensional modules will be denoted by $\Lambda_1 = -\frac{1}{2}\omega_2$ and $\Lambda_2 = \omega_1 - \frac{3}{2}\omega_2$.

As always, we work down the list of (standard) parabolic subalgebras (ignoring the Borel case that corresponds to highest-weight modules). Starting with $S = \{1, 2\}$, hence $\mathfrak{p} = \mathfrak{l} = \mathfrak{g}$, we check that both Λ_1 and Λ_2 satisfy all the conditions of Proposition 6.4, hence they belong to $\mathcal{B}_{\mathfrak{sp}_4}$. Since connected components for \mathfrak{sp}_4 always have two elements, by Proposition 6.5, there is a single connected component giving exactly one coherent family \mathcal{C} of $Z_{-1/2}$ -modules. It is characterised by its central character and contains both \mathcal{L}_{Λ_1} and \mathcal{L}_{Λ_2} .

If $S = \{1\}$, hence $\mathfrak{l} \simeq \mathfrak{sl}_2 \oplus \mathfrak{gl}_1$, then projecting onto the $\mathfrak{s} = \mathfrak{sl}_2$ weight space spanned by α_1 results in $\pi_{\mathfrak{s}}(\Lambda_1) = 0$ and $\pi_{\mathfrak{s}}(\Lambda_2) = \omega_1^{\mathfrak{s}}$. As both are dominant integral \mathfrak{sl}_2 -weights, they are not in $\mathcal{B}_{\mathfrak{sl}_2}$ and there are therefore no parabolic families corresponding to this S .

When $S = \{2\}$ however, we again have $\mathfrak{l} \simeq \mathfrak{sl}_2 \oplus \mathfrak{gl}_1$, but the projection this time gives two elements of $\mathcal{B}_{\mathfrak{sl}_2}$: $\pi_{\mathfrak{s}}(\Lambda_1) = -\frac{1}{2}\omega_1^{\mathfrak{s}}$ and $\pi_{\mathfrak{s}}(\Lambda_2) = -\frac{3}{2}\omega_1^{\mathfrak{s}}$. These represent a single connected component for \mathfrak{sl}_2 , hence it

corresponds to a single coherent family $\mathcal{C}_{-3/8} \otimes \mathcal{C}_{-\omega_1/2}$ of \mathfrak{l} -modules. There is thus a unique standard parabolic family $\mathcal{P} = \mathcal{S}_p(\mathcal{C}_{-3/8} \otimes \mathcal{C}_{-\omega_1/2})$ of $Z_{-1/2}$ -modules.

As always, the small Weyl group of the finite-dimensional modules is W . There are therefore only 2 finite-dimensional simple $Z_{-1/2}$ -modules. The small Weyl group of both \mathcal{L}_{Λ_1} and \mathcal{L}_{Λ_2} is $\langle s_1 \rangle$, hence W -twisting gives $|W/\langle s_1 \rangle| = 4$ modules each. The result is thus 8 infinite-dimensional simple highest-weight $Z_{-1/2}$ -modules. Once again, the small Weyl groups of the parabolic and coherent families are trivial. Hence, we get $|W/W_{\mathfrak{s}l_2}| = 4$ parabolic families and a single coherent family of $Z_{-1/2}$ -modules. We believe that the corresponding classification of simple relaxed highest-weight $L_{-1/2}(\mathfrak{sp}_4)$ -modules, with finite-dimensional weight spaces, is new.

9.4. Example: $L_{-5/3}(\mathfrak{g}_2)$. Our next example features a simple Lie algebra which is not of AC-type. Take $\mathfrak{g} = \mathfrak{g}_2$ and $k = -\frac{5}{3}$, an admissible level corresponding to the third member of the Deligne exceptional series of affine vertex operator algebras. Our convention is that α_1 denotes the short simple root and α_2 the long one. The fundamental weights are thus $\omega_1 = 2\alpha_1 + \alpha_2$ and $\omega_2 = 3\alpha_1 + 2\alpha_2$.

The classification of simple highest-weight $L_{-5/3}(\mathfrak{g}_2)$ -modules was carried out in [61]. At the level of the Zhu algebra $Z_{-5/3}$, there is a single simple finite-dimensional module \mathcal{L}_0 and two infinite-dimensional simple highest-weight modules \mathcal{L}_{Λ_1} and \mathcal{L}_{Λ_2} , where $\Lambda_1 = -\frac{2}{3}\omega_2$ and $\Lambda_2 = \omega_1 - \frac{4}{3}\omega_2$.

Because \mathfrak{g}_2 is not of AC-type, there can be no coherent families corresponding to $S = \{1, 2\}$. Moreover, $S = \{1\}$ (thus $\mathfrak{l} \simeq \mathfrak{sl}_2 \oplus \mathfrak{gl}_1$) does not yield any parabolic families because neither $\pi_{\mathfrak{s}}(\Lambda_1) = 0$ nor $\pi_{\mathfrak{s}}(\Lambda_2) = \omega_1^{\mathfrak{s}}$ belong to $\mathcal{B}_{\mathfrak{s}l_2}$. We therefore turn to $S = \{2\}$, for which $\mathfrak{l} \simeq \mathfrak{sl}_2 \oplus \mathfrak{gl}_1$, $\pi_{\mathfrak{s}}(\Lambda_1) = -\frac{2}{3}\omega_2^{\mathfrak{s}}$ and $\pi_{\mathfrak{s}}(\Lambda_2) = -\frac{4}{3}\omega_2^{\mathfrak{s}}$. Both projections belong to $\mathcal{B}_{\mathfrak{s}l_2}$ and constitute a single connected component. We therefore have only one standard parabolic family \mathcal{P} of $Z_{-5/3}$ -modules (and it corresponds to $S = \{2\}$). It contains both \mathcal{L}_{Λ_1} and \mathcal{L}_{Λ_2} and is constructed by applying \mathcal{S}_p to the coherent family $\mathcal{C}_{-4/9} \otimes \mathcal{C}_{-\omega_1}$ of $\mathfrak{sl}_2 \oplus \mathfrak{gl}_1$ -modules.

Summarising, there is just one finite-dimensional simple highest-weight $Z_{-5/3}$ -module. Because the small Weyl groups of \mathcal{L}_{Λ_1} and \mathcal{L}_{Λ_2} are both easily checked to be $\langle s_1 \rangle$, we get 6 distinct W -twists from each, resulting in 12 simple infinite-dimensional highest-weight $Z_{-5/3}$ -modules. The standard parabolic family found above again has trivial small Weyl group, so it also generates $|W/W_{\mathfrak{s}l_2}| = 6$ parabolic families under W -twisting. As mentioned above, there are no coherent families of $Z_{-5/3}$ -modules. Again, the corresponding classification of simple relaxed highest-weight $L_{-5/3}(\mathfrak{g}_2)$ -modules, with finite-dimensional weight spaces, is surely new.

9.5. Example: $L_{-2}(\mathfrak{so}_8)$. We finish up with a more challenging example, both to illustrate the power of our classification results but also to point out that non-admissible levels have some slightly different features.

The fourth member of the Deligne exceptional series of affine vertex operator algebras corresponds to $\mathfrak{g} = \mathfrak{so}_8$ at the non-admissible level $k = -2$. We choose simple roots $\alpha_1, \dots, \alpha_4$, ordered so that α_2 corresponds to the centre of the Dynkin diagram. The fundamental weights then have the form

$$\omega_2 = \alpha_2 + \sum_{i=1}^4 \alpha_i, \quad \omega_i = \frac{1}{2}(\alpha_i + \alpha_2) + \frac{1}{2} \sum_{i=1}^4 \alpha_i, \quad i = 1, 3, 4. \quad (9.5)$$

The simple highest-weight modules over the Zhu algebra Z_{-2} were classified in [62]. The result is that there is again a unique simple finite-dimensional module \mathcal{L}_0 , while now we have four infinite-dimensional simples: \mathcal{L}_{Λ_i} , $i = 1, 2, 3, 4$, with $\Lambda_2 = -\omega_2$ and $\Lambda_i = -2\omega_i$ for $i = 1, 3, 4$.

The subsets $S \subseteq \{1, 2, 3, 4\}$ that lead to AC-type parabolic subalgebras are easy to find:

- (a) $S = \{1\}, \{3\}, \{4\}$, giving $\mathfrak{l} \simeq \mathfrak{sl}_2 \oplus \mathfrak{gl}_1^{\oplus 3}$;
- (b) $S = \{2\}$, giving $\mathfrak{l} \simeq \mathfrak{sl}_2 \oplus \mathfrak{gl}_1^{\oplus 3}$;
- (c) $S = \{1, 2\}, \{2, 3\}, \{2, 4\}$, giving $\mathfrak{l} \simeq \mathfrak{sl}_3 \oplus \mathfrak{gl}_1^{\oplus 2}$;
- (d) $S = \{1, 3\}, \{1, 4\}, \{3, 4\}$, giving $\mathfrak{l} \simeq \mathfrak{sl}_2^{\oplus 2} \oplus \mathfrak{gl}_1^{\oplus 2}$;
- (e) $S = \{1, 3, 4\}$, giving $\mathfrak{l} \simeq \mathfrak{sl}_2^{\oplus 3} \oplus \mathfrak{gl}_1$.

(f) $S = \{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}$, giving $\mathfrak{l} \simeq \mathfrak{sl}_4 \oplus \mathfrak{gl}_1$.

Note that the subsets listed in each case are all related by the \mathbb{Z}_3 outer automorphism of \mathfrak{so}_8 , represented on the Dynkin node labels by $1 \rightarrow 3 \rightarrow 4 \rightarrow 1$. We shall therefore only need to analyse parabolic families from one representative of each case, the others then following by twisting by outer automorphisms.

We now classify the parabolic families of Z_{-2} -modules for each of the cases a–f above.

- (a) When $S = \{1\}$, projecting the Λ_i onto the $\mathfrak{s} \simeq \mathfrak{sl}_2$ weight space spanned by α_1 results in $\pi_{\mathfrak{s}}(\Lambda_1) = -2\omega_1^{\mathfrak{s}}$ and $\pi_{\mathfrak{s}}(\Lambda_i) = 0$ for $i = 2, 3, 4$. As $-2\omega_1^{\mathfrak{s}} \in \mathcal{B}_{\mathfrak{sl}_2}$ is integral, the connected component has a single element and so there is a single associated coherent family $\mathcal{C}^1 = \mathcal{C}_0 \otimes \mathcal{C}_{-\omega_2}$ of \mathfrak{l} -modules. It induces to a standard parabolic family $\mathcal{P}^1 = \mathcal{S}_p \mathcal{C}^1$ of Z_{-2} -modules that contains \mathcal{L}_{Λ_1} . Interestingly, because the coherent family \mathcal{C}_0 of \mathfrak{sl}_2 -modules contains the trivial \mathfrak{sl}_2 -module \mathbb{C}_0 , it follows that \mathcal{P}^1 contains $\mathcal{S}_p(\mathbb{C}_0 \otimes \mathcal{C}_{-\omega_2}) \simeq \mathcal{L}_{-\omega_2} = \mathcal{L}_{\Lambda_2}$. Twisting by \mathbb{Z}_3 , we generate two other standard parabolic families of Z_{-2} -modules which we shall denote by \mathcal{P}^3 and \mathcal{P}^4 . The family \mathcal{P}^i , $i = 1, 3, 4$, therefore corresponds to $S = \{i\}$ and contains both \mathcal{L}_{Λ_2} and \mathcal{L}_{Λ_i} .
- (b) When $S = \{2\}$, projecting gives $\pi_{\mathfrak{s}}(\Lambda_i) = 0$, for $i = 1, 3, 4$, and $\pi_{\mathfrak{s}}(\Lambda_2) = -\omega_1^{\mathfrak{s}} \in \mathcal{B}_{\mathfrak{sl}_2}$. This corresponds to a single coherent family $\mathcal{C}^2 = \mathcal{C}_{-1/2} \otimes \mathcal{C}_{\mu}$ of \mathfrak{l} -modules, where $\mu = -\frac{1}{2}(\omega_1 + \omega_3 + \omega_4)$. Inducing therefore gives a standard parabolic family \mathcal{P}^2 of Z_{-2} -modules that contains \mathcal{L}_{Λ_2} . Note that $\mathcal{C}_{-1/2}$ contains only one simple highest-weight \mathfrak{sl}_2 -module, hence \mathcal{P}^2 contains only one simple highest-weight \mathfrak{so}_8 -module.
- (c) When $S = \{1, 2\}$, projecting onto the $\mathfrak{s} \simeq \mathfrak{sl}_3$ weight space spanned by α_1 and α_2 gives $\pi_{\mathfrak{s}}(\Lambda_1) = -2\omega_1^{\mathfrak{s}}$ and $\pi_{\mathfrak{s}}(\Lambda_2) = -\omega_2^{\mathfrak{s}}$, while both Λ_3 and Λ_4 give 0. As $\pi_{\mathfrak{s}}(\Lambda_1)$ and $\pi_{\mathfrak{s}}(\Lambda_2)$ are both shifted-singular integral weights in $\mathcal{B}_{\mathfrak{sl}_3}$, the corresponding connected component has two elements. This thus yields precisely one coherent family $\mathcal{C}^{1,2} = \mathcal{C} \otimes \mathcal{C}_{\nu}$ of \mathfrak{l} -modules, where $\nu = -\frac{2}{3}(\omega_3 + \omega_4)$ and \mathcal{C} is determined by its \mathfrak{sl}_3 central character (which coincides with that of the \mathfrak{sl}_3 -modules of highest weights $\pi_{\mathfrak{s}}(\Lambda_1)$ and $\pi_{\mathfrak{s}}(\Lambda_2)$).

Note that the eigenvalue of the quadratic Casimir of \mathcal{C} is

$$(\pi_{\mathfrak{s}}(\Lambda_1), \pi_{\mathfrak{s}}(\Lambda_1) + 2\rho^{\mathfrak{s}}) = (\pi_{\mathfrak{s}}(\Lambda_2), \pi_{\mathfrak{s}}(\Lambda_2) + 2\rho^{\mathfrak{s}}) = -\frac{4}{3} < 0. \quad (9.6)$$

Because the Casimir eigenvalue on any finite-dimensional simple \mathfrak{sl}_3 -module is non-negative, it follows that \mathcal{C} contains no such modules. Its only highest-weight modules are thus the infinite-dimensional ones already found. We can therefore now conclude that the standard parabolic family $\mathcal{P}^{1,2}$ of Z_{-2} -modules that we obtain by inducing contains only two highest-weight \mathfrak{so}_8 -modules: \mathcal{L}_{Λ_1} and \mathcal{L}_{Λ_2} . There are thus two other inequivalent standard parabolic families $\mathcal{P}^{2,3}$ and $\mathcal{P}^{2,4}$ which may be obtained as \mathbb{Z}_3 -twists of $\mathcal{P}^{1,2}$.

- (d) When $S = \{1, 3\}$, we have $\mathfrak{s} = \mathfrak{s}_1 \oplus \mathfrak{s}_3$, where \mathfrak{s}_1 is the \mathfrak{sl}_2 subalgebra corresponding to α_1 and \mathfrak{s}_3 is the \mathfrak{sl}_2 subalgebra corresponding to α_3 . From case a, the only Λ_j whose \mathfrak{s}_i -projection lands in $\mathcal{B}_{\mathfrak{s}_i}$, for $i = 1$ or 3 , is Λ_i . As no Λ_j satisfies this boundedness criterion for both $i = 1$ and 3 , there is no coherent family of \mathfrak{l} -modules, hence no parabolic family of Z_{-2} -modules. The same is obviously true for $S = \{1, 4\}$ and $\{3, 4\}$.
- (e) When $S = \{1, 3, 4\}$, there are likewise no parabolic families of Z_{-2} -modules for the same reason as in the previous case.
- (f) Finally, when $S = \{1, 2, 3\}$, we have $\mathfrak{s} \simeq \mathfrak{sl}_4$ and the projections are $\pi_{\mathfrak{s}}(\Lambda_1) = -2\omega_1^{\mathfrak{s}}$, $\pi_{\mathfrak{s}}(\Lambda_2) = -\omega_2^{\mathfrak{s}}$, $\pi_{\mathfrak{s}}(\Lambda_3) = -2\omega_3^{\mathfrak{s}}$ and $\pi_{\mathfrak{s}}(\Lambda_4) = 0$. All but the last belong to $\mathcal{B}_{\mathfrak{sl}_4}$ and may be checked to be shifted-singular and integral. They therefore form a single connected component, hence we get one coherent family $\mathcal{C}^{1,2,3} = \mathcal{C} \otimes \mathcal{C}_{-\omega_4}$ of \mathfrak{l} -modules. Here, \mathcal{C} is determined by its central character which agrees with that of the \mathfrak{sl}_4 -modules of highest weights $\pi_{\mathfrak{s}}(\Lambda_i)$, $i = 1, 2, 3$. In particular, the common quadratic Casimir eigenvalue is -3 which again rules out \mathcal{C} containing any finite-dimensional \mathfrak{sl}_4 -modules. In this way, we arrive at one standard parabolic family $\mathcal{P}^{1,2,3}$ of Z_{-2} -modules that contains only the \mathcal{L}_{Λ_i} with $i = 1, 2, 3$. \mathbb{Z}_3 -twisting gives two more standard parabolic families which we shall denote by $\mathcal{P}^{1,2,4}$ and $\mathcal{P}^{2,3,4}$.

This gives us all the standard parabolic families of Z_{-2} -modules. It only remains to determine how many non-standard families there are. For this, we recall that the Weyl group W of \mathfrak{so}_8 is isomorphic to $S_4 \times Z_2^3$ and so has order $4! \cdot 2^3 = 192$. We tabulate the small Weyl groups of each family as well as the Weyl groups of the corresponding Levi factors l :

\mathcal{M}	\mathcal{L}_0	$\mathcal{L}_{\Lambda_1}, \mathcal{L}_{\Lambda_3}, \mathcal{L}_{\Lambda_4}$	\mathcal{L}_{Λ_2}	$\mathcal{P}^1, \mathcal{P}^3, \mathcal{P}^4$	\mathcal{P}^2	$\mathcal{P}^{1,2}, \mathcal{P}^{2,3}, \mathcal{P}^{2,4}$	$\mathcal{P}^{1,2,3}, \mathcal{P}^{1,2,4}, \mathcal{P}^{2,3,4}$
$W_{\mathcal{M}}$	W	S_4	Z_2^3	Z_2^2	1	1	1
W_l	1	1	1	Z_2	Z_2	S_3	S_4
$ W $	1	8	24	24	96	32	8
$\frac{ W_{\mathcal{M}} }{ W_l }$	1	8	24	24	96	32	8

We therefore have 1 finite-dimensional simple module, $3 \times 8 + 24 = 48$ infinite-dimensional highest-weight modules, $3 \times 24 + 96 = 168$ one-parameter parabolic families with $l \simeq \mathfrak{sl}_2$, $3 \times 32 = 96$ two-parameter parabolic families with $l \simeq \mathfrak{sl}_3$, and $3 \times 8 = 24$ three-parameter parabolic families with $l \simeq \mathfrak{sl}_4$. This gives the complete classification of simple weight Z_{-2} -modules with finite-dimensional weight spaces. We are quietly confident that the corresponding classification of simple relaxed highest-weight $L_{-2}(\mathfrak{so}_8)$ -modules (with finite-dimensional weight spaces) was unknown before now.

Note finally the unexpected, and therefore interesting, fact that the parabolic families \mathcal{P}^1 , \mathcal{P}^3 and \mathcal{P}^4 all contain \mathcal{L}_{Λ_2} . This is due to the fact that each of the corresponding coherent families of \mathfrak{sl}_2 -modules has a finite-dimensional highest-weight submodule. We did not observe integral highest weights upon projecting the admissible weights of the previous examples, so it is reasonable to conjecture that this is a feature of non-admissible levels. In this example, the projected weights were always shifted-singular, so the coherent families are completely characterised by their central characters. It would be very interesting, and perhaps a little alarming, to find an example with shifted-regular projections, hence coherent families that cannot be distinguished by their central characters alone.

10. AN APPLICATION TO CATEGORY \mathcal{O}

The previous section detailed many examples of applications of Main Theorem 1, hence Theorem 4.5. In this section, we outline an application of Main Theorem 2, hence Theorem 5.3.

For an admissible-level affine vertex operator algebra (like those studied in Sections 9.1 to 9.4), the modules in category \mathcal{O} are known to be semisimple. This was originally conjectured in [6] and proven in [32]. Here, we pose the question of whether a *quasilisse* affine vertex operator algebra can have a non-semisimple module in category \mathcal{O} . Recall that quasilisse vertex operator algebras are generalisations, introduced in [41], of the well-known lisse, or C_2 -cofinite, vertex operator algebras. Admissible-level affine vertex operator algebras are always quasilisse [41], but there are also quasilisse affine examples with non-admissible levels. We shall answer the question posed above in the affirmative by establishing that \mathcal{O} is non-semisimple for the quasilisse, but non-admissible-level, affine vertex operator algebra $L_{-2}(\mathfrak{so}_8)$, studied in Section 9.5.

To establish this, we shall construct a non-semisimple extension \mathcal{M} of two highest-weight $\text{Zhu}[L_{-2}(\mathfrak{so}_8)]$ -modules as a submodule of a non-semisimple parabolic family \mathcal{P} of \mathfrak{so}_8 -modules. We will then apply Main Theorem 2 to show that $\text{Ind}[\mathcal{P}]$, and therefore the submodule $\text{Ind}[\mathcal{M}]$, is an $L_{-2}(\mathfrak{so}_8)$ -module (see Proposition 10.2 below for the precise result). We use the same notation as in Section 9.5.

Recall that there are five simple highest-weight $L_{-2}(\mathfrak{so}_8)$ -modules, up to isomorphism, and that their highest weights are $\Lambda_0 = -2\omega_0$ (the vacuum module), $\Lambda_2 = -\omega_2$ and $\Lambda_i = -2\omega_i$, $i = 1, 3, 4$. The conformal weights of their highest-weight vectors are easily established to be 0, for the vacuum, and -1 otherwise. Let \mathfrak{p} be the standard parabolic subalgebra defined by the subset $\{1\}$ of simple root labels. The Levi factor of \mathfrak{p} is then $l \simeq \mathfrak{s} \oplus \mathfrak{gl}_1^{\oplus 3}$ with $\mathfrak{s} \simeq \mathfrak{sl}_2$.

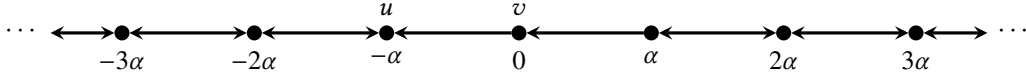


FIGURE 3. An illustration of the structure of the non-semisimple dense $\mathfrak{s} \simeq \mathfrak{sl}_2$ -module $\mathcal{D}_{\alpha,0}$. Weights are indicated with black circles and the corresponding weight spaces are all 1-dimensional. The highest-weight vectors u and v are also indicated above the circles representing their weights. Arrows pointing right (left) are drawn whenever the action of the \mathfrak{sl}_2 root vector $e = e^\alpha$ ($f = e^{-\alpha}$) is bijective. Note that the vector of weight α indeed generates the entire module.

Our first task is to construct the irreducible non-semisimple parabolic family \mathcal{P} of \mathfrak{so}_8 -modules. This parabolic family will be $-\alpha_1$ -bijective (see Definition 5.1) and shall contain the simple highest-weight \mathfrak{so}_8 -module \mathcal{L}_{Λ_1} as a submodule. This will allow us to apply Main Theorem 2 to conclude that $\text{Ind}[\mathcal{P}]$ is an $L_{-2}(\mathfrak{so}_8)$ -module. The key step in this construction is the existence of an irreducible $-\alpha$ -bijective coherent family \mathcal{C} of \mathfrak{s} -modules (α denoting the simple root of $\mathfrak{s} \simeq \mathfrak{sl}_2$) that contains the bounded highest-weight \mathfrak{s} -module $\mathcal{L}_{-\alpha}^{\mathfrak{s}}$ as a submodule. Tensoring with an appropriate $\mathfrak{gl}_1^{\oplus 3}$ -module and inducing with $\mathcal{S}_{\mathfrak{p}}$ will then give the desired parabolic family \mathcal{P} . We shall outline a construction of the coherent family \mathcal{C} using induction for completeness. Readers for whom the existence of \mathcal{C} is clear may safely skip the next two paragraphs.

Recall that one may construct a dense \mathfrak{s} -module $\mathcal{D}_{\lambda,q}$ by inducing the one-dimensional module $\mathbb{C}_{\lambda,q}$ of the centraliser $\mathcal{U}(\mathfrak{s})^{\mathfrak{h}_{\mathfrak{s}}} \simeq \mathbb{C}[h, \Omega]$:

$$\mathcal{D}_{\lambda,q} = \mathcal{U}(\mathfrak{s}) \otimes_{\mathcal{U}(\mathfrak{s})^{\mathfrak{h}_{\mathfrak{s}}}} \mathbb{C}_{\lambda,q}. \quad (10.1)$$

Here, $\mathfrak{h}_{\mathfrak{s}} = \text{span}\{h\}$ is the Cartan subalgebra of \mathfrak{s} , Ω is the quadratic Casimir of \mathfrak{s} , λ is the unique weight of $\mathbb{C}_{\lambda,q}$, and q is the Ω -eigenvalue. It is easy to show that $\mathcal{D}_{\lambda,q}$ has 1-dimensional weight spaces with weight support $\lambda + \mathbb{Q}_{\mathfrak{s}}$ (further details may be found, for example, in [36, Ex. 3.99] or [1, Sec. 3.2]). We set q to 0, noting that it follows that the weight of any highest-weight or lowest-weight vector in $\mathcal{D}_{\lambda,0}$ belongs to $\{0, \pm\alpha\}$. We conclude that the $\mathcal{D}_{\lambda,0}$ are simple and dense, hence $-\alpha$ -bijective, whenever λ does not lie in $\mathbb{Q}_{\mathfrak{s}}$.

Now, $\mathcal{D}_{\lambda,0}$ is not simple for $\lambda \in \mathbb{Q}_{\mathfrak{s}}$. In particular, setting λ to α results in a dense module $\mathcal{D}_{\alpha,0}$ containing highest-weight vectors u and v of \mathfrak{s} -weights $-\alpha$ and 0, respectively, satisfying

$$e^\alpha u = 0, \quad e^\alpha v = 0 \quad \text{and} \quad u = e^{-\alpha} v. \quad (10.2)$$

Here, $e^{\pm\alpha}$ denotes the root vector of \mathfrak{s} corresponding to the root $\pm\alpha$. The composition factors of $\mathcal{D}_{\alpha,0}$ are thus the simple highest-weight \mathfrak{s} -modules $\mathcal{L}_{-\alpha}^{\mathfrak{s}}$ and $\mathcal{L}_0^{\mathfrak{s}}$, along with the simple lowest-weight \mathfrak{s} -module $w(\mathcal{L}_{-\alpha}^{\mathfrak{s}})$, where w denotes the Weyl reflection of \mathfrak{s} . We illustrate the structure of $\mathcal{D}_{\alpha,0}$ in Figure 3. What is most important here, however, is that $\mathcal{D}_{\alpha,0}$ is $-\alpha$ -bijective. The direct sum

$$\mathcal{C} = \bigoplus_{0 < t \leq 1} \mathcal{D}_{t\alpha,0} \quad (10.3)$$

is therefore the desired irreducible $-\alpha$ -bijective coherent family of $\mathfrak{s} \simeq \mathfrak{sl}_2$ -modules. Note that the highest-weight submodule $\mathcal{L}_{-\alpha}^{\mathfrak{s}} \subset \mathcal{C}$ is clearly bounded.

Having constructed \mathcal{C} , we now lift it to a coherent family of modules over $\mathfrak{l} \simeq \mathfrak{sl}_2 \oplus \mathfrak{gl}_1^{\oplus 3} \subseteq \mathfrak{so}_8$ by identifying the \mathfrak{sl}_2 simple root α with its \mathfrak{so}_8 counterpart α_1 and tensoring with the one-dimensional $\mathfrak{gl}_1^{\oplus 3}$ -module $\mathbb{C}_{-\omega_2} = \text{span}\{w\}$ whose sole weight is $-\omega_2$. It follows that $\mathcal{C} \otimes \mathbb{C}_{-\omega_2}$ has highest-weight vectors $u \otimes w$ and $v \otimes w$ of weights $-\alpha_1 - \omega_2 = -2\omega_1 = \Lambda_1$ and $-\omega_2 = \Lambda_2$. Note that $u \otimes w$ generates a simple bounded highest-weight \mathfrak{l} -submodule of $\mathcal{C} \otimes \mathbb{C}_{-\omega_2}$.

Form the parabolic family $\mathcal{P} = \mathcal{S}_{\mathfrak{p}}(\mathcal{C} \otimes \mathbb{C}_{-\omega_2})$ of \mathfrak{so}_8 -modules. Because \mathfrak{p} contains the standard Borel, the image of any highest-weight vector under $\mathcal{C} \otimes \mathbb{C}_{-\omega_2} \hookrightarrow \mathcal{P}$ will again be a highest-weight vector. Moreover, the image of $u \otimes w$ generates a *simple* highest-weight \mathfrak{so}_8 -submodule of \mathcal{P} , by Proposition 3.1b and d. This submodule is obviously \mathfrak{l} -bounded and isomorphic to \mathcal{L}_{Λ_1} . The image $\text{Ind}[\mathcal{L}_{\Lambda_1}]$ under the Zhu-induction

functor is therefore the simple highest-weight $L_{-2}(\mathfrak{so}_8)$ -module of highest weight Λ_1 . By Main Theorem 2, we may now conclude that every subquotient of $\text{Ind}[\mathcal{P}]$ is also an $L_{-2}(\mathfrak{so}_8)$ -module.

In particular, the Zhu-induction of the highest-weight submodule \mathcal{M} of \mathcal{P} generated by (the image of) the highest-weight vector $v \otimes w$ is an $L_{-2}(\mathfrak{so}_8)$ -module. It follows from Theorem 1.1b that \mathcal{M} has \mathcal{L}_{Λ_2} as a composition factor (because $v \otimes w$ has weight Λ_2). However, $u \otimes w$ is also a highest-weight vector in \mathcal{M} , by (10.2), hence \mathcal{M} has \mathcal{L}_{Λ_1} as another composition factor. As \mathcal{M} is highest-weight, it is indecomposable and so we have proved the following proposition. We only need remark that the outer automorphisms of \mathfrak{so}_8 allow us to swap Λ_1 for Λ_3 or Λ_4 in this conclusion.

Proposition 10.1. *There exist non-simple highest-weight $L_{-2}(\mathfrak{so}_8)$ -modules. In particular, for each $i = 1, 3, 4$, there exists a highest-weight $L_{-2}(\mathfrak{so}_8)$ -module whose composition factors include $\text{Ind}[\mathcal{L}_{\Lambda_2}]$ and $\text{Ind}[\mathcal{L}_{\Lambda_i}]$.*

To the best of our knowledge, this is the first time that non-semisimplicity has been demonstrated in category \mathcal{O} for a quasilinear affine vertex operator algebra (non-quasilinear examples with \mathcal{O} non-semisimple are already known, see [63, Rem. 5.8] and [64, Thm. 7.2]). In this regard, it is also interesting to note that the character of the vacuum $L_{-2}(\mathfrak{so}_8)$ -module $\text{Ind}[\mathcal{L}_0]$ is quasimodular, but not modular [41]. See also [65] for a relation between non-admissible levels and semisimplicity.

It is in fact easy to show that the non-simple highest-weight $L_{-2}(\mathfrak{so}_8)$ -modules that we have constructed have no other composition factors. First, consideration of the conformal weights of the highest-weight vectors of the five possible isomorphism classes of composition factors lets us conclude that any composition factor of $\text{Ind}[\mathcal{M}]$ with non-zero highest weight will already be detected as a composition factor of $\mathcal{M} \subset \mathcal{P}$. Clearly, \mathcal{L}_{Λ_2} appears with multiplicity 1 in \mathcal{M} . Because $\Lambda_i = \Lambda_2 - \alpha_i$, for $i = 1, 3, 4$, the multiplicity of \mathcal{L}_{Λ_i} in \mathcal{M} is at most 1. Proposition 10.1 then shows that this multiplicity is exactly 1 for $i = 1$.

To determine the multiplicity of \mathcal{L}_{Λ_j} , $j = 3, 4$, it suffices to check the subsingularity of the vector $e^{-\alpha_j}(v \otimes w)$ in \mathcal{P} . However, acting with any simple root vector necessarily gives 0, so we conclude that this vector is either singular or zero. Either way, the \mathfrak{so}_8 -module it generates in \mathcal{P} has zero intersection with $\mathcal{R}_\mathfrak{p}\mathcal{P} = \mathcal{C} \otimes \mathbb{C}_{-\omega_2}$, hence $e^{-\alpha_j}v$ must be zero in \mathcal{P} by definition. As it must therefore also be zero in \mathcal{M} , we conclude that \mathcal{L}_{Λ_j} is not a composition factor of \mathcal{M} , hence that $\text{Ind}[\mathcal{L}_{\Lambda_j}]$ is not a composition factor of $\text{Ind}[\mathcal{M}]$, for $j = 3, 4$.

A similar argument rules out the vacuum module $\text{Ind}[\mathcal{L}_{\Lambda_0}]$ appearing as a composition factor. For this, we note that the highest root of \mathfrak{so}_8 is $\theta = \omega_2$. It follows that the multiplicity of $\text{Ind}[\mathcal{L}_{\Lambda_0}]$ in $\text{Ind}[\mathcal{M}]$ is also at most 1. This time, the precise multiplicity may be determined by studying the subsingularity of $e^\theta v \in \text{Ind}[\mathcal{M}]$. However, this is easily shown to be a singular vector or zero, using $k = -2$. Either way, the submodule it generates has zero intersection with \mathcal{M} , hence it vanishes by Theorem 1.1b. This gives the desired conclusion.

Proposition 10.2. *For $L_{-2}(\mathfrak{so}_8)$, there exist in category \mathcal{O} non-split extensions of $\text{Ind}[\mathcal{L}_{\Lambda_2}]$ by $\text{Ind}[\mathcal{L}_{\Lambda_i}]$, where $i = 1, 3, 4$. In other words, there exist indecomposable $L_{-2}(\mathfrak{so}_8)$ -modules $\text{Ind}[\mathcal{M}_i]$, $i = 1, 3, 4$, in \mathcal{O} such that*

$$0 \longrightarrow \text{Ind}[\mathcal{L}_{\Lambda_i}] \longrightarrow \text{Ind}[\mathcal{M}_i] \longrightarrow \text{Ind}[\mathcal{L}_{\Lambda_2}] \longrightarrow 0 \quad (10.4)$$

is exact.

As before, this follows for $i = 3, 4$ from the argument above for $\mathcal{M}_1 = \mathcal{M}$ by applying outer automorphisms.

We conclude by noting that the above arguments also establish the existence of non-split short exact sequences of modules over $U(\mathfrak{so}_8)/J$, where J is the Joseph ideal of \mathfrak{so}_8 . Recall that when \mathfrak{g} is not of type A, there is a unique completely prime primitive ideal of $U(\mathfrak{g})$, the *Joseph ideal*, whose associated variety is the closure of the minimal nilpotent orbit in \mathfrak{g}^* . As $\text{Zhu}[L_{-2}(\mathfrak{so}_8)] \simeq \mathbb{C} \times U(\mathfrak{so}_8)/J$ [66, Thm. 3.1], it follows that a $\text{Zhu}[L_{-2}(\mathfrak{so}_8)]$ -module with no composition factor isomorphic to \mathcal{L}_0 is automatically a $U(\mathfrak{so}_8)/J$ -module.

Corollary 10.3. *There exist non-split short exact sequences of $\mathcal{U}(\widehat{\mathfrak{so}}_8)/\mathcal{J}$ -modules, including*

$$0 \longrightarrow \mathcal{L}_{\Lambda_i} \longrightarrow \mathcal{M}_i \longrightarrow \mathcal{L}_{\Lambda_2} \longrightarrow 0, \quad i = 1, 3, 4. \quad (10.5)$$

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