

TENSOR CATEGORIES ARISING FROM THE VIRASORO ALGEBRA

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ABSTRACT. We show that there is a braided tensor category structure on the category of C_1 -cofinite modules for the (universal or simple) Virasoro vertex operator algebras of arbitrary central charge. In the generic case of central charge $c = 13 - 6(t + t^{-1})$, with $t \notin \mathbb{Q}$, we prove semisimplicity, rigidity and non-degeneracy and also compute the fusion rules of this tensor category.

1. INTRODUCTION

1.1. Conformal-field-theoretic tensor categories. Tensor categories arising in conformal field theory have been studied since the late 1980s. Moore and Seiberg were the first to realize that an axiomatic formulation of rational two-dimensional conformal field theory essentially leads to what we now call a modular tensor category [MS]. A vertex operator algebra is a mathematically rigorous formulation of the chiral algebra of a two dimensional conformal field theory and, thus, it is a natural general expectation that sufficiently nice categories of modules of a given vertex operator algebra form rigid braided tensor categories.

In particular, Moore and Seiberg predicted [MS] that the category of integrable highest-weight modules for an affine Lie algebra at a fixed positive-integral level should have the structure of a rigid braided tensor category. Kazhdan and Lusztig were then the first to construct such a structure on a certain category of modules for affine Lie algebras [KL1]–[KL5]. However, the level in their work was not a positive integer, but was restricted so that adding the dual Coxeter number did not give a positive rational number. Then, several works, including those of Beilinson–Feigin–Mazur [BFM] and Huang–Lepowsky [HL6], constructed rigid braided tensor categories at positive-integral levels; these were in fact shown to be modular tensor categories.

The Huang–Lepowsky construction of these categories is based on the tensor category theory for general vertex operator algebras which they developed in [HL1]–[HL6] and [H1], in the rational case, and in [HLZ1]–[HLZ9], for the logarithmic case (with the latter work joint with Zhang). This theory turns out to be a powerful tool to construct tensor category structures and prove results in vertex operator algebras, topology, mathematical physics and related areas. Currently, however, almost all known examples of braided tensor category structures on categories of modules of a vertex operator algebra are for algebras satisfying Zhu’s C_2 -cofiniteness condition.

We would like to thank Yi-Zhi Huang, Robert McRae and Antun Milas for many useful discussions. We also thank the referee for their valuable comments and suggestions. T. C. is supported by NSERC #RES0020460, C. J. is supported by NSFC grants 11771281 and 11531004, DR’s research is supported by the Australian Research Council Discovery Project DP160101520.

The most obvious non- C_2 -cofinite examples are the Heisenberg vertex operator algebras. The category generated by their highest-weight modules, with real highest weights, was shown to be a braided tensor category (in fact a vertex tensor category) in [CKLR]. The first substantial examples of non-rational and non- C_2 -cofinite vertex operator algebras are the affine vertex operator algebras at admissible levels where the category of ordinary modules has been shown to admit a braided tensor category structure [CHY]. Rigidity was also proven in the case that the Lie algebra is simply-laced [C1]. The non-generic but non-admissible-level case is the most challenging and here the tensor category structure is not yet understood (for recent progress, see [ACGY, CY]).

Here, we study the important family of vertex operator algebras coming from representations of the Virasoro algebra. For $c \in \mathbb{C}$, denote by $M(c, 0)$ and $L(c, 0)$ the universal Virasoro vertex operator algebra and its unique simple quotient, respectively [FZ1]. The representation theories of both have been studied intensively in the mathematics and physics literature. In particular, the Virasoro minimal models correspond to $L(c_{p,q}, 0)$, where

$$c_{p,q} = 1 - 6 \frac{(p-q)^2}{pq}, \quad \text{for } p, q \in \mathbb{Z}_{\geq 2} \quad \text{and} \quad \gcd\{p, q\} = 1. \quad (1.1)$$

The vertex operator algebras $L(c_{p,q}, 0)$ are rational [W] and C_2 -cofinite [DLM]. Huang's general theorems [H2, H3] for rational and C_2 -cofinite vertex operator algebras then show that the modules of $L(c_{p,q}, 0)$ form a modular tensor category.

1.2. The problem. In this paper, we aim to construct braided tensor structures on certain categories of $M(c, 0)$ -modules for arbitrary $c \in \mathbb{C}$ (note that $M(c, 0) = L(c, 0)$ unless $c = c_{p,q}$ as in (1.1)). As $M(c, 0)$ is neither rational nor C_2 -cofinite, constructing such tensor category structures for these types of vertex operator algebras is usually very hard due to the fact that they generally have infinitely many simple objects and may admit nontrivial extensions. Another difficult problem is to establish rigidity — even for semisimple categories this is usually very difficult.

Given $c \in \mathbb{C}$, let H_c denote the set of $h \in \mathbb{C}$ such that the Virasoro Verma module $V(c, h)$ of central charge c and conformal weight h is reducible. Let $L(c, h)$ denote the simple quotient of $V(c, h)$. We study the category $\mathcal{O}_c^{\text{fin}}$ of finite-length $M(c, 0)$ -modules whose composition factors are of the form $L(c, h)$, with $h \in H_c$. It turns out that the objects of this category are all C_1 -cofinite. It is actually very natural to consider the category \mathcal{C}_1 of lower-bounded C_1 -cofinite modules, not only because the C_1 -cofiniteness condition is a very minor restriction and is relatively easy to verify for familiar families of vertex operator algebras, but also because the fusion product of two C_1 -cofinite modules is again C_1 -cofinite [Mi] (see also [N]). However, this category is not abelian in general because it need not be closed under taking submodules and contragredient duals. Furthermore, the associativity isomorphism that plays a key role in the tensor category theory of Huang, Lepowsky and Zhang is also hard to verify for \mathcal{C}_1 .

In [H3], Huang studied the category of grading-restricted generalized modules for C_1 -cofinite vertex operator algebras (in the sense of Li [Li1]) and proved that they form a braided tensor

category if the simple objects are \mathbb{R} -graded, C_1 -cofinite and there exists a positive integer N such that the differences between the real parts of the highest conformal weights of the simple objects are bounded by N and the level- N Zhu algebra is finite-dimensional. In particular, if the vertex operator algebra is of positive-energy (that is, if $V_{(n)} = 0$ for $n < 0$ and $V_{(0)} = \mathbb{C}\mathbf{1}$) and has only finitely many simple modules (as is the case when it is C_2 -cofinite), then the category of grading-restricted generalized modules has a braided tensor category structure. Unfortunately, Huang's conditions do not hold for the Virasoro vertex operator algebras, mainly because of the existence of infinitely many simple modules.

1.3. The main results. We first show that the category \mathcal{C}_1 of lower-bounded C_1 -cofinite generalized $M(c, 0)$ -modules is actually the same as the category $\mathcal{O}_c^{\text{fin}}$ of finite-length $M(c, 0)$ -modules with composition factors $L(c, h)$ for $h \in H_c$. Since it is obvious that the category $\mathcal{O}_c^{\text{fin}}$ is closed under taking submodules and contragredient duals, this bypasses the difficulty of showing that \mathcal{C}_1 is abelian directly. The identification of these two categories, together with the applicability result in [H4], also verifies the associativity isomorphism needed to invoke the logarithmic tensor category theory of Huang, Lepowsky and Zhang. As a result, the category $\mathcal{O}_c^{\text{fin}}$ has a braided tensor category structure.

We also prove rigidity and non-degeneracy for the category $\mathcal{O}_c^{\text{fin}}$ when c is generic, that is, when $c = 13 - 6(t + t^{-1})$ with $t \notin \mathbb{Q}$. A full subcategory \mathcal{O}_c^{L} of this category has been studied in [FZ2] by I. Frenkel and M. Zhu. We first show that the category $\mathcal{O}_c^{\text{fin}}$ is semisimple, with infinitely many simple objects, and then compute the fusion rules of these simples. As a consequence of these computations and the coset realization of the Virasoro vertex operator algebra [GKO, ACL], we show that there is a braided tensor equivalence between the full subcategory \mathcal{O}_c^{L} and a simple current twist of the category of ordinary modules for the affine vertex operator algebra $V_\ell(\mathfrak{sl}_2)$ at a certain non-rational level ℓ depending on t . This shows that the objects in \mathcal{O}_c^{L} are rigid. Since the category $\mathcal{O}_c^{\text{fin}}$ is generated by the simple objects in \mathcal{O}_c^{L} and a similar full subcategory \mathcal{O}_c^{R} (by the fusion rules of Theorem 5.2.5), the category $\mathcal{O}_c^{\text{fin}}$ is rigid.

To summarize, the main theorem of the paper is as follows.

Main Theorem. *Let $\mathcal{O}_c^{\text{fin}}$ denote the category of finite-length $M(c, 0)$ -modules of central charge $c = 13 - 6(t + t^{-1})$ whose composition factors are C_1 -cofinite. Then,*

- (1) $\mathcal{O}_c^{\text{fin}}$ has a braided tensor category structure (Theorem 4.2.6).
- (2) For $t \notin \mathbb{Q}$, $\mathcal{O}_c^{\text{fin}}$ is semisimple, rigid and non-degenerate (Theorems 5.1.2 and 5.5.3 and Proposition 5.5.5).

Recently, the tensor structures of the categories $\mathcal{O}_c^{\text{fin}}$ with $t = 1/p, p \in \mathbb{Z}_{\geq 1}$, were studied by McRae and the last author in [MY], where the fusion rules and rigidity of $\mathcal{O}_c^{\text{fin}}$ are established.

1.4. Applications and future work. Rigid braided vertex tensor categories are relevant in various modern problems. For example, the quantum geometric Langlands program can be related to equivalences of tensor categories of modules of W -algebras and affine vertex algebras, see [AFO,

Sec. 6] for example. In fact, our rigidity proof (Theorem 5.5.3) also proves one case of Conjecture 6.4 in [AFO] for $G = SU(2)$. From a physics perspective, this relates to the S -duality and again vertex tensor categories are crucial as they describe categories of line defects ending on topological boundary conditions [CGai, FGai].

Vertex tensor categories also allow one to construct module categories of vertex algebras out of those of certain subalgebras [KO, HKL, CKM1]. Our results are very useful from this point of view for various important vertex algebras as they all contain the Virasoro vertex operator algebra as their subalgebra. For example, in the context of S -duality, vertex superalgebras that are extensions in a completion of $\mathcal{O}_c^{\text{fin}}$ appear in [CGL]. As an instructive example, the simple affine vertex operator superalgebra $L_k(\mathfrak{osp}(1|2))$ at positive-integer levels [CFK] and, more generally at all admissible levels [CKLiR], has been well-understood by viewing it as an extension of the tensor product of $L_k(\mathfrak{sl}(2))$ and a Virasoro minimal model. The results of our work allow us to extend these studies to generic levels.

More importantly, the vertex algebras with the best understood non-semisimple representation categories are associated to extensions of the Virasoro vertex operator algebra at central charge $c_{1,p}$. These are the triplet algebras $\mathcal{W}(p)$ [Ka, GKa2, AM1, NT, TW, FGST], the singlet algebras $\mathcal{M}(p)$ [A1, AM2, CM, RW1] and the logarithmic $\mathcal{B}(p)$ algebras [A2, CRW, ACKR]. Note that the $\mathcal{B}(p)$ -algebras are extensions of the Virasoro algebra times a rank one Heisenberg algebra and that $\mathcal{B}(2)$ is the rank one $\beta\gamma$ vertex algebra whose even subalgebra is $L_{-1/2}(\mathfrak{sl}(2))$ [R].

An illustration of the usefulness of tensor categories is the recent work [ACGY]. There, uniqueness results of certain vertex operator algebra extensions of the Virasoro algebra at central charge $c_{1,p}$ are derived [ACGY, Thm. 8 and Cor. 14]. These results could only be proven because of our Main Theorem. Moreover, the uniqueness statements were then employed to resolve affirmatively the conjectures (see [CRW, C2, ACKR]) that the $\mathcal{B}(p)$ algebra is a subregular W -algebra of type \mathfrak{sl}_{p-1} and also a chiral algebra of Argyres-Douglas theories of type (A_1, A_{2p-3}) .

Another important application in this spirit is to construct and study rigid braided tensor category structures for these extensions of Virasoro vertex operator algebras. While braided tensor structures on the category of generalized modules for the triplet algebras $\mathcal{W}(p)$ have been proven to exist [H3] and the rigidity and fusion rules are known [TW], rigid braided tensor structures on the corresponding module categories for the singlet vertex algebras $\mathcal{M}(p)$ and the $\mathcal{B}(p)$ -algebras have only recently been constructed and studied in [CMY2]. This study combined the Virasoro tensor categories constructed in the present paper with direct limit completion technology for vertex and braided tensor categories [CMY1]. We expect to apply the techniques and methodology developed in the present paper, as well as in [CMY1, CMY2], to study the vertex and braided tensor structures of the representation categories of many more vertex operator algebras.

2. PRELIMINARIES

2.1. Vertex operator algebras. Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra. We first recall the definitions of various types of V -modules.

Definition 2.1.1.

(1) A *weak V -module* is a vector space W equipped with a vertex operator map

$$\begin{aligned} Y_W: V &\rightarrow (\text{End } W)[[x, x^{-1}]], \\ v &\mapsto Y_W(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}, \end{aligned} \quad (2.1)$$

satisfying the following axioms:

- (i) The lower truncation condition: for $u, v \in V$, $Y_W(u, x)v$ has only finitely many terms with negative powers in x .
- (ii) The vacuum property: $Y_W(\mathbf{1}, x)$ is the identity endomorphism 1_W of W .
- (iii) The Jacobi identity: for $u, v \in V$,

$$\begin{aligned} &x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_W(u, x_1) Y_W(v, x_2) \\ &- x_0^{-1} \delta \left(\frac{-x_2 + x_1}{x_0} \right) Y_W(v, x_2) Y_W(u, x_1) \\ &= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y_W(Y(u, x_0)u, x_2). \end{aligned} \quad (2.2)$$

- (iv) The Virasoro algebra relations: if we write $Y_W(\omega, x) = \sum_{n \in \mathbb{Z}} L_n x^{-n-2}$, then for any $m, n \in \mathbb{Z}$, we have

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12} \delta_{m+n, 0} c 1_W, \quad (2.3)$$

where $c \in \mathbb{C}$ is the central charge of V .

- (v) The L_{-1} -derivative property: for any $v \in V$,

$$Y_W(L_{-1}v, x) = \frac{d}{dx} Y_W(v, x). \quad (2.4)$$

(2) A *generalized V -module* is a weak V -module (W, Y_W) with a \mathbb{C} -grading

$$W = \coprod_{n \in \mathbb{C}} W_{[n]} \quad (2.5)$$

such that $W_{[n]}$ is a generalized eigenspace for the operator L_0 with eigenvalue n .

- (3) A *lower-bounded generalized V -module* is a generalized V -module such that for any $n \in \mathbb{C}$, $W_{[n+m]} = 0$ for $m \in \mathbb{Z}$ sufficiently negative.
- (4) A *grading-restricted generalized V -module* is a lower-bounded generalized V -module such that $\dim W_{[n]} < \infty$ for any $n \in \mathbb{C}$.
- (5) An *ordinary V -module* (sometimes just called a V -module for brevity) is a grading-restricted generalized V -module such that the $W_{[n]}$ in (2.5) are eigenspaces for the operator L_0 .
- (6) A generalized V -module W has *length l* if there exists a filtration $W = W_1 \supset \cdots \supset W_{l+1} = 0$ of generalized V -submodules such that each W_i/W_{i+1} is irreducible. A *finite-length* generalized V -module is one whose length is finite.

Definition 2.1.2. A vertex operator algebra is *rational* if every weak module is a direct sum of simple ordinary modules. We say that a category of ordinary V -modules is *semisimple* if every ordinary module is a direct sum of simple ordinary modules.

Let V be a vertex operator algebra and let (W, Y_W) be a lower-bounded generalized V -module, graded as in (2.5). Its *contragredient module* is then the vector space

$$W' = \prod_{n \in \mathbb{C}} (W_{[n]})^*, \quad (2.6)$$

equipped with the vertex operator map Y' defined by

$$\langle Y'(v, x)w', w \rangle = \langle w', Y_W^\circ(v, x)w \rangle \quad (2.7)$$

for any $v \in V$, $w' \in W'$ and $w \in W$, where

$$Y_W^\circ(v, x) = Y_W(e^{xL(1)}(-x^{-2})^{L_0}v, x^{-1}), \quad (2.8)$$

for any $v \in V$, is the *opposite vertex operator* (see [FHL]). We also use the standard notation

$$\overline{W} = \prod_{n \in \mathbb{C}} W_{[n]}, \quad (2.9)$$

for the formal completion of W with respect to the \mathbb{C} -grading.

The notion of a logarithmic intertwining operator [M1, HLZ3] plays a key role in the study of the representations of vertex operator algebras that are not rational, such as the Virasoro vertex operator algebras discussed in this paper. Let $W\{x\}$ denote the space of formal power series in arbitrary complex powers of x with coefficients in W .

Definition 2.1.3. Let (W_1, Y_1) , (W_2, Y_2) and (W_3, Y_3) be generalized V -modules. A *logarithmic intertwining operator of type* $\binom{W_3}{W_1 W_2}$ is a linear map

$$\begin{aligned} \mathcal{Y}: W_1 \otimes W_2 &\rightarrow W_3\{x\}[\log x], \\ w_{(1)} \otimes w_{(2)} &\mapsto \mathcal{Y}(w_{(1)}, x)w_{(2)} = \sum_{k=0}^K \sum_{n \in \mathbb{C}} \left(w_{(1)} \mathcal{Y}_{n,k} w_{(2)} \right) x^{-n-1} (\log x)^k, \end{aligned} \quad (2.10)$$

satisfying the following conditions:

- (i) The lower truncation condition: given any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $k = 0, \dots, K$ and $n \in \mathbb{C}$,

$$w_{(1)} \mathcal{Y}_{n+m,k} w_{(2)} = 0 \quad \text{for } m \in \mathbb{Z} \text{ sufficiently large.} \quad (2.11)$$

(ii) The Jacobi identity:

$$\begin{aligned}
 & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_3(v, x_1) \mathcal{Y}(w_{(1)}, x_2) w_{(2)} \\
 & - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \mathcal{Y}(w_{(1)}, x_2) Y_2(v, x_1) w_{(2)} \\
 & = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \mathcal{Y}(Y_1(v, x_0) w_{(1)}, x_2) w_{(2)},
 \end{aligned} \tag{2.12}$$

for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$.

(iii) The L_{-1} -derivative property: for any $w_{(1)} \in W_1$,

$$\mathcal{Y}(L_{-1}w_{(1)}, x) = \frac{d}{dx} \mathcal{Y}(w_{(1)}, x). \tag{2.13}$$

A logarithmic intertwining operator \mathcal{Y} is called an *intertwining operator* if no $\log x$ appears in $\mathcal{Y}(w_{(1)}, x)w_{(2)}$, for any $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. The dimension of the space $\mathcal{V}_{W_1 W_2}^{W_3}$ of all logarithmic intertwining operators is called the *fusion coefficient* or *fusion rule* of the same type; it is denoted by $\mathcal{N}_{W_1 W_2}^{W_3}$.

We remark that the term ‘‘fusion rule’’ commonly has a different meaning in the literature, referring instead to an explicit identification of the isomorphism class of a ‘‘fusion product’’.

The following finiteness condition plays an important role in this paper:

Definition 2.1.4. Let V be a vertex operator algebra and let W be a weak V -module. Let $C_1(W)$ be the subspace of W spanned by elements of the form $u_{-1}w$, where $u \in V_+ := \coprod_{n \in \mathbb{Z}_{>0}} V_n$ and $w \in W$. We say that W is C_1 -cofinite if the C_1 -quotient $W/C_1(W)$ is finite-dimensional.

We shall also need the following facts.

Lemma 2.1.5 ([H1]). *Let V be a vertex operator algebra.*

(1) *For a (generalized) V -module W and a (generalized) V -submodule U of W ,*

$$C_1(W/U) = \frac{C_1(W) + U}{U} \quad \text{and so} \quad \frac{W/U}{C_1(W/U)} \cong \frac{W}{C_1(W) + U}. \tag{2.14}$$

It follows that W/U is C_1 -cofinite if W is.

(2) *If W is a finite-length V -module with C_1 -cofinite composition factors, then W is also C_1 -cofinite.*

2.2. Representations of Virasoro algebra. The Virasoro algebra is the Lie algebra

$$\mathcal{L} = \bigoplus_{n=-\infty}^{+\infty} \mathbb{C}L_n \oplus \mathbb{C}\underline{c} \tag{2.15}$$

with the commutation relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12} \delta_{m+n, 0} \underline{c}, \quad [\mathcal{L}, \underline{c}] = 0. \tag{2.16}$$

Set

$$\mathcal{L}_{\geq 0} = \bigoplus_{n \geq 0} \mathbb{C}L_n \oplus \mathbb{C}\mathcal{L} \quad (2.17)$$

and recall that for $c, h \in \mathbb{C}$, the Verma module $V(c, h)$ for \mathcal{L} is defined to be

$$V(c, h) = \mathcal{U}(\mathcal{L}) \otimes_{\mathcal{U}(\mathcal{L}_{\geq 0})} \mathbb{C}\mathbf{1}_{c,h}, \quad (2.18)$$

where $\mathcal{U}(-)$ denotes a universal enveloping algebra and the $\mathcal{L}_{\geq 0}$ -module structure of $\mathbb{C}\mathbf{1}_{c,h}$ is given by $L_0\mathbf{1}_{c,h} = h\mathbf{1}_{c,h}$, $\mathcal{L}\mathbf{1}_{c,h} = c\mathbf{1}_{c,h}$ and $L_n\mathbf{1}_{c,h} = 0$ for $n > 0$. As usual, the Verma module $V(c, h)$ has a unique (possibly trivial) maximal proper submodule. We denote its unique simple quotient by $L(c, h)$.

When $h = 0$, $L_{-1}\mathbf{1}_{c,0}$ is a singular vector in $V(c, 0)$ (the tensor product symbol is omitted for brevity). It was shown in [FZ1] that

$$M(c, 0) = V(c, 0) / \langle L_{-1}\mathbf{1}_{c,0} \rangle \quad (2.19)$$

admits the structure of a vertex operator algebra. It is called the *universal Virasoro vertex operator algebra* of central charge c . The simple quotient $L(c, 0)$ of $M(c, 0)$ therefore admits a vertex operator algebra structure as well. Note that all $L(c, 0)$ - and $M(c, 0)$ -modules are \mathcal{L} -modules. Another important result of [FZ1] is that every highest-weight \mathcal{L} -module of central charge c is an $M(c, 0)$ -module.

We recall the existence criterion of Feigin and Fuchs for singular vectors in Verma \mathcal{L} -modules. Useful expositions may be found in [IK, KRa]. Note that a Verma module is reducible if and only if it possesses a non-trivial singular vector (by which we mean one that is not proportional to the cyclic highest-weight vector).

Proposition 2.2.1 ([FF]). *For $r, s \in \mathbb{Z}_{\geq 1}$ and $t \in \mathbb{C} \setminus \{0\}$, define*

$$c = c(t) = 13 - 6t - 6t^{-1}, \quad h = h_{r,s}(t) = \frac{r^2 - 1}{4}t - \frac{rs - 1}{2} + \frac{s^2 - 1}{4}t^{-1}. \quad (2.20)$$

- (1) *If there exist $r, s \in \mathbb{Z}_{\geq 1}$ and $t \in \mathbb{C} \setminus \{0\}$ such that c and h satisfy (2.20), then there is a singular vector of weight $h + rs$ in the Verma module $V(c, h)$.*
- (2) *Conversely, if $V(c, h)$ possesses a non-trivial singular vector, then there exist $r, s \in \mathbb{Z}_{\geq 1}$ and $t \in \mathbb{C} \setminus \{0\}$ such that (2.20) holds.*
- (3) *For each N , there exists at most one singular vector of weight $h + N$ in $V(c, h)$, up to scalar multiples.*

Of course, any singular vector in $V(c, h)$ may be expressed as a linear combination of Poincaré–Birkhoff–Witt-ordered monomials in the L_i , $i < 0$, acting on the highest-weight vector $\mathbf{1}_{c,h}$. A crucial fact for what follows is that the coefficient of L_{-1}^N is never 0 (irrespective of the chosen order). Here, N is the conformal weight of the singular vector minus that of $\mathbf{1}_{c,h}$.

Proposition 2.2.2 ([As]). *If $V(c, h)$ has a singular vector v of conformal weight $h + N$, then*

$$v = \sum_{|I|=N} a_I(c, h) L_{-I} \mathbf{1}_{c,h}, \quad (2.21)$$

where $L_{-I} = L_{-i_1} \cdots L_{-i_n}$ and the sum is over sequences $I = \{i_1, \dots, i_n\}$ of ordered n -tuples $i_1 \geq \dots \geq i_n$ with $|I| = i_1 + \dots + i_n = N$. Moreover, the coefficients $a_I(c, h)$ depend polynomially on c and h and the coefficient $a_{\{1, \dots, 1\}}(c, h)$ of L_{-1}^N may be chosen to be 1.

Notation 2.2.3. For $c = c(t) \in \mathbb{C}$, set

$$H_c = \{h \mid V(c, h) \text{ is reducible}\}. \quad (2.22)$$

By Proposition 2.2.1, we have $H_c = \{h_{r,s}(t) \mid r, s \in \mathbb{Z}_{\geq 1}\}$. For brevity, we shall also denote the Verma module $V(c(t), h_{r,s}(t))$ and its simple quotient by $V_{r,s}$ and $L_{r,s}$, respectively.

The set H_c of conformal weights is often referred to as the *extended Kac table* because the original Kac table of the Virasoro minimal models corresponds to the subset of $h_{r,s}(t)$ with $t = \frac{p}{q}$ ($p, q \in \mathbb{Z}_{\geq 2}$ and $\gcd\{p, q\} = 1$), $r = 1, \dots, p-1$ and $s = 1, \dots, q-1$. This subset consists precisely of the conformal weights of the simple $L(c(t), 0)$ -modules [W, RW2].

The embedding structure of Virasoro Verma modules is also due to Feigin and Fuchs. A convenient summary appears in [IK, Ch. 5].

Theorem 2.2.4 ([FF]). *The embedding structures of the reducible Verma \mathcal{L} -modules $V_{r,s}$, where $r, s \in \mathbb{Z}_{\geq 1}$, are as follows:*

- (1) *If $t = \frac{q}{p}$, for $p, q \in \mathbb{Z}_{\geq 1}$ and $\gcd\{p, q\} = 1$, then there are two possible “shapes” for the embedding diagrams (see [IK] for further details): If r is a multiple of p or s is a multiple of q , then one has the embedding chain*

$$V_{r,s} \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \dots \quad (2.23)$$

Otherwise, the embedding diagram is as follows:

$$\begin{array}{ccccccc} & & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \dots \\ & & \swarrow & & \swarrow & & \swarrow & & \swarrow & & \dots \\ V_{r,s} & & \bullet & & \bullet & & \bullet & & \bullet & & \dots \\ & & \nwarrow & & \nwarrow & & \nwarrow & & \nwarrow & & \dots \\ & & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \dots \end{array} \quad (2.24)$$

- (2) *If $t = -\frac{q}{p}$ for $p, q \in \mathbb{Z}_{\geq 1}$ and $\gcd\{p, q\} = 1$, then there are again two possible “shapes”, similar to those in (1) except that the diagrams are now finite (again the details may be found in [IK]):*

If r is a multiple of p or s is a multiple of q , then one has the embedding chain

$$V_{r,s} \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \dots \leftarrow \bullet \quad (2.25)$$

Otherwise, the embedding diagram is as follows.

$$(2.26)$$

(3) If $t \notin \mathbb{Q}$, then the embedding diagram is $V_{r,s} \leftarrow V_{-r,s}$.

Corollary 2.2.5. *It follows from these embedding structures that:*

- (1) Every non-zero submodule of a Verma \mathcal{L} -module is either a Verma module itself or the sum of two Verma modules.
- (2) Every non-Verma highest-weight \mathcal{L} -module has finite length.
- (3) Every Verma module $V(c, h)$ in the embedding diagram of a reducible Verma \mathcal{L} -module satisfies $h \in H_c$, with the exception of the socle which, if it exists, has the form $V(c, h')$ for some $h' \notin H_c$.
- (4) $M(c, 0) = L(c, 0)$ unless the central charge satisfies $c = c(\frac{p}{q}) = c_{p,q}$, where $p, q \in \mathbb{Z}_{\geq 2}$ and $\gcd\{p, q\} = 1$ (see (1.1)).

In light of Corollary 2.2.5(4) and the fact that the $L(c_{p,q}, 0)$ -modules ($p, q \in \mathbb{Z}_{\geq 2}$ and $\gcd\{p, q\} = 1$) form a modular tensor category [H2, H3], we shall restrict our considerations in what follows to modules of the universal Virasoro vertex operator algebras $M(c, 0)$ (for arbitrary $c \in \mathbb{C}$).

Remark 2.2.6. The representations of $M(c, 0)$ have been investigated in both the mathematics and physics literature. We list some relevant work here.

- (1) The representation theory of $M(c, 0)$ has been explored in detail by physicists under the moniker “logarithmic minimal models” [PRZ, RS, MRR]. In particular, fusion products were studied [GKa1, EF, MR] in order to determine the types of non-semisimple modules that appeared at central charges of the form $c_{p,q}$, $p, q \in \mathbb{Z}_{\geq 1}$, especially the so-called *staggered modules* [Ro, KyR] that arise in logarithmic conformal field theories [RoS, G, F2, Ga, CR].
- (2) The famous triplet algebras $\mathcal{W}(p)$ [Ka, F1, GKa2, AM1, NT, TW, FGST] and singlet algebras $\mathcal{M}(p)$ [A1, AM2, CM, RW1] have been extensively studied as extensions of $M(c_{1,p}, 0)$. The $\mathcal{B}(p)$ -algebras [A2, CRW, ACKR] are similarly extensions of $M(c_{1,p}, 0)$ times a rank one Heisenberg vertex operator algebra.
- (3) The representations and fusion rules of the Virasoro algebra of central charge $c_{1,1} = 1$ have been studied in [M2]. In [Mc], these fusion rules were used to prove that the semisimple full subcategory of $M(1, 0)$ -modules generated by the $L(1, \frac{n^2}{4})$, $n \in \mathbb{Z}_{\geq 0}$, is braided-tensor and tensor equivalent to a modification of the category of finite-dimensional modules of \mathfrak{sl}_2 involving 3-cocycle on $\mathbb{Z}/2\mathbb{Z}$.
- (4) For the case $t \notin \mathbb{Q}$, the category of modules generated by the $V_{1,s}$ with $s \in \mathbb{Z}_{\geq 1}$ was studied in [FZ2] and the fusion rules of this category were determined. These rules will be crucial to our results, see Section 5.2 below.

The vertex operator algebra $M(c, 0)$ is neither rational nor C_2 -cofinite because it has infinitely many inequivalent simple modules. It is therefore natural to study the C_1 -cofiniteness of an $M(c, 0)$ -module W . One can check that

$$C_1(W) = \sum_{n=2}^{\infty} L_{-n}W, \quad (2.27)$$

hence the C_1 -quotient of a highest-weight $M(c, 0)$ -module is spanned by the powers of L_{-1} acting on the cyclic highest-weight vector. We therefore have the following consequences of Proposition 2.2.2 and Corollary 2.2.5.

Corollary 2.2.7.

- (1) A highest-weight $M(c, 0)$ -module is C_1 -cofinite if and only if it is not isomorphic to a Verma module. In particular, $L(c, h)$ is C_1 -cofinite if and only if $h \in H_c$.
- (2) A C_1 -cofinite highest-weight $M(c, 0)$ -module has finite length.
- (3) The composition factors of a C_1 -cofinite highest-weight $M(c, 0)$ -module are of the form $L(c, h)$ with $h \in H_c$.

3. EQUIVALENCE OF TWO CATEGORIES

Recall that \mathcal{C}_1 denotes the category of lower-bounded C_1 -cofinite generalized $M(c, 0)$ -modules and that $\mathcal{O}_c^{\text{fin}}$ denotes the category of finite-length $M(c, 0)$ -modules with composition factors $L(c, h)$ for $h \in H_c$. In this section, we will prove that these two categories are the same.

From Lemma 2.1.5(2) and Corollary 2.2.7, we have $\mathcal{O}_c^{\text{fin}} \subseteq \mathcal{C}_1$. We therefore only need to prove the reverse inclusion in what follows.

3.1. The reverse inclusion.

Lemma 3.1.1. *Let V be a vertex operator algebra and let W be a generalized V -module.*

- (1) Suppose that $0 \neq w \in W$ has L_0 -eigenvalue $h \in \mathbb{C}$ and that $h - n$ is not an eigenvalue of L_0 for any $n \in \mathbb{Z}_{\geq 1}$. Then, $w \notin C_1(W)$.
- (2) If W is a lower-bounded generalized V -module satisfying $W = C_1(W)$, then $W = 0$.

Proof. If $w \in C_1(W)$, then there would exist homogeneous elements $u^i \in M(c, 0)$, with $\text{wt } u_i \in \mathbb{Z}_{\geq 1}$, and $w^i \in W$ such that $w = \sum_i u_{-1}^i w^i$. As $w \neq 0$, there exists at least one non-zero term $u_{-1}^k w^k$ in this decomposition. But, $\text{wt } w^k = \text{wt } w - \text{wt } u^k \in h - \mathbb{Z}_{\geq 1}$, a contradiction. This proves (1).

For (2), assume that $W \neq 0$. As W is lower-bounded, there exists an L_0 -eigenvalue $h \in \mathbb{C}$ such that $h - n$ is not an eigenvalue of L_0 for any $n \in \mathbb{Z}_{\geq 1}$. If w is the corresponding eigenvector, then $w \in W = C_1(W)$, contradicting (1). \square

Proposition 3.1.2. *If $W \in \mathcal{C}_1$, then there exists a finite filtration*

$$0 = W_0 \subset W_1 \subset \cdots \subset W_{n-1} \subset W_n = W \quad (3.1)$$

such that the quotients W_i/W_{i-1} , $i = 1, \dots, n$, are highest-weight $M(c, 0)$ -modules.

Proof. Since W is lower-bounded, we can choose an L_0 -eigenvector $w_1 \in W$ whose eigenvalue h is such that there are no L_0 -eigenvalues of the form $h - n$, $n \in \mathbb{Z}_{\geq 1}$. It follows that w_1 is a highest-weight vector; moreover, $w_1 \notin C_1(W)$ by Lemma 3.1.1(1). Denote by W_1 the highest-weight submodule of W generated by w_1 . From Lemma 2.1.5(1), W/W_1 is also a lower-bounded C_1 -cofinite generalized module. Moreover, $w_1 \notin C_1(W)$ gives $C_1(W) \subset C_1(W) + W_1$ and hence

$$\dim \frac{W/W_1}{C_1(W/W_1)} = \dim \frac{W}{C_1(W) + W_1} < \dim \frac{W}{C_1(W)} < \infty. \quad (3.2)$$

The same argument now shows that W/W_1 has a highest-weight vector

$$\overline{w_2} \notin C_1(W/W_1) = \frac{C_1(W) + W_1}{W_1} \Rightarrow w_2 \notin C_1(W) + W_1, \quad (3.3)$$

where $w_2 \in W$ is any element whose image in W/W_1 is $\overline{w_2}$. Now, $\overline{w_2}$ generates a highest-weight module $\overline{W_2} \subseteq W/W_1$. It follows that there exists $W_2 \subseteq W$ such that $W_2/W_1 \cong \overline{W_2}$ is highest-weight and $W_1 \subset W_2$ gives $C_1(W) + W_1 \subset C_1(W) + W_2$, by (3.3). In other words, we have obtained a sequence of epimorphisms

$$W \twoheadrightarrow W/W_1 \twoheadrightarrow (W/W_1)/(W_2/W_1) \cong W/W_2 \quad (3.4)$$

satisfying $\dim W/(C_1(W) + W_2) < \dim W/(C_1(W) + W_1)$, hence

$$\dim \frac{W/W_2}{C_1(W/W_2)} < \dim \frac{W/W_1}{C_1(W/W_1)} < \dim \frac{W}{C_1(W)} < \infty. \quad (3.5)$$

Continuing in this manner, we obtain sequences

$$W \twoheadrightarrow W/W_1 \twoheadrightarrow W/W_2 \twoheadrightarrow W/W_3 \twoheadrightarrow \dots \twoheadrightarrow W/W_{n-1} \twoheadrightarrow W/W_n \quad (3.6)$$

in which each W/W_i is lower-bounded, C_1 -cofinite and generalized. Moreover, the dimensions of the C_1 -quotients of the W/W_i are strictly decreasing as i increases. As $\dim W/C_1(W)$ is finite, there exists n such that this dimension is 0. By Lemma 3.1.1, we therefore have $W/W_n = 0$. Thus, we have obtained a sequence of submodules

$$0 \hookrightarrow W_1 \hookrightarrow W_2 \hookrightarrow \dots \hookrightarrow W_{n-1} \hookrightarrow W_n = W \quad (3.7)$$

such that $W_i/W_{i-1} \cong \overline{W_i}$ is a highest-weight $M(c, 0)$ -module. \square

Proposition 3.1.2 shows that an arbitrary $W \in \mathcal{C}_1$ is composed of finitely many highest-weight modules, but this does not guarantee that W has finite length because the length of one of the highest-weight modules might be infinite. We therefore need the following stronger result whose proof is deferred until the following section.

Proposition 3.1.3. *Let $W \in \mathcal{C}_1$ have a finite filtration (3.1) such that the quotients W_i/W_{i-1} , for $i = 1, \dots, n$, are highest-weight $M(c, 0)$ -modules. Then, each W_i , and hence each W_i/W_{i-1} , is C_1 -cofinite.*

Given this proposition, the main theorem of this section, which plays the key role in the construction of the tensor category structure on $\mathcal{O}_c^{\text{fin}}$, is easily proven.

Theorem 3.1.4. *As categories of $M(c, 0)$ -modules, $\mathcal{C}_1 = \mathcal{O}_c^{\text{fin}}$.*

Proof. As remarked above, we only need to prove that $\mathcal{C}_1 \subseteq \mathcal{O}_c^{\text{fin}}$. Given $W \in \mathcal{C}_1$, the highest-weight modules of Proposition 3.1.2 are \mathcal{C}_1 -cofinite, by Proposition 3.1.3, so they are finite-length, by Corollary 2.2.7(2). It follows that W is also finite-length. Moreover, the composition factors of the highest-weight modules will all have the form $L(c, h)$, with $h \in H_c$, by Corollary 2.2.7(3), hence so will those of W . We conclude that $W \in \mathcal{O}_c^{\text{fin}}$, completing the proof. \square

It only remains to prove Proposition 3.1.3.

3.2. Proof of Proposition 3.1.3. The hard work needed for this proof is isolated below as Lemma 3.2.1. For this, we consider \mathcal{L} -modules $\widetilde{W} \subseteq W$ for which W/\widetilde{W} is highest-weight. We prepare some convenient notation for what follows.

Let $\bar{w} \in W/\widetilde{W}$ be the cyclic highest-weight vector and let $w \in W$ be any element whose image in W/\widetilde{W} is \bar{w} . Since $W/\widetilde{W} = \mathcal{U}(\mathcal{L}_{<0})\bar{w}$, where $\mathcal{L}_{<0} = \bigoplus_{n<0} \mathbb{C}L_n$ is the negative Virasoro subalgebra, we have

$$W = \mathcal{U}(\mathcal{L}_{<0})w + \widetilde{W}, \quad (3.8a)$$

as vector spaces, hence

$$C_1(W) = \sum_{n=2}^{\infty} L_{-n}\mathcal{U}(\mathcal{L}_{<0})w + C_1(\widetilde{W}), \quad (3.8b)$$

by (2.27).

Lemma 3.2.1. *Given $W \in \mathcal{C}_1$, $\widetilde{W} \subseteq W$ and $w \in W$ as above, every element of $\mathcal{U}(\mathcal{L}_{<0})w \cap \widetilde{W}$ of sufficiently large conformal weight is in $C_1(\widetilde{W})$.*

Proof. Because $W \in \mathcal{C}_1$, we have $W/\widetilde{W} \in \mathcal{C}_1$, by Lemma 2.1.5(1), and so W/\widetilde{W} cannot be isomorphic to a Verma module, by Corollary 2.2.7(1). Theorem 2.2.4 therefore allows only two possibilities: either W/\widetilde{W} is isomorphic to $V(c, h)/V(c, h')$ or $V(c, h)/(V(c, h_1) + V(c, h_2))$, for some $h, h', h_1, h_2 \in \mathbb{C}$.

Assume first that $W/\widetilde{W} \cong V(c, h)/V(c, h')$. Then, there exists $U \in \mathcal{U}(\mathcal{L}_{<0})$ of weight $h' - h$ such that $U\mathbf{1}_{c,h}$ is singular in $V(c, h)$. Thus, there exists $w \in W$ such that $U\bar{w} = 0$ and so $Uw \in \widetilde{W}$. Moreover, we have

$$\mathcal{U}(\mathcal{L}_{<0})w \cap \widetilde{W} = \mathcal{U}(\mathcal{L}_{<0})Uw. \quad (3.9)$$

For convenience, we shall normalise U so that the coefficient of $L_{-1}^{h'-h}$ is 1, as in Proposition 2.2.2.

Since $W \in \mathcal{C}_1$, we have $L_{-1}^M w \in C_1(W)$ for all M sufficiently large. By (3.8b), there exist $U^{(n)} \in \mathcal{U}(\mathcal{L}_{<0})$ and $w' \in C_1(\widetilde{W})$ such that

$$L_{-1}^M w = \sum_{n=2}^{\infty} L_{-n} U^{(n)} w + w' \quad (3.10)$$

$$\Rightarrow w' = L_{-1}^M w - \sum_{n=2}^{\infty} L_{-n} U^{(n)} w \in \mathcal{U}(\mathcal{L}_{<0}) w \cap \widetilde{W} = \mathcal{U}(\mathcal{L}_{<0}) U w, \quad (3.11)$$

by (3.9). It follows that $w' = U' U w \in C_1(\widetilde{W})$, for some $U' \in \mathcal{U}(\mathcal{L}_{<0})$ of weight $N = M - h' + h$.

As $U' U w = w' = L_{-1}^M w + \dots$, the coefficient of L_{-1}^N in U' is 1 and so we may write U' in the form $L_{-1}^N + U''$, where $U'' \in \sum_{n=2}^{\infty} L_{-n} \mathcal{U}(\mathcal{L}_{<0})$. Since $U'' U w$ is obviously in $C_1(\widetilde{W})$, it then follows that $L_{-1}^N U w = w' - U'' U w \in C_1(\widetilde{W})$ for all sufficiently large N . Consequently, every element of the form $U_N U w \in \mathcal{U}(\mathcal{L}_{<0}) U w = \mathcal{U}(\mathcal{L}_{<0}) w \cap \widetilde{W}$ is guaranteed to be in $C_1(\widetilde{W})$ if the weight of $U_N \in \mathcal{U}(\mathcal{L}_{<0})$ is sufficiently large, as desired.

It remains to describe the modifications needed when $W/\widetilde{W} \cong V(c, h)/(V(c, h_1) + V(c, h_2))$. First, the role of U is now played by two elements $U_i = L_{-1}^{h_i-h} + \dots \in \mathcal{U}(\mathcal{L}_{<0})$, $i = 1, 2$, so that

$$\mathcal{U}(\mathcal{L}_{<0}) w \cap \widetilde{W} = \mathcal{U}(\mathcal{L}_{<0}) U_1 w + \mathcal{U}(\mathcal{L}_{<0}) U_2 w. \quad (3.12)$$

The element $w' \in C_1(\widetilde{W})$, defined by (3.10), therefore has the form $U'_1 U_1 w + U'_2 U_2 w$, where $U'_i \in \mathcal{U}(\mathcal{L}_{<0})$ has weight $N_i = M - h_i + h$, $i = 1, 2$.

The next step is slightly different because comparing with (3.10) leads to $U'_i = a_i L_{-1}^{N_i} + U''_i$, $i = 1, 2$, where $a_1, a_2 \in \mathbb{C}$ satisfy $a_1 + a_2 = 1$. Thus, we can only conclude that

$$a_1 L_{-1}^{N_1} U_1 w + a_2 L_{-1}^{N_2} U_2 w \in C_1(\widetilde{W}), \quad (3.13)$$

for all sufficiently large M . However, the embedding diagrams of Theorem 2.2.4 show that when a submodule of $V(c, h)$ is not Verma, then the two generating singular vectors (here of weights h_1 and h_2) have a common descendant singular vector (of weight h_3 say). Since Verma modules are free as $\mathcal{U}(\mathcal{L}_{<0})$ -modules, this means that there exist $T_1, T_2 \in \mathcal{U}(\mathcal{L}_{<0})$ such that $T_1 U_1 = T_2 U_2$. Moreover, Proposition 2.2.2 gives $T_i = L_{-1}^{h_3-h_i} + \dots$ as usual.

Assuming that M is taken sufficiently large, it now follows that $a_2 L_{-1}^{h_3-h_2} U_2 w$ may be replaced in (3.13) by $a_2 T_2 U_2 w = a_2 T_1 U_1 w$, modulo terms in $C_1(\widetilde{W})$. In other words, we arrive at $L_{-1}^N U_1 w \in C_1(\widetilde{W})$ for all N sufficiently large and, by swapping the indices 1 and 2 in this argument, also $L_{-1}^N U_2 w \in C_1(\widetilde{W})$ for all N sufficiently large. By virtue of (3.12), the proof is complete. \square

We can now prove Proposition 3.1.3. In the filtration (3.1), $W_n = W \in \mathcal{C}_1$, so it will suffice to show that $W_i \in \mathcal{C}_1$ implies that $W_{i-1} \in \mathcal{C}_1$. We therefore assume that $W_i \in \mathcal{C}_1$. As above, let $\overline{w}_j \in W_j/W_{j-1}$ be the cyclic highest-weight vector, for each $1 \leq j \leq n$, and choose $w_j \in W_j$ so that its image in W_j/W_{j-1} is \overline{w}_j .

As $w_j \in W_i$ for each $j < i$, we have $L_{-1}^{N_j} w_j \in C_1(W_i)$ for all sufficiently large N_j . By (3.8b), we may therefore write

$$L_{-1}^{N_j} w_j = \sum_{n \geq 2} L_{-n} U_j^{(n)} w_i + w'_j, \quad (3.14)$$

where $U_j^{(n)} \in \mathcal{U}(\mathcal{L}_{<0})$ and $w'_j \in C_1(W_{i-1})$. The first term on the right-hand side is clearly in $\mathcal{U}(\mathcal{L}_{<0})w_i$. However, it is also in W_{i-1} because the second term is, as is the left-hand side (because $j < i$). By Lemma 3.2.1 (with $W = W_i$, $\widetilde{W} = W_{i-1}$ and $w = w_i$), this first term therefore belongs to $C_1(W_{i-1})$ for sufficiently large N_j . But, the second term does too, hence we have

$$L_{-1}^{N_j} w_j \in C_1(W_{i-1}) \quad (3.15)$$

for all $j < i$ and sufficiently large N_j .

Iterating (3.8), with $W = W_{i-1}$, down the filtration (3.1) now gives

$$W_{i-1} = \sum_{j=1}^{i-1} \mathcal{U}(\mathcal{L}_{<0})w_j \quad \text{and} \quad C_1(W_{i-1}) = \sum_{j=1}^{i-1} \sum_{n=2}^{\infty} L_{-n} \mathcal{U}(\mathcal{L}_{<0})w_j. \quad (3.16)$$

It follows that $W_{i-1}/C_1(W_{i-1})$ is spanned by the (images of the) $L_{-1}^m w_j$, with $j < i$ and $m \in \mathbb{Z}_{\geq 0}$. By (3.15), we have $\dim W_{i-1}/C_1(W_{i-1}) < \infty$ and the proof is complete.

4. TENSOR CATEGORIES ASSOCIATED TO THE VIRASORO ALGEBRA

Recall that $\mathcal{O}_c^{\text{fin}}$ denotes the category of finite length $M(c, 0)$ -modules with composition factors $L(c, h)$ for $h \in H_c$ and note that $\mathcal{O}_c^{\text{fin}}$ is closed under taking direct sums, generalized submodules, quotient generalized modules and contragredient duals. In this section, we will construct a tensor category structure on $\mathcal{O}_c^{\text{fin}}$ by verifying that all of the conditions needed in the Huang-Lepowsky-Zhang logarithmic tensor theory in [HLZ1]–[HLZ9] hold for $\mathcal{O}_c^{\text{fin}}$. For convenience, we first recall the general constructions and main results in [HLZ1]–[HLZ9].

4.1. $P(z)$ -tensor product. In the tensor category theory for vertex operator algebras, the tensor product bifunctors are not built on the classical tensor product bifunctor for vector spaces. Instead, the central concept underlying the constructions is the notion of $P(z)$ -tensor product [HL3, HLZ4, HLZ5], where z is a nonzero complex number and $P(z)$ is the Riemann sphere $\overline{\mathbb{C}}$ with one negatively oriented puncture at ∞ and two ordered positively oriented punctures at z and 0 , with local coordinates $1/w$, $w - z$ and w , respectively. We refer to [KaR] for an expository account that motivates the definition of this tensor product, also known as the fusion product.

Definition 4.1.1. Let W_1 , W_2 and W_3 be generalized modules for a vertex operator algebra V . A $P(z)$ -intertwining map of type $\binom{W_3}{W_1 W_2}$ is a linear map

$$I: W_1 \otimes W_2 \longrightarrow \overline{W_3}, \quad (4.1)$$

satisfying the following conditions:

(i) The *lower truncation condition*. For any element $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $n \in \mathbb{C}$,

$$\pi_{n-m}(I(w_{(1)} \otimes w_{(2)})) = 0 \quad \text{for } m \in \mathbb{Z}_{\geq 0} \text{ sufficiently large,} \quad (4.2)$$

where π_n is the canonical projection of \overline{W} to the weight subspace $W_{(n)}$

(ii) The *Jacobi identity*. For $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$,

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) Y_3(v, x_1) I(w_{(1)} \otimes w_{(2)}) \\ &= z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) I(Y_1(v, x_1) w_{(1)} \otimes w_{(2)}) \\ &+ x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) I(w_{(1)} \otimes Y_2(v, x_1) w_{(2)}). \end{aligned} \quad (4.3)$$

Remark 4.1.2. The vector space of $P(z)$ -intertwining maps of type $\binom{W_3}{W_1 W_2}$ is isomorphic to the space $\mathcal{V}_{W_1 W_2}^{W_3}$ of logarithmic intertwining operators (Definition 2.1.3) of the same type [HLZ4, Prop. 4.8].

Definition 4.1.3. Let W_1 and W_2 be generalized V -modules. A $P(z)$ -product of W_1 and W_2 is a generalized V -module (W_3, Y_3) together with a $P(z)$ -intertwining map I_3 of type $\binom{W_3}{W_1 W_2}$. We denote it by $(W_3, Y_3; I_3)$ or simply by (W_3, I_3) . Let $(W_4, Y_4; I_4)$ be another $P(z)$ -product of W_1 and W_2 . A *morphism* from $(W_3, Y_3; I_3)$ to $(W_4, Y_4; I_4)$ is a module map η from W_3 to W_4 such that

$$I_4 = \bar{\eta} \circ I_3, \quad (4.4)$$

where $\bar{\eta}$ is the natural map from $\overline{W_3}$ to $\overline{W_4}$ which extends η .

We recall the definition of a $P(z)$ -tensor product for a category \mathcal{C} of V -modules. The notion of a $P(z)$ -tensor product of W_1 and W_2 in \mathcal{C} is defined in terms of a universal property as follows.

Definition 4.1.4. For $W_1, W_2 \in \mathcal{C}$, a $P(z)$ -tensor product of W_1 and W_2 in \mathcal{C} is a $P(z)$ -product $(W_0, Y_0; I_0)$ with $W_0 \in \mathcal{C}$ such that for any $P(z)$ -product $(W, Y; I)$ with $W \in \mathcal{C}$, there is a unique morphism from $(W_0, Y_0; I_0)$ to $(W, Y; I)$. Clearly, a $P(z)$ -tensor product of W_1 and W_2 in \mathcal{C} , if it exists, is unique up to isomorphism. We denote the $P(z)$ -tensor product $(W_0, Y_0; I_0)$ by

$$(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)}) \quad (4.5)$$

and call the object

$$(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}) \quad (4.6)$$

the $P(z)$ -tensor product of W_1 and W_2 .

Recall that fusion rules are defined as the dimension of the space of intertwining operators $\mathcal{V}_{W_1 W_2}^{W_3}$ ([FHL], cf. Definition 2.1.3) and can be computed independently of the existence of tensor structures for a vertex algebra [FZ1, FZ2, Li2, HY]. However, as one would expect, in the case that the tensor structure of Lepowsky–Huang–Zhang does exist, the space of intertwining operators is isomorphic to the space of vertex operator algebra homomorphisms from the $P(z)$ -tensor product

of two modules to a third module. That is, the fusion rules coincide with the fusion product coefficients of the tensor product of two modules. We recall the following result in [HLZ4] that establishes this correspondence:

Proposition 4.1.5. [HLZ4, Prop. 4.17] *Let W_1 and W_2 be generalized V -modules. Suppose that the $P(z)$ -tensor product $W_1 \boxtimes_{P(z)} W_2$ exists. Then, for $p \in \mathbb{Z}$, we have a natural isomorphism*

$$\mathrm{Hom}_V(W_1 \boxtimes_{P(z)} W_2, W_3) \rightarrow \mathcal{V}_{W_1 W_2}^{W_3}, \quad \eta \mapsto \mathcal{Y}_{\eta, p}, \quad (4.7)$$

where $\mathcal{Y}_{\eta, p} = \mathcal{Y}_{I, p}$ with $I = \bar{\eta} \circ \boxtimes_{P(z)}$ and, for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$,

$$\mathcal{Y}_{I, p}(w_{(1)}, x)w_{(2)} = y^{L(0)}x^{L(0)}I(y^{-L(0)}x^{-L(0)}w_{(1)} \otimes y^{-L(0)}x^{-L(0)}w_{(2)})\Big|_{y=e^{-\log z - 2\pi i p}}. \quad (4.8)$$

We now recall the construction of the $P(z)$ -tensor product in [HLZ5]. Let $v \in V$ and let

$$Y_t(v, x) = \sum_{n \in \mathbb{Z}} (v \otimes t^n) x^{-n-1} \in (V \otimes \mathbb{C}[t, t^{-1}])[[x, x^{-1}]]. \quad (4.9)$$

Denote by $\tau_{P(z)}$ the action of

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}] \quad (4.10)$$

on the vector space $(W_1 \otimes W_2)^*$, where ι_+ is the operation of expanding a rational function in the formal variable t in the direction of positive powers of t , given by

$$\begin{aligned} & \left(\tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\ &= z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) \lambda (Y_1(e^{x_1 L(1)} (-x_1^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \\ &+ x_0^{-1} \delta \left(\frac{z - x_1^{-1}}{-x_0} \right) \lambda (w_{(1)} \otimes Y_2^\circ(v, x_1) w_{(2)}), \end{aligned} \quad (4.11)$$

for $v \in V$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. Denote by $Y'_{P(z)}$ the action of $V \otimes \mathbb{C}[t, t^{-1}]$ on $(W_1 \otimes W_2)^*$ defined by

$$Y'_{P(z)}(v, x) = \tau_{P(z)}(Y_t(v, x)). \quad (4.12)$$

Then, we have the operators $L'_{P(z)}(n)$ for $n \in \mathbb{Z}$ defined by

$$Y'_{P(z)}(\omega, x) = \sum_{n \in \mathbb{Z}} L'_{P(z)}(n) x^{-n-2}. \quad (4.13)$$

Given two V -modules W_1 and W_2 , let $W_1 \boxtimes_{P(z)} W_2$ be the vector space consisting of all the elements $\lambda \in (W_1 \otimes W_2)^*$ satisfying the following two conditions.

(1) $P(z)$ -compatibility condition:

- (a) *Lower truncation condition:* For all $v \in V$, the formal Laurent series $Y'_{P(z)}(v, x)\lambda$ involves only finitely many negative powers of x .

(b) The following formula holds:

$$\begin{aligned} & \tau_{P(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) \lambda \\ &= z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y'_{P(z)}(v, x_0) \lambda \quad \text{for all } v \in V. \end{aligned} \quad (4.14)$$

(2) $P(z)$ -local grading restriction condition:

- (a) *Grading condition:* λ is a (finite) sum of generalized eigenvectors of $(W_1 \otimes W_2)^*$ for the operator $L'_{P(z)}(0)$.
- (b) The smallest subspace W_λ of $(W_1 \otimes W_2)^*$ containing λ and stable under the component operators $\tau_{P(z)}(v \otimes t^n)$ of the operators $Y'_{P(z)}(v, x)$, for $v \in V$ and $n \in \mathbb{Z}$, satisfies $\dim(W_\lambda)_{[n]} < \infty$ and $(W_\lambda)_{[n+k]} = 0$ for $k \in \mathbb{Z}$ sufficiently negative and any $n \in \mathbb{C}$. Here, the subscripts denote the \mathbb{C} -grading given by the $L'_{P(z)}(0)$ -eigenvalues.

Theorem 4.1.6 ([HLZ5]). *The vector space $W_1 \boxtimes_{P(z)} W_2$ is closed under the action $Y'_{P(z)}$ of V and the Jacobi identity holds on $W_1 \boxtimes_{P(z)} W_2$. Furthermore, the $P(z)$ -tensor product of $W_1, W_2 \in \mathcal{C}$ exists if and only if $W_1 \boxtimes_{P(z)} W_2$, equipped with $Y'_{P(z)}$, is an object of \mathcal{C} . In this case, the $P(z)$ -tensor product is the contragredient of $(W_1 \boxtimes_{P(z)} W_2, Y'_{P(z)})$.*

To construct a tensor category structure on $\mathcal{O}_c^{\text{fin}}$, we first need to show that $\mathcal{O}_c^{\text{fin}}$ is closed under $P(z)$ -tensor products. This is an immediate corollary of the following result of Miyamoto.

Theorem 4.1.7 ([Mi]). *Let $W_1, W_2 \in \mathcal{C}_1$. If W_3 is a lower-bounded generalized module such that there exists a surjective intertwining operator of type $\binom{W_3}{W_1 W_2}$, then W_3 is also an object of \mathcal{C}_1 . In particular, the $P(z)$ -tensor product $W_1 \boxtimes_{P(z)} W_2$ is an object in \mathcal{C}_1 .*

Corollary 4.1.8. *The category $\mathcal{O}_c^{\text{fin}}$ is closed under taking $P(z)$ -tensor products. Namely, if $W_1, W_2 \in \mathcal{O}_c^{\text{fin}}$, then $W_1 \boxtimes_{P(z)} W_2 \in \mathcal{O}_c^{\text{fin}}$.*

Proof. By Theorem 3.1.4, W_1 and $W_2 \in \mathcal{O}_c^{\text{fin}}$ are lower-bounded and C_1 -cofinite, so using Theorem 4.1.7 we have that the $P(z)$ -tensor product of W_1 and W_2 is also lower-bounded and C_1 -cofinite. By Theorem 3.1.4 again, we have that $W_1 \boxtimes_{P(z)} W_2 \in \mathcal{O}_c^{\text{fin}}$. \square

4.2. Associativity isomorphism. The associativity isomorphism is the most important ingredient of the tensor category theory of Huang–Lepowsky–Zhang. To prove it, one needs the following convergence and extension property introduced in [HLZ8].

Definition 4.2.1. Let A be an abelian group and \tilde{A} an abelian group containing A as a subgroup. Let V be a strongly A -graded conformal vertex algebra. We say that the product of the intertwining operators \mathcal{Y}_1 and \mathcal{Y}_2 satisfies the *convergence and extension property for products* if for any doubly homogeneous elements $w_{(1)} \in W_1^{(\beta_1)}$ and $w_{(2)} \in W_2^{(\beta_2)}$, with $\beta_1, \beta_2 \in \tilde{A}$, and any $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$, there exist $r_1, \dots, r_M, s_1, \dots, s_M \in \mathbb{R}$, $i_1, \dots, i_M, j_1, \dots, j_M \in \mathbb{Z}_{\geq 0}$ and analytic functions $f_1(z), \dots, f_M(z)$ on $|z| < 1$ (for some $M \in \mathbb{Z}_{\geq 0}$) satisfying

$$\text{wt } w_{(1)} + \text{wt } w_{(2)} + s_k > N, \quad \text{for each } k = 1, \dots, M, \quad (4.15)$$

where $N \in \mathbb{Z}$ depends only on $\mathcal{Y}_1, \mathcal{Y}_2$ and $\beta_1 + \beta_2$, such that

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2} \quad (4.16)$$

is absolutely convergent on $|z_1| > |z_2| > 0$ and may be analytically extended to the multivalued analytic function

$$\sum_{k=1}^M z_2^{\tilde{r}_k} (z_1 - z_2)^{\tilde{s}_k} (\log z_2)^{i_k} (\log(z_1 - z_2))^{j_k} f_k \left(\frac{z_1 - z_2}{z_2} \right) \quad (4.17)$$

in the region $|z_2| > |z_1 - z_2| > 0$.

Definition 4.2.2. We say that the iterate of the intertwining operators \mathcal{Y}^1 and \mathcal{Y}^2 satisfies the *convergence and extension property for iterates* if for any doubly homogeneous elements $w_{(2)} \in W_2^{(\beta_2)}$, $w_{(3)} \in W_3^{(\beta_3)}$ with $\beta_2, \beta_3 \in \tilde{A}$, and any $w_{(1)} \in W_1$ and $w'_{(4)} \in W'_4$, there exist $\tilde{r}_1, \dots, \tilde{r}_M, \tilde{s}_1, \dots, \tilde{s}_M \in \mathbb{R}$, $\tilde{i}_1, \dots, \tilde{i}_M, \tilde{j}_1, \dots, \tilde{j}_M \in \mathbb{Z}_{\geq 0}$ and analytic functions $\tilde{f}_1(z), \dots, \tilde{f}_M(z)$ on $|z| < 1$ (for some $\tilde{M} \in \mathbb{Z}_{\geq 0}$) satisfying

$$\text{wt } w_{(2)} + \text{wt } w_{(3)} + \tilde{s}_k > \tilde{N}, \quad \text{for each } k = 1, \dots, \tilde{M}, \quad (4.18)$$

where $\tilde{N} \in \mathbb{Z}$ depends only on $\mathcal{Y}^1, \mathcal{Y}^2$ and $\beta_2 + \beta_3$, such that

$$\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0) w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_0=z_1-z_2, x_2=z_2} \quad (4.19)$$

is absolutely convergent on $|z_2| > |z_1 - z_2| > 0$ and may be analytically extended to the multivalued analytic function

$$\sum_{k=1}^{\tilde{M}} z_1^{\tilde{r}_k} z_2^{\tilde{s}_k} (\log z_1)^{\tilde{i}_k} (\log z_2)^{\tilde{j}_k} \tilde{f}_k \left(\frac{z_2}{z_1} \right) \quad (4.20)$$

in the region $|z_1| > |z_2| > 0$.

The convergence and extension property is part of the known sufficient conditions [HLZ7, Thm. 10.3], [HLZ8, Thm. 11.4] and [H4, Thm. 3.1] for the existence of the associativity isomorphism.

Theorem 4.2.3 ([HLZ7, HLZ8, H4]). *Let V be a vertex operator algebra satisfying the following conditions:*

- (1) *For any two modules W_1 and W_2 in \mathcal{C} and any $z \in \mathbb{C}^\times$, if the generalized V -module W_λ is generated by a generalized $L'_{P(z)}(0)$ -eigenvector $\lambda \in (W_1 \otimes W_2)^*$ satisfying the $P(z)$ -compatibility condition is lower-bounded, then W_λ is an object of \mathcal{C} .*
- (2) *The convergence and extension property holds for either the product or the iterates of intertwining operators for V .*

Then, for any V -modules W_1, W_2 and W_3 and any $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$, there is a unique isomorphism

$$\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)} : W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \rightarrow (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 \quad (4.21)$$

such that for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$, one has

$$\overline{\mathcal{A}}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}(w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)) = (w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3, \quad (4.22)$$

where

$$\overline{\mathcal{A}}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)} : \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \rightarrow \overline{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3} \quad (4.23)$$

is the canonical extension of $\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}$.

Condition (2) in Theorem 4.2.3 is guaranteed by the C_1 -cofiniteness condition.

Proposition 4.2.4. *The convergence and extension property for products and iterates holds for $\mathcal{O}_c^{\text{fin}}$.*

Proof. It follows from [HLZ8, Thm. 11.8] that if all the objects $W \in \mathcal{O}_c^{\text{fin}}$ are C_1 -cofinite and satisfy $\dim \prod_{\mathbb{R}(n) < r} W_{[n]} < \infty$, for any $r \in \mathbb{R}$, then the convergence and extension properties for products and iterates of intertwining operators hold. The C_1 -cofiniteness is guaranteed by Theorem 3.1.4 while the second condition is obvious because objects in $\mathcal{O}_c^{\text{fin}}$ have finite lengths. \square

Theorem 4.2.5. *The associativity isomorphism holds for the category $\mathcal{O}_c^{\text{fin}}$.*

Proof. Given Proposition 4.2.4, we only need to show that Condition (1) in Theorem 4.2.3 holds in $\mathcal{O}_c^{\text{fin}}$. Let W_λ be the lower-bounded generalized V -module constructed in Theorem 4.2.3.

The inclusion map $W_\lambda \subset (W_1 \otimes W_2)^*$ intertwines the action of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ given by $\tau_{P(z)}$. Therefore, by [HLZ5, Prop. 5.24], it corresponds to a $P(z)$ -intertwining map I of type $\binom{W'_\lambda}{W_1 W_2}$ such that

$$\lambda(w_1 \otimes w_2) = \langle \lambda, I(w_1 \otimes w_2) \rangle. \quad (4.24)$$

By [HLZ4, Prop. 4.8], there is an intertwining operator \mathcal{Y} of type $\binom{W'_\lambda}{W_1 W_2}$ such that

$$\mathcal{Y}(w_1, z)w_2 = I(w_1 \otimes w_2). \quad (4.25)$$

Note that this intertwining operator is surjective since

$$\langle \lambda, \mathcal{Y}(w_1, z)w_2 \rangle = \langle \lambda, I(w_1 \otimes w_2) \rangle = \lambda(w_1 \otimes w_2). \quad (4.26)$$

By Theorem 4.1.7, W'_λ is C_1 -cofinite since W_1 and W_2 are C_1 -cofinite. Then, since W'_λ is C_1 -cofinite and lower-bounded, it follows from Theorem 3.1.4 that $W'_\lambda \in \mathcal{O}_c^{\text{fin}}$ and thus $W_\lambda \in \mathcal{O}_c^{\text{fin}}$. \square

By Corollary 4.1.8, Theorem 4.2.5 and [HLZ9, Thm. 12.15], we obtain the main theorem of this section.

Theorem 4.2.6. *The category $\mathcal{O}_c^{\text{fin}}$ has a braided tensor category structure.*

Remark 4.2.7. There are many open conjectures about the representation categories of the singlet and triplet algebras [CGan, CM, CMR, CGR, RW1], but the most basic one is the existence of a rigid vertex tensor category structure on the category of finite-length modules [CMR]. Since the singlet (and triplet) vertex operator algebras are objects in the Ind-completion of $\mathcal{O}_c^{\text{fin}}$ for

$c = 1 - 6(p - 1)^2/p$, so $t = 1/p$, our results may be used to study the vertex tensor category of modules containing all known indecomposable but reducible modules for the singlet algebra.

5. RIGIDITY FOR GENERIC CENTRAL CHARGE

In this section, we will study the category $\mathcal{O}_c^{\text{fin}}$ for generic central charges $c = 13 - 6t - 6t^{-1}$, meaning that $t \notin \mathbb{Q}$, and prove that it is rigid. The simple objects of this category have the form $L(c, h_{r,s})$ for $r, s \in \mathbb{Z}_{\geq 1}$. Set $t = k + 2$, so that $k \notin \mathbb{Q}$, and recall the notation $L_{r,s} = L(c, h_{r,s})$ for $r, s \in \mathbb{Z}_{\geq 1}$ from Section 2.2. The condition that c is generic is understood to be in force for the rest of the section unless otherwise noted.

5.1. Semisimplicity. We start by establishing that $\mathcal{O}_c^{\text{fin}}$ is semisimple for generic central charges.

Lemma 5.1.1. *If $h \neq h' \in H_c$, then*

$$\text{Ext}_{\mathcal{O}_c^{\text{fin}}}^1(L(c, h'), L(c, h)) = 0. \tag{5.1}$$

Proof. Suppose there is a short exact sequence

$$0 \rightarrow L(c, h) \rightarrow M \rightarrow L(c, h') \rightarrow 0. \tag{5.2}$$

If $\mathfrak{S}(h) \neq \mathfrak{S}(h')$, then this sequence obviously splits, so we may assume that $\mathfrak{S}(h) = \mathfrak{S}(h')$.

If $\mathfrak{R}(h') < \mathfrak{R}(h)$, then there is a highest-weight vector of conformal weight h' in M . From Theorem 2.2.4, the singular vector $v_{h'}$ either generates a Verma module $V(c, h')$ or its simple quotient $L(c, h')$. But, it has to be $L(c, h')$ because $V(c, h')$ is not in $\mathcal{O}_c^{\text{fin}}$. The exact sequence (5.2) therefore splits.

If $\mathfrak{R}(h') > \mathfrak{R}(h)$, we consider the contragredient dual of M , recalling that $L(c, h)$ and $L(c, h')$ are self-dual, arriving at the short exact sequence

$$0 \rightarrow L(c, h') \rightarrow M' \rightarrow L(c, h) \rightarrow 0. \tag{5.3}$$

The previous argument then gives $M' \cong L(c, h) \oplus L(c, h')$, hence $M \cong L(c, h) \oplus L(c, h')$ too. \square

Theorem 5.1.2. *The category $\mathcal{O}_c^{\text{fin}}$ is semisimple for generic central charges.*

Proof. Lemma 5.1.1 shows that there are no extensions between non-isomorphic simple objects. But, [Ro, Thm. 6.4] and [KyR, Prop. 7.5] (see also [GK, Lem. 5.2.2] for a high-powered approach) have shown that there are no self-extensions of $L(c, h)$ for generic central charges. \square

Despite claims to the contrary in the literature, it is interesting that certain simples do admit self-extensions for non-generic central charges. The $L(c, h)$ that do are classified in [KyR].

5.2. Fusion rules. We next determine the fusion rules of the category $\mathcal{O}_c^{\text{fin}}$, for generic c , using the Zhu algebra tools developed in [FZ1, FZ2], see also [Li2]. From [FZ1, W], we know that the Zhu algebra of $M(c, 0)$ is $A(M(c, 0)) \cong \mathbb{C}[x]$, where $x = [\omega]$. Moreover, we have $A(V(c, h)) \cong \mathbb{C}[x, y]$ as a $\mathbb{C}[x]$ -bimodule [DMZ, Li2].

The fusion rules involving the simple $M(c, 0)$ -modules $L_{1,s}$ were computed by I. Frenkel and M. Zhu in [FZ2, Prop. 2.24] using the explicit singular vector formula of Benoit and Saint-Aubin [BSA].

Remark 5.2.1. Note that it follows from Proposition 4.1.5, and the discussion that precedes it, that the fusion rules computed by I. Frenkel and M. Zhu in [FZ2] are indeed, in light of Theorems 4.2.6 and 5.1.2, the fusion product coefficients in the vertex tensor category.

Combined with our Theorems 4.2.6 and 5.1.2, their result may be phrased as follows.

Theorem 5.2.2 ([FZ2]). *Let c be generic. Then, for $s_1, s_2, s_3 \in \mathbb{Z}_{\geq 1}$, we have*

$$L_{1,s_1} \boxtimes L_{1,s_2} \cong \bigoplus_{\substack{s_3=|s_1-s_2|+1 \\ s_3=s_1+s_2-1 \pmod 2}}^{s_1+s_2-1} L_{1,s_3}. \quad (5.4)$$

The obvious analogue for $L_{r,1} \boxtimes L_{r,1}$ also holds.

Our interest, however, is in the fusion of $L_{r,1}$ with $L_{1,s}$. This can also be attacked using the same methods, in particular the results [FZ2, Lems. 2.9 and 2.14] which we quote for convenience.

Proposition 5.2.3 ([FZ2]).

(1) *As an $A(M(c, 0))$ -bimodule, we have*

$$A(L_{r,s}) \cong \frac{\mathbb{C}[x, y]}{\langle f_{r,s}(x, y) \rangle}, \quad (5.5)$$

where $f_{r,s}(x, y)$ is the image of the non-cyclic singular vector of $V_{r,s}$ in $A(V_{r,s})$.

(2) *Let \bar{n} denote the residue of $n \in \mathbb{Z}$ modulo 2. Then, for $r, s \in \mathbb{Z}_{\geq 1}$, we have*

$$f_{r,1} = g'_r g'_{r-2} \dots g'_r \quad \text{and} \quad f_{1,s} = g_s g_{s-2} \dots g_{\bar{s}}, \quad (5.6)$$

where $g_1(x, y) = g'_1(x, y) = x - y$ and, for $r, s \in \mathbb{Z}_{\geq 2}$,

$$\begin{aligned} g'_r(x, y) &= (x - y - h_{r,1})(x - y - h_{-(r-2),1}) - (r-1)^2 t y \\ \text{and } g_s(x, y) &= (x - y - h_{1,s})(x - y - h_{1,-(s-2)}) - (s-1)^2 t^{-1} y. \end{aligned} \quad (5.7)$$

Specialising [FZ2, Lem. 2.22] to the case we are interested in, we arrive at the following characterisation of the fusion rules.

Proposition 5.2.4 ([FZ2]). $\mathcal{N}_{L_{r,1}L_{1,s}}^{L_{r',s'}} = 0$ unless

$$f_{r,1}(h_{r',s'}, h_{1,s}) = 0, \quad (5.8a)$$

$$f_{1,s}(h_{r',s'}, h_{r,1}) = 0 \quad (5.8b)$$

$$\text{and } f_{r',s'}(h_{1,s}, h_{r,1}) = 0, \quad (5.8c)$$

in which case, $\mathcal{N}_{L_{r,1}L_{1,s}}^{L_{r',s'}} = 1$.

Fix now $r, s \in \mathbb{Z}_{\geq 1}$ and consider first the $h_{r',s'}$ that solve (5.8a). Direct calculation shows that the solutions of $g'_r(x, h_{1,s}) = 0$ are $x = h_{r,s}$ and, if $r > 1$, $x = h_{-r+2,s}$. An easy calculation verifies that

$$h_{r,s} = h_{r',s'} \iff (r, s) = (r', s') \text{ or } (-r', -s'), \quad (5.9)$$

since $t \notin \mathbb{Q}$. It follows that $h_{-r+2,s} \notin H_c$, for $r > 1$, and so these solutions of (5.8a) correspond to simple Verma modules $V_{-r+2,s} = L_{-r+2,s} \notin \mathcal{O}_c^{\text{fin}}$ and hence cannot appear in the fusion of $L(r, 1)$ and $L(1, s)$, by Theorem 4.2.6. The viable solutions are therefore $h_{r',s'} = h_{r,s}, h_{r-2,s}, \dots, h_{\bar{r},s}$.

However, the same analysis gives the viable solutions of (5.8b) as $h_{r',s'} = h_{r,s}, h_{r,s-2}, \dots, h_{r,\bar{s}}$. Appealing to (5.9), we conclude that there is a unique solution to Equations (5.8a) and (5.8b): $h_{r',s'} = h_{r,s}$. We can now state our main fusion rule.

Theorem 5.2.5. *Let c be generic. Then, for $r, s \in \mathbb{Z}_{\geq 1}$, we have*

$$L_{r,1} \boxtimes L_{1,s} = L_{r,s}. \quad (5.10)$$

Proof. Since $\mathcal{O}_c^{\text{fin}}$ is tensor (Theorem 4.2.6) and semisimple (Theorem 5.1.2), $L_{r,1} \boxtimes L_{1,s}$ decomposes as a finite direct sum of simples. The previous arguments establish that the only possibilities for this decomposition are 0 or $L_{r,s}$, with the deciding condition whether (5.8c) is satisfied (for $r' = r$ and $s' = s$). Unfortunately, computing the left-hand side of (5.8c) directly is difficult.

Instead, note that combining $L_{r,1} \boxtimes L_{1,s} = 0$ with Theorem 5.2.2 (and associativity) gives

$$\begin{aligned} 0 &\cong (L_{r,1} \boxtimes L_{1,s}) \boxtimes L_{1,s} \cong L_{r,1} \boxtimes (L_{1,s} \boxtimes L_{1,s}) \cong L_{r,1} \boxtimes (L_{1,1} \oplus \dots \oplus L_{1,2s-1}) \\ &\cong L_{r,1} \oplus \dots, \end{aligned} \quad (5.11)$$

a contradiction. It follows that $L_{r,1} \boxtimes L_{1,s} \neq 0$ and the proof is complete. \square

We remark that this theorem proves, for generic c , a well-known conjecture of physicists. Of course, physicists are more interested in the non-generic version of this conjecture [MRR, Eq. (4.34)].

5.3. Coset realizations of the Virasoro algebra. The next ingredient for our proof of the rigidity of $\mathcal{O}_c^{\text{fin}}$ is the well-known coset realization of the Virasoro vertex operator algebra [GKO, ACL]. We review this here for general simply laced Lie algebras before specializing to \mathfrak{sl}_2 .

Let P_+ be the set of dominant integral weights of a simple complex Lie algebra \mathfrak{g} and let Q be its root lattice. Let $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ be the affinization of \mathfrak{g} . For $\ell \in \mathbb{C}$ and $\lambda \in P_+$, let $V_\ell(\lambda) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} E^\lambda$, where E^λ is the finite-dimensional \mathfrak{g} -module of highest weight λ regarded as a $\mathfrak{g}[t] \oplus \mathbb{C}K$ -module on which $t\mathfrak{g}[t]$ acts trivially and K acts as multiplication by the level ℓ . Denote by $L_\ell(\lambda)$ the simple quotient of $V_\ell(\lambda)$. The modules $V_\ell(0)$ and $L_\ell(0)$ admit, for $\ell \neq -h^\vee$, a vertex operator algebra structure [FZ1]. To emphasize the dependence of these vertex operator algebras on \mathfrak{g} , we denote them by $V_\ell(\mathfrak{g})$ and $L_\ell(\mathfrak{g})$.

It is known that the category KL_k of ordinary $L_k(\mathfrak{g})$ -modules has a rigid braided tensor category structure for all k such that $k + h^\vee \notin \mathbb{Q}_+$, $k \in \mathbb{Z}_{\geq 0}$, or k is admissible [KL1]–[KL5], [HL5, CHY]. When $k \in \mathbb{Z}_{\geq 0}$, the simple objects of KL_k are rather the $L_k(\lambda)$ with

$\lambda \in P_+^k = \{\lambda \in P_+ \mid (\lambda, \theta) \leq k\}$, where θ is the longest root of \mathfrak{g} . Here, we study the generic case $k \notin \mathbb{Q}$, for which $V_k(\mathfrak{g}) = L_k(\mathfrak{g})$ and the simple objects of KL_k are the $L_k(\lambda)$ with $\lambda \in P_+$.

Let $W^k(\mathfrak{g})$ be the W -algebra associated with \mathfrak{g} and a principal nilpotent element of \mathfrak{g} at level k . Let $W_k(\mathfrak{g})$ be the unique simple quotient of $W^k(\mathfrak{g})$. We denote by χ_λ the central character associated to the weight $\lambda \in P_+$ and let $\mathbf{M}_k(\chi_\lambda)$ be the Verma module of $W^k(\mathfrak{g})$ with highest weight χ_λ and $\mathbf{L}_k(\chi_\lambda)$ its unique simple (graded) quotient.

By specializing to $\mathfrak{g} = \mathfrak{sl}_2$ and identifying P_+ for \mathfrak{sl}_2 with $\mathbb{Z}_{\geq 0}$, we have $W^k(\mathfrak{sl}_2) = M(c, 0)$ and $W_k(\mathfrak{sl}_2) = L(c, 0)$, for $c = 13 - 6t - 6t^{-1}$ with $t = k + 2$. For $\mu, \nu \in P_+$, we moreover have

$$\mathbf{L}_k(\chi_{\mu-2(k+h^\vee)\nu}) = L_{\mu+1, \nu+1} \in \mathcal{O}_c^{\text{fin}}. \quad (5.12)$$

Theorem 5.3.1 ([ACL]). *Suppose that \mathfrak{g} is a simply-laced Lie algebra and $\ell \notin \mathbb{Q}$. Define $k \notin \mathbb{Q}$ by*

$$k + h^\vee = \frac{\ell + h^\vee}{\ell + h^\vee + 1}. \quad (5.13)$$

Then, for $\lambda \in P_+^1$ and $\mu \in P_+$, we have

$$L_1(\lambda) \otimes V_\ell(\mu) \cong \bigoplus_{\substack{\nu \in P_+, \\ \lambda + \mu - \nu \in Q}} V_{\ell+1}(\nu) \otimes \mathbf{L}_k(\chi_{\mu-2(k+h^\vee)\nu}) \quad (5.14a)$$

as $V_{\ell+1}(\mathfrak{g}) \otimes W^k(\mathfrak{g})$ -modules. In particular,

$$L_1(\mathfrak{g}) \otimes V_\ell(\mathfrak{g}) \cong \bigoplus_{\nu \in P_+ \cap Q} V_{\ell+1}(\nu) \otimes \mathbf{L}_k(\chi_{-2(k+h^\vee)\nu}). \quad (5.14b)$$

By specializing Theorem 5.3.1 to \mathfrak{sl}_2 , the decompositions (5.14) become

$$L_1(\lambda) \otimes V_\ell(\mu) \cong \bigoplus_{\substack{\nu=0 \\ \lambda+\mu=\nu \pmod{2}}}^{\infty} V_{\ell+1}(\nu) \otimes L_{\mu+1, \nu+1}, \quad (5.15a)$$

for $\lambda \in P_+^1 = \{0, 1\}$ and $\mu \in P_+ = \mathbb{Z}_{\geq 0}$, and

$$L_1(\mathfrak{sl}_2) \otimes V_\ell(\mathfrak{sl}_2) \cong \bigoplus_{\nu=0}^{\infty} V_{\ell+1}(2\nu) \otimes L_{1, 2\nu+1}. \quad (5.15b)$$

5.4. Categories of vertex operator algebra extensions. The last ingredient we need is the theory of tensor categories of vertex operator algebra extensions [CKM1]. Let \mathcal{C} be a braided tensor category with tensor unit $\mathbf{1}$, braiding c and associativity isomorphism a .

Definition 5.4.1. An *associative algebra* in \mathcal{C} is a triple (A, μ_A, ι_A) with $A \in \mathcal{C}$ and $\mu_A: A \boxtimes A \rightarrow A$, $\iota_A: \mathbf{1} \rightarrow A$ morphisms in \mathcal{C} satisfying the following axioms.

- (1) Unit: $A = \mathbf{1} \boxtimes A \xrightarrow{\iota_A \boxtimes \text{id}_A} A \boxtimes A \xrightarrow{\mu_A} A$ and $A = A \boxtimes \mathbf{1} \xrightarrow{\text{id}_A \boxtimes \iota_A} A \boxtimes A \xrightarrow{\mu_A} A$ are both id_A .
- (2) Associativity: $\mu_A \circ (\text{id}_A \boxtimes \mu_A) = \mu_A \circ (\mu_A \boxtimes \text{id}_A) \circ a_{A, A, A}: A \boxtimes (A \boxtimes A) \rightarrow A$.

An associative algebra $A \in \mathcal{C}$ is said to be *commutative* if $\mu_A \circ c_{A, A} = \mu_A: A \boxtimes A \rightarrow A$.

Definition 5.4.2. For an associative algebra A in \mathcal{C} , define \mathcal{C}_A to be the category of pairs (X, μ_X) for which $X \in \mathcal{C}$ and $\mu_X \in \text{Hom}_{\mathcal{C}}(A \otimes X, X)$ satisfy the following.

- (1) Unit: $X = \mathbf{1} \boxtimes X \xrightarrow{\iota_A \boxtimes \text{id}_X} A \boxtimes X \xrightarrow{\mu_X} X$ is id_X .
 (2) Associativity: $\mu_X \circ (\text{id}_A \boxtimes \mu_X) = \mu_X \circ (\mu_A \boxtimes \text{id}_X) \circ a_{A,A,X}: A \otimes (A \boxtimes X) \rightarrow X$.

Define \mathcal{C}_A^0 to be the full subcategory of \mathcal{C}_A consisting of *local modules*: those objects (X, μ_X) such that $\mu_X \circ c_{X,A} \circ c_{A,X} = \mu_X$.

When A is commutative, the category \mathcal{C}_A is naturally a tensor category with tensor product \boxtimes_A and unit object A , while the subcategory \mathcal{C}_A^0 is braided tensor (see for example [KO] — the case in which A is an object of the direct sum completion of \mathcal{C} , that is a countably infinite direct sum of objects, is addressed in [AR, CGR, CKM2]).

Let V be a vertex operator algebra and \mathcal{C} be a category of V -modules with a natural tensor category structure. The following theorem allows one to study vertex operator algebra extensions using abstract tensor category theory.

Theorem 5.4.3 ([HKL, CKM1]). *A vertex operator algebra extension $V \subset A$ in \mathcal{C} is equivalent to a commutative associative algebra in the braided tensor category \mathcal{C} with trivial twist and injective unit. Moreover, the category of modules in \mathcal{C} for the extended vertex operator algebra A is braided tensor equivalent to the category of local \mathcal{C} -algebra modules \mathcal{C}_A^0 via the induction functor \mathcal{F}_A .*

We recall that the *twist* θ_X of an object X in a category of modules over a vertex operator algebra is given by the action of $e^{2\pi i L_0}$. To say that A has trivial twist above therefore means that L_0 acts semisimply on A with integer conformal weights.

5.5. Rigidity. It is time to put all these ingredients together to prove the rigidity of $\mathcal{O}_c^{\text{fin}}$ for all generic central charges. We first use Theorem 5.4.3 to prove a braided tensor equivalence between the full subcategory $\mathcal{O}_c^L \subset \mathcal{O}_c^{\text{fin}}$, generated by the simple Virasoro modules $L_{\mu,1}$ with $\mu \in \mathbb{Z}_{\geq 1}$, and a simple current twist of the category KL_ℓ of ordinary $L_\ell(\mathfrak{sl}_2)$ -modules. Here, ℓ is related to k by (5.13), hence to $t = k + 2$ and the (generic) central charge $c = 13 - 6t - 6t^{-1}$.

Let

$$A = \bigoplus_{\nu=0}^{\infty} V_{\ell+1}(2\nu) \otimes L_{1,2\nu+1} \cong L_1(\mathfrak{sl}_2) \otimes V_\ell(\mathfrak{sl}_2). \quad (5.16)$$

Then, A is a commutative associative algebra object in the direct sum completion of $KL_{\ell+1} \boxtimes \mathcal{O}_c^{\text{fin}}$. Denote by $\mathcal{F}_A: KL_{\ell+1} \boxtimes \mathcal{O}_c^{\text{fin}} \rightarrow (KL_{\ell+1} \boxtimes \mathcal{O}_c^{\text{fin}})_A$ the induction functor.

Lemma 5.5.1. *The restriction of the induction functor \mathcal{F}_A to the full subcategory \mathcal{O}_c^L is fully faithful.*

Proof. Using the Virasoro fusion rule (5.10) and the specializations (5.15) of Theorem 5.3.1, we obtain

$$\begin{aligned} \mathcal{F}_A(V_{\ell+1}(\mathfrak{sl}_2) \otimes L_{\mu+1,1}) &= \left[\bigoplus_{\nu=0}^{\infty} V_{\ell+1}(2\nu) \otimes L_{1,2\nu+1} \right] \boxtimes \left[V_{\ell+1}(\mathfrak{sl}_2) \otimes L_{\mu+1,1} \right] \\ &\cong \bigoplus_{\nu=0}^{\infty} V_{\ell+1}(2\nu) \otimes L_{\mu+1,2\nu+1} \cong L_1(\bar{\mu}) \otimes V_\ell(\mu), \end{aligned} \quad (5.17)$$

where $\bar{\mu}$ is the residue of $\mu \in \mathbb{Z}_{\geq 0}$ modulo 2. *A priori*, the last isomorphism is a $KL_{\ell+1} \boxtimes \mathcal{O}_c^{\text{fin}}$ -morphism. However, Frobenius reciprocity gives, for $\nu \in \mathbb{Z}_{\geq 0}$ and $\rho \in \{0, 1\}$,

$$\begin{aligned} & \text{Hom}_{(KL_{\ell+1} \boxtimes \mathcal{O}_c^{\text{fin}})_A} \left(\mathcal{F}_A(V_{\ell+1}(\mathfrak{sl}_2) \otimes L_{\mu+1,1}), L_1(\rho) \otimes V_{\ell}(\nu) \right) \\ & \cong \text{Hom}_{KL_{\ell+1} \boxtimes \mathcal{O}_c^{\text{fin}}} \left(V_{\ell+1}(\mathfrak{sl}_2) \otimes L_{\mu+1,1}, \bigoplus_{\substack{\lambda=0 \\ \lambda=\nu+\rho \pmod{2}}}^{\infty} V_{\ell+1}(\lambda) \otimes L_{\nu+1,\lambda+1} \right) \\ & \cong \begin{cases} \mathbb{C} & \text{if } \nu = \mu \text{ and } \rho = \bar{\mu}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (5.18)$$

In other words, the isomorphism (5.17) is actually an isomorphism in $KL_1 \boxtimes KL_{\ell}$. \square

Restricting \mathcal{F}_A to \mathcal{O}_c^L , its image is the full subcategory of $KL_1 \boxtimes KL_{\ell}$ whose simple objects have the form $L_1(\bar{\mu}) \otimes V_{\ell}(\mu)$. Denote this category by $(KL_1 \boxtimes KL_{\ell})_0$. This category is a rigid semisimple tensor category with the same simple objects and fusion rules as the category of finite dimensional \mathfrak{sl}_2 -modules ([KL1]-[KL5], [L]). This proves the following proposition.

Proposition 5.5.2. \mathcal{O}_c^L and $(KL_1 \boxtimes KL_{\ell})_0$ are braided-equivalent tensor categories. In particular, \mathcal{O}_c^L is rigid.

We now come to the main theorem of this section.

Theorem 5.5.3. The category $\mathcal{O}_c^{\text{fin}}$ is rigid for generic central charges.

Proof. As \mathcal{O}_c^L is rigid, the simple objects $L_{\mu,1}$, with $\mu \in \mathbb{Z}_{\geq 1}$ are rigid. An identical argument shows that the $L_{1,\nu}$ with $\nu \in \mathbb{Z}_{\geq 1}$ are likewise rigid. It follows now from Theorem 5.2.5 that $L_{\mu,\nu} \cong L_{\mu,1} \boxtimes L_{1,\nu}$ is rigid. Since $\mathcal{O}_c^{\text{fin}}$ is semisimple (Theorem 5.1.2), this completes the proof. \square

Remark 5.5.4. Braided tensor equivalences between categories of modules for W -algebras and affine vertex operator algebras may be viewed as reformulations of problems in the quantum geometric Langlands program. For example, Proposition 5.5.2 is the case $\mathfrak{g} = \mathfrak{sl}_2$, $N = 1$ and β generic of [AFO, Conj. 6.4], up to a simple current twist (see also [C1, Rem. 7.2]).

We finish by demonstrating that $\mathcal{O}_c^{\text{fin}}$ is moreover non-degenerate. Recall that in a rigid braided tensor category \mathcal{C} , an object $T \in \mathcal{C}$ is called *transparent* if it has trivial monodromy with every other object of \mathcal{C} , that is if $c_{X,T} \circ c_{T,X} = \text{id}_{T \boxtimes X}$ for all $X \in \mathcal{C}$. \mathcal{C} is said to be *non-degenerate* if the only transparent objects are finite direct sums of the tensor unit. In our case, non-degeneracy follows easily from the balancing axiom

$$c_{X,T} \circ c_{T,X} = (\theta_X^{-1} \boxtimes \theta_T^{-1}) \circ \theta_{X \boxtimes T}, \quad (5.19)$$

where we recall that θ_X denotes the twist of $X \in \mathcal{C}$.

Proposition 5.5.5. The category $\mathcal{O}_c^{\text{fin}}$ is non-degenerate for generic central charges.

Proof. Set $T = L_{r,s}$ with $s \neq 1$. We then take $X = L_{1,2}$ so that the fusion rules (5.4) and (5.10) give

$$\begin{aligned} T \boxtimes X &\cong (L_{r,1} \boxtimes L_{1,s}) \boxtimes L_{1,2} \cong L_{r,1} \boxtimes (L_{1,s} \boxtimes L_{1,2}) \cong L_{r,1} \boxtimes (L_{1,s-1} \oplus L_{1,s+1}) \\ &\cong L_{r,s-1} \oplus L_{r,s+1}. \end{aligned} \tag{5.20}$$

As the twist coincides with the action of $e^{2\pi i L_0}$, (5.19) shows that $c_{X,T} \circ c_{T,X}$ acts as multiplication by $e^{2\pi i(h_{r,s\pm 1} - h_{r,s} - h_{1,2})}$ on $L_{r,s\pm 1}$. The ratio $e^{2\pi i s/t}$ of these eigenvalues is never 1, hence T is not transparent. The case $r \neq 1$ is completely analogous. \square

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