

A REALISATION OF THE BERSHADSKY–POLYAKOV ALGEBRAS AND THEIR RELAXED MODULES

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ABSTRACT. We present a realisation of the universal/simple Bershadsky–Polyakov vertex algebras as subalgebras of the tensor product of the universal/simple Zamolodchikov vertex algebras and an isotropic lattice vertex algebra. This generalises the realisation of the universal/simple affine vertex algebras associated to \mathfrak{sl}_2 and $\mathfrak{osp}(1|2)$ given in [2]. Relaxed highest-weight modules are likewise constructed, conditions for their irreducibility are established, and their characters are explicitly computed, generalising the character formulae of [40].

1. INTRODUCTION

Let \mathfrak{g} be a finite-dimensional complex basic classical simple Lie superalgebra and let $V^k(\mathfrak{g})$ denote the corresponding universal affine vertex superalgebra of level k . Associated to every nilpotent element $f \in \mathfrak{g}$, or rather to every orbit of nilpotent elements, there is a vertex superalgebra $W^k(\mathfrak{g}; f)$ called a (universal) W-algebra. It is defined [29, 36] as the cohomology of the tensor product of $V^k(\mathfrak{g})$ and a certain ghost vertex operator superalgebra. An important problem is to understand the representation theory of $W^k(\mathfrak{g}; f)$ and that of its simple quotient $W_k(\mathfrak{g}; f)$.

There are certain cases in which the representation theory of $W_k(\mathfrak{g}; f)$ is relatively well-understood. In particular, $W^k(\mathfrak{g}; 0) = V^k(\mathfrak{g})$ and so $W_k(\mathfrak{g}; 0)$ is the simple affine vertex superalgebra $L_k(\mathfrak{g})$. For admissible levels, the highest-weight modules of the latter were classified in [14] for \mathfrak{g} a simple Lie algebra. On the other hand, when f is principal and k is admissible and nondegenerate, the representation theory of $W_k(\mathfrak{g}; f)$ was completely determined in [13], again for \mathfrak{g} nonsuper. Other results in this direction may be found in [11, 17].

Our interest here is in the so-called relaxed highest-weight modules [30, 48] that play an important role in the representation theory of certain classes of nonrational non- C_2 -cofinite vertex superalgebras including the universal W-algebras and many of their simple quotients. The first classification result of this type addressed the simple relaxed highest-weight modules of $L_k(\mathfrak{sl}_2)$ for k admissible [8]. Recently, similar classifications for other affine vertex superalgebras have started to appear [16, 21, 23, 32, 33, 41, 46, 52]. Moreover, [39] explains how one can rigorously derive, for general affine vertex algebras, the relaxed classification from the highest-weight one. A natural question now is how to obtain relaxed classifications for nonrational non- C_2 -cofinite W-algebras.

One answer to this question is to use explicit singular vector formulae [6]. However, this is limited to a very small subset of W-algebras and levels. Another is to realise the simple W-algebras using a coset construction, if one is available. Such constructions are generally very difficult to prove, see [15] for example, and are thus far limited to a small class of nilpotents (notably the principal ones). Nevertheless, coset constructions provide powerful tools to analyse the representation theory of certain W-algebras. A particularly tractable, but still important, special case concerns cosets by a Heisenberg subalgebra for which there are general tools available [20]. This includes, for example, the nonunitary minimal models of the $N = 2$ superconformal algebras [22].

A somewhat more general approach is to use quantum hamiltonian reduction functors to construct W-algebra modules from affine vertex superalgebra modules, when the latter are well-understood. More precisely, one can restrict the functor to category \mathcal{O} , try to prove that the reduction functor is surjective onto category \mathcal{O} for the W-algebra, and then use the methods of [39] to extend this to a classification of relaxed modules.

This approach is currently being tested in [28] for the simplest nonrational non- C_2 -cofinite W-algebras, the Bershadsky–Polyakov algebras $\text{BP}_k = W_k(\mathfrak{sl}_3; f_{\min})$ [19, 45] with k nondegenerate admissible and f_{\min} minimal (take the lowest root vector of \mathfrak{sl}_3 for definiteness). This uses a detailed understanding [9, 36, 37] of the minimal reduction functor. However, generalising this approach to other W-algebras will require a far better understanding of the corresponding reduction functors than is currently available.

Here, we study the representation theory of the Bershadsky–Polyakov algebras using a promising alternative approach that has the benefit of constructing the relaxed highest-weight modules directly, by “inverting” the quantum hamiltonian reduction functor (see [49]). This method was pioneered in [2] for $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{osp}(1|2)$. In particular, the simple affine vertex algebra $L_k(\mathfrak{sl}_2)$ was, for $k \notin \mathbb{Z}_{\geq 0}$, there realised as a subalgebra of the tensor product of a simple Virasoro vertex algebra L_c^{Vir} and a lattice vertex algebra Π of indefinite type. Similarly, $L_k(\mathfrak{osp}(1|2))$ was realised in the tensor product of a simple $N = 1$ superconformal vertex algebra $L_c^{N=1}$, a free fermion and another lattice vertex algebra (closely related to Π).

Moreover, the known irreducible $L_k(\mathfrak{sl}_2)$ -modules were constructed in [2] as submodules of the tensor product of an irreducible L_c^{Vir} -module M and an irreducible Π -module $\Pi_r(\lambda)$. An especially nice observation is that the irreducible relaxed highest-weight modules were realised directly as $M \otimes \Pi_{-1}(\lambda)$, where M is some irreducible highest-weight L_c^{Vir} -module. This neatly explains why the characters of these relaxed modules, proposed in [24] and proven in [40], have the well-known irreducible Virasoro characters as factors. Analogous realisations for irreducible $L_k(\mathfrak{osp}(1|2))$ -modules at arbitrary levels also appear in [2], again explaining the factorisations of their characters [40].

Of course, L_c^{Vir} and $L_c^{N=1}$ are the principal quantum hamiltonian reductions $W_k(\mathfrak{sl}_2; f_{\text{pr}})$ and $W_k(\mathfrak{osp}(1|2); f_{\text{pr}})$, respectively. It is in this sense that tensoring with an appropriate vertex operator superalgebra inverts the reduction functor. In this paper, we extend the results of [2] to the Bershadsky–Polyakov algebras BP^k and BP_k . The role of the quantum hamiltonian reductions will be played by the Zamolodchikov algebra $Z^k = W^k(\mathfrak{sl}_3; f_{\text{pr}})$ and its simple quotient Z_k [53]. Our results may therefore be regarded as not inverting the principal (or minimal) reduction functor, but rather as inverting an (as yet undefined) affine version of the quantum hamiltonian reduction by stages functors introduced (for type A) in [44].

Recall the lattice vertex algebra Π and its irreducible modules $\Pi_{-1}(\lambda)$ (see Section 3.2 for precise definitions). We shall prove the following results.

- For all k , the universal Bershadsky–Polyakov algebra BP^k is a vertex subalgebra of $Z^k \otimes \Pi$ (Theorems 3.6 and 7.3).
- For all k such that $2k + 3 \notin \mathbb{Z}_{\geq 0} \cup \{-3\}$, the simple Bershadsky–Polyakov algebra BP_k is a vertex subalgebra of $Z_k \otimes \Pi$ (Theorem 6.2).
- If M is an irreducible highest-weight Z^k -module, then $M \otimes \Pi_{-1}(\lambda)$ is an indecomposable relaxed highest-weight BP^k -module that is irreducible for almost all λ (Theorem 5.12).
- If $2k + 3 \notin \mathbb{Z}_{\geq 0} \cup \{-3\}$, then the previous assertion holds with Z^k and BP^k replaced by Z_k and BP_k (Theorem 6.3).
- Every nonordinary irreducible (conjugate) highest-weight BP^k -module may be constructed as an explicitly given submodule of some $M \otimes \Pi_{-1}(\lambda)$ (Proposition 5.14).

In this paper, we will not delve deeper into the representations of the Bershadsky–Polyakov algebras, leaving logarithmic (staggered) modules and Whittaker modules for a sequel. We will also not prove that our relaxed construction produces all the relaxed highest-weight BP_k -modules, up to isomorphism, noting only that this follows by comparing with the classification results of [28]. Nevertheless, it would be very satisfying to prove this completeness using the framework developed here and we intend to also address this in the sequel.

Our success in generalising the results of [2] not only lends weight to the conjectural existence of affine reduction by stages functors, it also suggests a general program for elucidating the representation theory of a given (simple) W-algebra $W_k(\mathfrak{g}; f)$. We shall describe this program, initially assuming that k is nondegenerate admissible for simplicity. The principal W-algebra $W_k(\mathfrak{g}; f_{\text{pr}})$ is then rational [13] and its representation theory is, in principle, known. Nilpotent orbits admit a natural partial ordering via inclusions of their closures with the principal orbit being the largest and the zero orbit the smallest. The program then amounts to iteratively inverting the affine reduction by stages functors to construct the representations of $W_k(\mathfrak{g}; f)$ from those of $W_k(\mathfrak{g}; f_{\text{pr}})$.

For $\mathfrak{g} = \mathfrak{sl}_3$, the poset of nilpotent orbits is totally ordered, with the minimal orbit lying between the zero and principal ones. Following the program described above means choosing a nondegenerate admissible level k and using the known representation theory of the rational vertex operator algebra $Z_k = W_k(\mathfrak{sl}_3; f_{\text{pr}})$ to construct the representations of $\text{BP}_k = W_k(\mathfrak{sl}_3; f_{\text{min}})$. This is the content of this paper. Continuing this program, we should next attempt to use these results to construct the representation theory of $L_k(\mathfrak{sl}_3) = W_k(\mathfrak{sl}_3; 0)$. We intend to return to this in the future [4], comparing with the results of the complementary approach of [14, 39].

For degenerate admissible levels, we expect that the privileged role of the principal W-algebra as the starting point of the program will be replaced by the simple exceptional W-algebras of [12, 27, 38], many of which have been recently proven to be rational [17]. For example, if $\mathfrak{g} = \mathfrak{sl}_3$ and $k \in -\frac{3}{2} + \mathbb{Z}_{\geq 0}$, then the principal reduction of $L_k(\mathfrak{sl}_3)$ is zero and the exceptional W-algebra is BP_k . The latter is rational [11] and so our program begins here. If instead $k \in \mathbb{Z}_{\geq 0}$, then both the principal and minimal reductions of $L_k(\mathfrak{sl}_3)$ are zero and the exceptional W-algebra is $L_k(\mathfrak{sl}_3)$ itself (which is also rational).

We conclude this introduction with a brief outline of the contents of the paper. In Sections 2 and 3, we first introduce our conventions for the Bershadsky–Polyakov algebras, as well as the Zamolodchikov algebras and the lattice vertex algebra Π . We then verify that we have a homomorphism from the universal Bershadsky–Polyakov algebra to the tensor product of the universal Zamolodchikov algebra and Π in Section 3.3. The fact that this is an embedding is proven in Section 4.

We next construct relaxed highest-weight modules for the Bershadsky–Polyakov algebras, arguing in Section 5 that these modules are “almost irreducible” which means, among other things, that for almost all values of the parameters that naturally specify these modules, they are irreducible. We also give a precise criterion for irreducibility. This allows us to realise the simple Bershadsky–Polyakov algebra as a submodule of such a relaxed module in Section 6 when the level satisfies $2k + 3 \notin \mathbb{Z}_{\geq 0}$, thereby proving the simple analogue of the embedding of Sections 3.3 and 4 for these levels. Finally, Section 7 establishes critical-level analogues of these results.

Notation. Given a homogeneous field $A(z)$ of conformal weight Δ_A , we define operators A_n and $A_{(n)}$, $n \in \mathbb{Z}$, by the expansions

$$(1.1) \quad A(z) = \sum_{n \in \mathbb{Z} - \Delta_A} A_n z^{-n - \Delta_A} = \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n - 1}.$$

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2. BERSHADSKY–POLYAKOV ALGEBRAS

In this section, we introduce the Bershadsky–Polyakov algebras as vertex algebras. They were originally defined independently by Polyakov [45] and Bershadsky [19] as non-principal quantum hamiltonian reductions of the universal affine vertex algebras $V^k(\mathfrak{sl}_3)$ associated to \mathfrak{sl}_3 . In the framework of Kac–Roan–Wakimoto [36], they are simultaneously the minimal and subregular reductions.

We start by defining the universal Bershadsky–Polyakov vertex operator algebras BP^k in terms of generators and relations. The Bershadsky–Polyakov vertex algebra at the critical level $k = -3$ will be studied in Section 7.

Definition 2.1. For $k \neq -3$, the universal Bershadsky–Polyakov vertex operator algebra BP^k is the universal vertex algebra generated by fields L , J , G^+ and G^- subject to the following operator product expansions:

$$\begin{aligned}
(2.1) \quad & J(z)J(w) \sim \frac{2k+3}{3(z-w)^2}, & J(z)G^\pm(w) &\sim \pm \frac{G^\pm(w)}{z-w}, \\
& L(z)G^+(w) \sim \frac{G^+(w)}{(z-w)^2} + \frac{\partial G^+(w)}{z-w}, & L(z)G^-(w) &\sim \frac{2G^-(w)}{(z-w)^2} + \frac{\partial G^-(w)}{z-w}, \\
& L(z)J(w) \sim -\frac{2k+3}{3(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}, & G^\pm(z)G^\pm(w) &\sim 0, \\
& L(z)L(w) \sim \frac{c_k^{\text{BP}}}{2(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial L(w)}{z-w}, \\
& G^+(z)G^-(w) \sim \frac{(k+1)(2k+3)}{(z-w)^3} + \frac{3(k+1)J(w)}{(z-w)^2} \\
& \quad + \frac{3:J(w)J(w): + (2k+3)\partial J(w) - (k+3)L(w)}{z-w}.
\end{aligned}$$

The central charge is

$$(2.2) \quad c_k^{\text{BP}} = -\frac{4(k+1)(2k+3)}{k+3}.$$

As always, BP^k has a unique simple quotient and we shall denote it by BP_k .

Remark 2.2. Both BP^k and BP_k are $\mathbb{Z}_{\geq 0}$ -graded by the eigenvalue of the zero mode L_0 because the conformal weights of $G^\pm(z)$ are 1 and 2, respectively. This asymmetry is also reflected in the fact that $J(z)$ fails to be quasiprimary. This failure may be rectified by instead choosing the conformal vector to be

$$(2.3) \quad \tilde{L} = L - \frac{1}{2}\partial J.$$

Both G^+ and G^- will then have conformal weight $\frac{3}{2}$. The only downside is that BP^k and BP_k are now $\frac{1}{2}\mathbb{Z}_{\geq 0}$ -graded by the eigenvalue of \tilde{L}_0 . Unless otherwise indicated, we shall keep L as the conformal vector.

The commutation relations of the modes are easily computed from the operator product expansions (2.1). We record them for convenience.

$$\begin{aligned}
(2.4) \quad & [J_m, J_n] = \frac{2k+3}{3}m\delta_{m+n,0}, \quad [J_m, G_n^\pm] = \pm G_{m+n}^\pm, \\
& [L_m, G_n^+] = -nG_{m+n}^+, \quad [L_m, G_n^-] = (m-n)G_{m+n}^-, \\
& [L_m, J_n] = -nJ_{m+n} - \frac{2k+3}{3}\frac{m^2-m}{2}\delta_{m+n,0}, \quad [G_m^\pm, G_n^\pm] = 0, \\
& [L_m, L_n] = (m-n)L_{m+n} - \frac{(2k+3)(k+1)}{k+3}\frac{m^3-m}{3}\delta_{m+n,0}, \\
& [G_m^+, G_n^-] = 3:JJ_{:m+n} - (k+3)L_{m+n} + (km - (2k+3)(n+1))J_{m+n} \\
& \quad + (k+1)(2k+3)\frac{m^2-m}{2}\delta_{m+n,0}.
\end{aligned}$$

The associative algebra of modes specified by these relations admits a useful family of automorphisms called *spectral flow automorphisms*. These may be lifted to maps on BP^k , following [43], by introducing

$$(2.5) \quad \Lambda(\ell J, z) = z^{-\ell J_0} \prod_{n=1}^{\infty} \exp\left(\frac{(-1)^n}{n}\ell J_n z^{-n}\right)$$

and defining the result of acting with the spectral flow map σ^ℓ , $\ell \in \mathbb{Z}$, on a field $A(z)$ of BP^k to be

$$(2.6) \quad \sigma^\ell(A(z)) = Y(\Lambda(\ell J, z)A, z),$$

where Y is the vertex map of BP^k . In particular, we have

$$\begin{aligned}
(2.7) \quad & \sigma^\ell(G^\pm(z)) = z^{\mp\ell}G^\pm(z), \quad \sigma^\ell(J(z)) = J(z) - \frac{2k+3}{3}\ell z^{-1}, \\
& \sigma^\ell(L(z)) = L(z) - \ell z^{-1}J(z) + \frac{2k+3}{3}\frac{\ell(\ell+1)}{2}z^{-2}.
\end{aligned}$$

One can check explicitly that spectral flow preserves the defining operator product expansions (2.1) (and the vacuum) of BP^k . These spectral flows may, moreover, be extended to $\ell \in \frac{1}{2}\mathbb{Z}$ if we allow half-integer modes for $G^\pm(z)$ as when acting on twisted BP^k -modules.

Let M be a BP^k -module. By twisting the action of BP^k on M by the spectral flow map $\sigma^{-\ell}$, we may give M a new structure as a BP^k -module. We shall denote this new BP^k -module by $\sigma^\ell(M)$. Denoting its elements by $\sigma^\ell(v)$, where $v \in M$, the twisted action is explicitly realised as

$$(2.8) \quad A \cdot \sigma^\ell(v) = \sigma^\ell(\sigma^{-\ell}(A)v).$$

In this way, spectral flow lifts to invertible endofunctors on the category of BP^k -modules. Consequently, $\sigma^\ell(M)$ is irreducible if and only if M is.

3. REALISATION OF BP^k

Here, we realise the universal Bershadsky–Polyakov algebra BP^k as a vertex subalgebra of the tensor product of the principal quantum hamiltonian reduction of $V^k(\mathfrak{sl}_3)$, which we shall refer to as the Zamolodchikov algebra, and an isotropic lattice vertex algebra Π . We first introduce these vertex algebras and some of their modules.

3.1. The Zamolodchikov algebras Z^k and Z_k . As with the universal Bershadsky–Polyakov algebras, the universal Zamolodchikov algebras [53] may also be defined in terms of generators and relations.

Definition 3.1. For $k \neq -3$, the universal Zamolodchikov vertex operator algebra Z^k is the universal vertex algebra generated by fields T and W subject to the following operator product expansions:

$$(3.1) \quad \begin{aligned} T(z)T(w) &\sim \frac{c_k^Z}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \\ T(z)W(w) &\sim \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w}, \\ W(z)W(w) &\sim \frac{(k+3)^3}{3} \left[\frac{2\Lambda(w)}{(z-w)^2} + \frac{\partial\Lambda(w)}{z-w} \right] \\ &\quad + A \left[\frac{c_k^Z}{3(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} + \frac{\frac{3}{10}\partial^2 T(w)}{(z-w)^2} + \frac{\frac{1}{15}\partial^3 T(w)}{z-w} \right]. \end{aligned}$$

Here, the central charge is

$$(3.2) \quad c_k^Z = -\frac{2(3k+5)(4k+9)}{k+3}$$

and we have defined

$$(3.3) \quad A = -\frac{(k+3)^2(3k+4)(5k+12)}{6} \quad \text{and} \quad \Lambda = :TT: - \frac{3}{10}\partial^2 T,$$

for convenience. The unique simple quotient of Z^k will be denoted by Z_k .

Remark 3.2. The definition of Z^k given above differs from the standard one in that we have renormalised the field $W(z)$ by a factor of \sqrt{A} . This removes a singularity in the standard definition when $c_k^Z = -\frac{22}{5}$, hence $k = -\frac{4}{3}$ or $-\frac{12}{5}$. At this central charge, W and Λ belong to the maximal ideal of $Z^{-4/3} = Z^{-12/5}$. The simple quotient is in fact the Virasoro minimal model $M(2, 5)$ (also known as the Yang–Lee model).

3.2. The vertex algebra Π and its modules. Consider the abelian Lie algebra $\mathfrak{h} = \text{span}_{\mathbb{C}}\{a, b\}$, equipped with the bilinear form defined by

$$(3.4) \quad \langle a, a \rangle = -\langle b, b \rangle = 1 \quad \text{and} \quad \langle a, b \rangle = 0.$$

For convenience, we let

$$(3.5) \quad c = a - b \quad \text{and} \quad d = a + b,$$

noting that these elements of \mathfrak{h} are isotropic: $\langle c, c \rangle = \langle d, d \rangle = 0$. The group algebra $\mathbb{C}[\mathbb{Z}c] = \text{span}_{\mathbb{C}}\{e^{nc} : n \in \mathbb{Z}\}$ then becomes an \mathfrak{h} -module with action

$$(3.6) \quad he^{nc} = \langle h, c \rangle e^{nc}, \quad h \in \mathfrak{h}.$$

Let H denote the Heisenberg vertex algebra associated to \mathfrak{h} .

Definition 3.3. Let Π denote the lattice vertex algebra $H \otimes \mathbb{C}[\mathbb{Z}c]$, where the action of $h \in \mathfrak{h}$ on $\mathbb{C}[\mathbb{Z}c]$ is identified with the action of the zero mode h_0 of $h(z) \in H$. We equip Π with the conformal structure given by

$$(3.7) \quad t(z) = \frac{1}{2}:c(z)d(z): + \frac{2k+3}{3}\partial c(z) - \frac{1}{2}\partial d(z),$$

so that a and b both have conformal weight 1, whilst the weight of e^{nc} , $n \in \mathbb{Z}$, is n . The central charge is

$$(3.8) \quad c_k^\Pi = 2 + 8(2k+3).$$

Vertex algebras like Π were studied in [18], under the name “half-lattice vertex algebras”, as were their representation theories. We refer to [2, Sec. 4] for a convenient summary. Here, we record the following operator product expansion for future convenience:

$$(3.9) \quad e^c(z)e^{-c}(w) = \mathbb{1} + c(w)(z-w) + \frac{1}{2}(:c(w)c(w): + \partial c(w))(z-w)^2 + \dots$$

Before introducing the Π -modules relevant to what follows, we discuss spectral flow for Π . In analogy with (2.5), we set [43]

$$(3.10) \quad \Lambda(\ell j, z) = z^{-\ell j_0} \prod_{n=1}^{\infty} \exp\left(\frac{(-1)^n}{n} \ell j_n z^{-n}\right), \quad j = b + \frac{k+3}{3}c,$$

and define the action of the spectral flow map γ^ℓ , $\ell \in \mathbb{Z}$, on a field $A(z)$ of Π by

$$(3.11) \quad \gamma^\ell(A(z)) = Y(\Lambda(\ell j, z)A, z),$$

where Y is now the vertex map of Π . The reason for taking the particular element $j \in \mathfrak{h}$ will become clear in Section 3.3.

With this, it is easy to verify that spectral flow acts on the generators of Π as follows:

$$(3.12) \quad \begin{aligned} \gamma^\ell(a(z)) &= a(z) - \frac{k+3}{3}\ell z^{-1}, & \gamma^\ell(e^{nc}(z)) &= z^{-\ell n} e^{nc}(z) \quad (n \in \mathbb{Z}). \\ \gamma^\ell(b(z)) &= b(z) - \frac{k}{3}\ell z^{-1}, \end{aligned}$$

The γ^ℓ thus preserve operator product expansions (and the vacuum) of Π , but do not preserve its conformal structure:

$$(3.13) \quad \gamma^\ell(t(z)) = t(z) - \frac{1}{2}\ell z^{-1}j(z) + \frac{2k+3}{3}\frac{\ell(\ell+1)}{2}z^{-2}.$$

As in Section 2, twisting by spectral flow defines invertible functors on the category of Π -modules. This therefore associates to every $\ell \in \mathbb{Z}$ and every (irreducible) Π -module M a new (irreducible) Π -module $\gamma^\ell(M)$. Again, this generalises to $\ell \in \frac{1}{2}\mathbb{Z}$ and twisted modules.

The Π -modules of interest here may be obtained by considering the $\mathbb{C}[\mathbb{Z}c]$ -module generated by $e^h \in \mathbb{C}[\mathfrak{h}]$ and inducing. For later convenience, we shall generally write h as a linear combination of the basis vectors j , defined in (3.10), and c . It will also be convenient to introduce

$$(3.14) \quad i = d - j = a - \frac{k+3}{3}c.$$

We then have $\langle j, j \rangle = \frac{2k+3}{3} = -\langle i, i \rangle$, $\langle j, c \rangle = 1 = \langle i, c \rangle$ and $\langle i, j \rangle = 0$.

For each $r \in \frac{1}{2}\mathbb{Z}$ and $\lambda \in \mathbb{C}$, define

$$(3.15) \quad \Pi_r(\lambda) = \Pi \cdot e^{rj+\lambda c}.$$

Recall that an indecomposable module is positive-energy, with respect to a given conformal structure, if its conformal weights are bounded below. Its top space is then the eigenspace corresponding to the minimal conformal weight (should it exist).

The irreducible weight Π -modules were constructed and classified in [18]. In particular, this work implies that $\Pi_r(\lambda)$ is an irreducible Π -module for $r \in \mathbb{Z}$. One can modify this result and show that $\Pi_r(\lambda)$ is an irreducible twisted Π -module for $r \in \mathbb{Z} + \frac{1}{2}$. Thus we have (see also [2, Prop. 4.1]):

Proposition 3.4.

- For $r \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$, $\Pi_r(\lambda)$ is an irreducible (untwisted) \mathbb{Z} -graded Π -module.
- For $r \in \mathbb{Z} + \frac{1}{2}$ and $\lambda \in \mathbb{C}$, $\Pi_r(\lambda)$ is an irreducible $\frac{1}{2}\mathbb{Z}$ -graded ($e^{\pi\sqrt{-1}i_0}$ -twisted) Π -module.

- In both cases, we have $\Pi_r(\lambda) \cong \Pi_r(\lambda + n)$ for all $n \in \mathbb{Z}$. Otherwise, the $\Pi_r(\lambda)$ are mutually inequivalent.
- c_0 acts on $\Pi_r(\lambda)$ as r times the identity and the $e_{-n(r+1)}^{nc}$, $n \in \mathbb{Z}$, act injectively.
- $\Pi_r(\lambda)$ is positive-energy if and only if $r = -1$. $\Pi_{-1}(\lambda)$ is thus a relaxed highest-weight Π -module and its top space $\Pi_{-1}(\lambda)_{\text{top}}$ has conformal weight $\frac{2k+3}{3}$.
- Twisting the action of Π by the spectral flow maps γ^ℓ gives (see [3, Prop. 4.1] for a similar calculation)

$$(3.16) \quad \gamma^\ell(\Pi_r(\lambda)) \cong \Pi_{r+\ell}(\lambda).$$

Here, a *relaxed highest-weight Π -module* is a module that is generated by a single relaxed highest-weight vector, the latter being a weight vector that is annihilated by the h_n , with $h \in \mathfrak{h}$ and $n > 0$, and the e_n^{mc} with $m \in \mathbb{Z}$ and $n > 0$. Note that the parameter r in $\Pi_r(\lambda)$ is a spectral flow index while λ represents the eigenvalue of i_0 . The isomorphism class of $\Pi_r(\lambda)$ thus only depends on r and the image of λ in \mathbb{C}/\mathbb{Z} . Obviously, the vacuum module is $\Pi_0(0)$.

It is also straightforward to determine the characters of the relaxed highest-weight modules $\Pi_{-1}(\lambda)$.

Proposition 3.5. *Let $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ as usual. Then, the character of $\Pi_{-1}(\lambda)$ is*

$$(3.17) \quad \text{ch}[\Pi_{-1}(\lambda)](y, z; q) = \text{tr}_{\Pi_{-1}(\lambda)} y^{c_0} z^{i_0} q^{t_0 - c^\Pi/24} = \frac{y^{-1} z^\lambda}{\eta(q)^2} \delta(z).$$

3.3. Realisation. The (easily verified) central charge relation $c_k^{\text{BP}} = c_k^{\text{Z}} + c_k^\Pi$ suggests that the three vertex operator algebras BP^k , Z^k and Π might be related. The following result determines this relation precisely.

Theorem 3.6. *For $k \neq -3$, there is an injective vertex operator algebra homomorphism $\phi^k: \text{BP}^k \rightarrow \text{Z}^k \otimes \Pi$, uniquely determined by*

$$(3.18) \quad \begin{aligned} G^+ &\mapsto \mathbb{1} \otimes e^c, & J &\mapsto \mathbb{1} \otimes j, & L &\mapsto T \otimes \mathbb{1} + \mathbb{1} \otimes t, \\ G^- &\mapsto \left(W + \frac{1}{2}(k+2)(k+3)\partial T\right) \otimes e^{-c} + (k+3)T \otimes i_{-1}e^{-c} \\ &\quad - \mathbb{1} \otimes (i_{-1}^3 + 3(k+2)i_{-2}i_{-1} + 2(k+2)^2i_{-3})e^{-c}. \end{aligned}$$

Here, i and j were defined in (3.14) and (3.10), respectively.

Sketch of proof that ϕ^k is a vertex operator algebra homomorphism. Because BP^k is universal, it suffices to show that the operator product expansions (2.1) of the generators J , L and G^\pm match those of their ϕ^k -images. This can be checked explicitly from the defining operator product expansions (3.1) of Z^k and those, for example (3.9), of Π . We used the computer algebra package OPEDEFS [50] for this purpose, but the computations are also easily performed by hand.

For example, the coefficient of the third-order pole of $G^+(z)G^-(w)$ in (2.1) is determined by $G_2^+G^- = (k+1)(2k+3)\mathbb{1}$ whilst the corresponding calculation for their ϕ^k -images proceeds as follows. First, note that

$$(3.19) \quad \phi^k(G^+)_2\phi^k(G^-) = -\mathbb{1} \otimes [e_2^c, i_{-1}^3 + 3(k+2)i_{-2}i_{-1} + 2(k+2)^2i_{-3}]e^{-c},$$

because e_2^c annihilates both e^{-c} and $i_{-1}e^{-c}$. Since $[e_m^c, i_n] = -e_{m+n}^c$ and $e_{-1}^c e^{-c} = \mathbb{1}$, this indeed evaluates to

$$(3.20) \quad \begin{aligned} \phi^k(G^+)_2\phi^k(G^-) &= \mathbb{1} \otimes (e_1^c i_{-1}^2 + 3(k+2)e_0^c i_{-1} + 2(k+2)^2 e_{-1}^c) e^{-c} \\ &= \mathbb{1} \otimes (-e_0^c i_{-1} - 3(k+2)e_{-1}^c + 2(k+2)^2 e_{-1}^c) e^{-c} \\ &= \mathbb{1} \otimes (1 - 3(k+2) + 2(k+2)^2) e_{-1}^c e^{-c} \end{aligned}$$

$$= (k+1)(2k+3)\mathbb{1}.$$

Similar computations determine that all the singular coefficients match, hence that ϕ^k is a homomorphism. \square

Remark 3.7. Comparing the spectral flow maps σ^ℓ of BP^k and γ^ℓ of Π , we see that the explicit realisation (3.18) requires $\sigma^\ell = \text{id} \otimes \gamma^\ell$. In the definition (3.10) of the spectral flow maps of Π , we could have replaced j by any $h \in \mathfrak{h}$ and still preserved the operator product expansions. However, the above realisation singles out $h = j$ as being particularly useful for our purposes.

Remark 3.8. The universal Zamolodchikov algebras Z^k admit free field realisations inside a rank-2 Heisenberg vertex operator algebra H' , equipped with a positive-definite bilinear form [54]. Tensoring H' with Π thus gives free field realisations of the universal Bershadsky–Polyakov algebras BP^k . This is not the standard free field realisation [19], but is obtained from it by bosonising the $\beta\gamma$ ghost system. Free field realisations of the simple Bershadsky–Polyakov algebras BP_k are less common. However, such realisations are known for $k = -\frac{9}{4}$ [25] and $k = -\frac{5}{3}$ [6]. At other levels, we expect that BRST-type realisations exist instead.

We will prove that ϕ^k is injective in Section 4. Granting this, it follows from Theorem 3.6 that for $k \neq -3$, any $Z^k \otimes \Pi$ -module is a BP^k -module, by restriction. Combining this with Proposition 3.4, we obtain a construction of many positive-energy BP^k -modules. Proposition 3.5 then gives their characters.

Corollary 3.9. *Let $k \neq -3$ and suppose that M is a Z^k -module that admits a character: $\text{ch}[M](q) = \text{tr}_M q^{T_0 - c_k^Z/24}$. Then, the BP^k -module $R_M(\lambda) = M \otimes \Pi_{-1}(\lambda)$ is relaxed highest-weight with character*

$$\begin{aligned} (3.21) \quad \text{ch}[R_M(\lambda)](z; q) &= \text{tr}_{M \otimes \Pi_{-1}(\lambda)} z^{J_0} q^{L_0 - c_k^{\text{BP}}/24} \\ &= \text{ch}[M](q) \text{ch}[\Pi_{-1}(\lambda)](z^{(2k+3)/3}, z; q) \\ &= \frac{z^{\lambda - (2k+3)/3} \text{ch}[M](q)}{\eta(q)^2} \delta(z). \end{aligned}$$

Recall that a *relaxed highest-weight* BP^k - or BP_k -module is a module generated by a single relaxed highest-weight vector, the latter being a weight vector that is annihilated by the L_n, J_n, G_n^+ and G_n^- with $n > 0$. When M is irreducible, we shall show in Section 5 that the $R_M(\lambda)$ are always indecomposable and, in fact, are irreducible for almost all λ .

4. INJECTIVITY OF ϕ^k

In this section, we show that the homomorphism $\phi^k: \text{BP}^k \rightarrow Z^k \otimes \Pi$ is injective, for $k \neq -3$, and thereby prove Theorem 3.6. Recall that a partition is a finite sequence of positive integers $\mu = (\mu_1, \mu_2, \dots, \mu_\ell)$ of length $\ell = \ell(\mu) \in \mathbb{Z}_{\geq 0}$ satisfying

$$(4.1) \quad \mu_1 \geq \mu_2 \geq \dots \geq \mu_\ell.$$

The weight of the partition μ is defined to be $|\mu| = \mu_1 + \mu_2 + \dots + \mu_\ell$. Let \mathcal{P} denote the set of all partitions.

Given a partition $\mu \in \mathcal{P}$ of length ℓ and an element A of a vertex algebra, we introduce (whenever it makes sense) the convenient notation

$$(4.2) \quad \begin{aligned} A_{+\mu} &= A_{\mu_\ell} \cdots A_{\mu_2} A_{\mu_1}, & A_{-\mu} &= A_{-\mu_1} A_{-\mu_2} \cdots A_{-\mu_\ell}, \\ A_{(+\mu)} &= A_{(\mu_\ell)} \cdots A_{(\mu_2)} A_{(\mu_1)}, & A_{(-\mu)} &= A_{(-\mu_1)} A_{(-\mu_2)} \cdots A_{(-\mu_\ell)}, \end{aligned}$$

recalling the conventions for mode indices in (1.1). We shall also write $\mu+n$ (and $-\mu-n$) in the above to indicate that every part of $\mu \in \mathcal{P}$ should be increased by $n \in \mathbb{Z}_{\geq 0}$. The following lemma is now clear from universality (see [37, Thm. 4.1(b)]).

Lemma 4.1.

- The universal Bershadsky–Polyakov algebra BP^k has a Poincaré–Birkhoff–Witt-type basis $B_{\text{BP}} = \{J_{(-\mu)}G_{(-\nu)}^+L_{(-\rho)}G_{(-\sigma)}^- \mathbb{1} : \mu, \nu, \rho, \sigma \in \mathcal{P}\}$.
- The universal Zamolodchikov algebra \mathbb{Z}^k likewise has a Poincaré–Birkhoff–Witt-type basis $B_{\mathbb{Z}} = \{T_{(-\mu)}W_{(-\nu)} \mathbb{1} : \mu, \nu \in \mathcal{P}\}$.

Because the lattice vertex operator algebra Π restricts, as an H -module, to an infinite direct sum of Fock modules (one for each $e^{nc} \in \Pi$), we get our third basis.

Lemma 4.2. *The lattice vertex operator algebra Π has a Poincaré–Birkhoff–Witt-type basis $B'_{\Pi} = \{j_{(-\mu)}c_{(-\nu)}e^{nc} : \mu, \nu \in \mathcal{P} \text{ and } n \in \mathbb{Z}\}$.*

This basis will, however, need some finessing. Let $S_m(c)$, $m \in \mathbb{Z}_{\geq 0}$, denote the Schur function in the (commuting) variables $c_{(-n)}$, $n \in \mathbb{Z}_{\geq 1}$, corresponding to the partition (m) . Equivalently, these functions may be defined by the following special case of the Cauchy identity:

$$(4.3) \quad \prod_{n=1}^{\infty} \exp\left(\frac{c_{(-n)}}{n} z^n\right) = \sum_{m=0}^{\infty} S_m(c) z^m.$$

In particular, we have

$$(4.4) \quad S_0(c) = 1, \quad S_1(c) = c_{(-1)} \quad \text{and} \quad S_2(c) = \frac{1}{2} (c_{(-2)} + c_{(-1)}^2).$$

For general $m \in \mathbb{Z}_{\geq 1}$, the $S_m(c)$ have the form

$$(4.5) \quad S_m(c) = \frac{1}{m} c_{(-m)} + [\text{terms quadratic and higher in the } c_{(-n)} \text{ with } n < m].$$

Proposition 4.3. *The set $B_{\Pi} = \{j_{(-\mu)}e_{(-\nu-1)}^c e^{nc} : \mu, \nu \in \mathcal{P} \text{ and } n \in \mathbb{Z}\}$ is also a basis of Π .*

Proof. It follows easily from (4.3) and c being isotropic that

$$(4.6) \quad e_{(-m-1)}^c e^{nc} = \begin{cases} S_m(c) e^{(n+1)c} & \text{if } m \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Setting $\ell = \ell(\nu)$, we therefore have

$$(4.7) \quad \begin{aligned} e_{(-\nu-1)}^c e^{nc} &= e_{(-\nu_1-1)}^c \cdots e_{(-\nu_\ell-1)}^c e^{nc} = S_{\nu_1}(c) \cdots S_{\nu_\ell}(c) e^{(n+\ell)c} \\ &= \frac{c_{(-\nu_1)} \cdots c_{(-\nu_\ell)} e^{(n+\ell)c}}{\nu_1 \cdots \nu_\ell} + \cdots = \frac{c_{(-\nu)} e^{(n+\ell)c}}{\nu_1 \cdots \nu_\ell} + \cdots, \end{aligned}$$

where $+\cdots$ indicates terms whose c -degrees are greater than ℓ . Composing with $j_{(-\mu)}$, it follows that the elements of B_{Π} are linearly independent. Moreover, they span Π because an obvious inductive triangularity argument shows that (4.7) may be inverted and thus solved for the basis elements of B'_{Π} . \square

Remark 4.4. Note that an element of the form $j_{(-\mu)}e_{(-\nu')}^c e^{n'c} \in \Pi$, $\mu, \nu' \in \mathcal{P}$ and $n' \in \mathbb{Z}$, may be identified with one of the basis vectors of B_{Π} by isolating any parts of ν' equal to 1. If there are m such parts, let ν'' be the partition obtained from ν' by removing them. Then, ν'' has no part equal to 1 so it may be written as $\nu + 1$ for some (unique) partition ν . Setting $n = m + n'$, we get the desired form:

$$(4.8) \quad j_{(-\mu)}e_{(-\nu')}^c e^{n'c} = j_{(-\mu)}e_{(-\nu'')}^c (e_{(-1)}^c)^m e^{n'c} = j_{(-\mu)}e_{(-\nu-1)}^c e^{nc}.$$

We now prove that ϕ^k is injective.

Proof of Theorem 3.6. Recall from the explicit realisation (3.18) that

$$(4.9) \quad \begin{aligned} \phi^k(J_{(-n)}) &= \mathbb{1} \otimes j_{(-n)}, \\ \phi^k(G_{(-n)}^+) &= \mathbb{1} \otimes e_{(-n)}^c, \\ \phi^k(L_{(-n)}) &= T_{(-n)} \otimes \mathbb{1} + [\text{terms not involving } T] \\ \text{and } \phi^k(G_{(-n)}^-) &= \sum_{m=0}^{\infty} W_{(-n+m)} \otimes e_{(-m-1)}^{-c} + [\text{terms not involving } W]. \end{aligned}$$

We will show that the images of the BP^k basis vectors in B_{BP} are linearly independent. These images have the form

$$(4.10) \quad \begin{aligned} \phi^k(J_{(-\mu)}G_{(-\nu)}^+L_{(-\rho)}G_{(-\sigma)}^-\mathbb{1}) &= T_{(-\rho)}W_{(-\sigma)}\mathbb{1} \otimes j_{(-\mu)}e_{(-\nu)}^c(e_{(-1)}^{-c})^{\ell(\sigma)}\mathbb{1} + \dots \\ &= T_{(-\rho)}W_{(-\sigma)}\mathbb{1} \otimes j_{(-\mu)}e_{(-\nu)}^ce^{-\ell(\sigma)c} + \dots, \end{aligned}$$

where $+\dots$ indicates a linear combination of similar terms that have either fewer T -modes, fewer W -modes, or have the same number of T - and W -modes but also have some $e_{(-m-1)}^{-c}$ -modes with $m \geq 1$. In the latter case, the W -partition σ is replaced by another of the same length but strictly lower weight.

The $c \rightarrow -c$ analogues of (4.6) and (4.7) show that the action of any $e_{(-m-1)}^{-c}$ may be expressed in terms of the action of c -modes, hence in terms of actions of e^c -modes. We can thereby rewrite any term with an $e_{(-m-1)}^{-c}$ as a linear combination of basis terms from $B_Z \otimes B_{\Pi}$. The point is that this rewriting will not change the j -, T - and W -modes, in particular it will not change the fact that the corresponding W -partition has weight strictly lower than $|\sigma|$.

The term exhibited on the right-hand side of (4.10) is therefore the unique term, when expressed in the basis $B_Z \otimes B_{\Pi}$, with $\ell(\mu)$ j -modes, $\ell(\rho)$ T -modes and $\ell(\sigma)$ W -modes corresponding to a partition of weight $|\sigma|$. The images on the left-hand side are therefore linearly independent, as desired, hence ϕ^k is injective. \square

5. ALMOST-IRREDUCIBILITY

Our next aim is to study the irreducibility of the relaxed highest-weight BP^k -modules $R_M(\lambda)$, introduced in Corollary 3.9. More specifically, we wish to show that $R_M(\lambda)$ is ‘‘almost irreducible’’ (to be defined shortly) when M is an irreducible highest-weight Z^k -module. Here, we first prepare the groundwork for this by proving that the Π -modules $\Pi_{-1}(\lambda)$ are almost irreducible as modules over a certain vertex subalgebra U .

Definition 5.1 ([26]). Let V be a vertex operator algebra and M a $\mathbb{Z}_{\geq 0}$ -graded V -module with top component M_{top} .

- We say that M is *top-generated* if M is generated by M_{top} .
- We say that M has *only top-submodules* if every nonzero submodule of M has a nonzero intersection with M_{top} .
- We say that M is *almost irreducible* if it is top-generated and has only top-submodules.

One motivation for introducing almost-irreducibility is to isolate a class of modules whose irreducibility is determined by its top space. Recall that the top space M_{top} of a module over a vertex operator algebra V is naturally a module over the Zhu algebra $A(V)$ [55]. The action of $A(V)$ is of course nothing but the action of the zero modes A_0 , $A \in V$, on the top space.

Proposition 5.2. *If M is an almost-irreducible V -module and M_{top} is an irreducible $A(V)$ -module, then M is irreducible.*

Proof. Let N be a nonzero submodule of M . Since M has only top-submodules, $N \cap M_{\text{top}}$ is a nonzero $A(V)$ -module. Since M_{top} is irreducible, we have $M_{\text{top}} \subseteq N$. Finally, M being top-generated forces $M \subseteq N$, hence $N = M$. \square

Another motivation is to model modules obtained by Zhu-induction [42, 55]. More precisely, this induction functor constructs a vertex operator algebra module from a Zhu algebra module in such a way that the top space of the former coincides with the latter. If we now quotient the former by the sum of all submodules whose intersection with the top space is zero, then the result has the same top space but is now almost irreducible. In this sense, almost-irreducibility captures the notion of the “smallest” vertex operator algebra module with a given top space [26].

5.1. Almost-irreducibility of $\Pi_{-1}(\lambda)$. Let U be the vertex subalgebra of Π generated by $j = b + \frac{k+3}{3}c$ and e^c . Our aim is to show that the $\Pi_{-1}(\lambda)$ are almost irreducible as U -modules. Recall that

$$(5.1) \quad \Pi_{-1}(\lambda)_{\text{top}} = \text{span}_{\mathbb{C}}\{e^{-j+(\lambda+n)c} : n \in \mathbb{Z}\}.$$

We first prove top-generation. This follows easily by determining an appropriate basis using the method of Proposition 4.3. An obvious basis (the analogue of that of Lemma 4.2) is

$$(5.2) \quad \{j_{-\mu}c_{-\nu}e^{-j+(\lambda+n)c} : \mu, \nu \in \mathcal{P} \text{ and } n \in \mathbb{Z}\},$$

where we recall the notation of (4.2).

Lemma 5.3. *For every $\lambda \in \mathbb{C}$, the set*

$$(5.3) \quad \{j_{-\mu}e_{-\nu}^c e^{-j+(\lambda+n)c} : \mu, \nu \in \mathcal{P} \text{ and } n \in \mathbb{Z}\}$$

is a basis of $\Pi_{-1}(\lambda)$.

Proof. First, generalise (4.6) to

$$(5.4) \quad e_{-m}^c e^{-j+(\lambda+n)c} = \begin{cases} S_m(c)e^{-j+(\lambda+n+1)c} & \text{if } m \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where the $S_m(c)$ are the Schur functions defined in (4.3). The assertion now follows using the same argument as in the proof of Proposition 4.3. \square

Proposition 5.4. *For every $\lambda \in \mathbb{C}$, $\Pi_{-1}(\lambda)$ is top-generated as a U -module.*

To show that $\Pi_{-1}(\lambda)$ has only top-submodules, we need a preparatory lemma. It follows easily from the commutation relations $[j_m, c_n] = m\delta_{m+n,0}$, $[e_m^c, j_n] = -e_{m+n}^c$ and $[e_m^c, c_n] = [e_m^c, e_n^c] = 0$ ($m, n \in \mathbb{Z}$), as well as the formula

$$(5.5) \quad e_n^c e^{-j+\lambda c} = \delta_{n,0} e^{-j+(\lambda+1)c} \quad (n \in \mathbb{Z}_{\geq 0}).$$

Lemma 5.5.

• *For every $\mu, \nu \in \mathcal{P}$ and $\lambda \in \mathbb{C}$, we have*

$$(5.6) \quad e_{+\mu}^c j_{-\mu} c_{-\nu} e^{-j+\lambda c} = (-1)^{\ell(\mu)} c_{-\nu} e^{-j+(\lambda+\ell(\mu))c} \neq 0.$$

Moreover, if $\mu' \neq \mu$ satisfies either $\ell(\mu') > \ell(\mu)$ or $\ell(\mu') = \ell(\mu)$ and $|\mu'| \geq |\mu|$, then $e_{+\mu'}^c j_{-\mu} c_{-\nu} e^{-j+\lambda c} = 0$.

- For every $\nu \in \mathcal{P}$ and $\lambda \in \mathbb{C}$, we have

$$(5.7) \quad j_{+\nu}c_{-\nu}e^{-j+\lambda c} = \prod_{i=1}^{\ell(\nu)} \nu_i \cdot e^{-j+\lambda c} \neq 0.$$

Moreover, if $\nu' \neq \nu$ satisfies $|\nu'| \geq |\nu|$, then $j_{+\nu'}c_{-\nu}e^{-j+\lambda c} = 0$.

Proposition 5.6. For every $\lambda \in \mathbb{C}$, the U -module $\Pi_{-1}(\lambda)$ has only top-submodules.

Proof. Let w be a nonzero element of $\Pi_{-1}(\lambda)$. It suffices to show that there exists a linear combination of products of modes of U mapping w to a nonzero element of $\Pi_{-1}(\lambda)_{\text{top}}$. Since i_0 acts semisimply on $\Pi_{-1}(\lambda)$, we may assume that $i_0 w = (\lambda + n)w$ for some $n \in \mathbb{Z}$. Then, w has the form

$$(5.8) \quad w = \sum_{\mu, \nu \in \mathcal{P}} C_{\mu, \nu} j_{-\mu} c_{-\nu} e^{-j+(\lambda+n)c},$$

for some $C_{\mu, \nu} \in \mathbb{C}$ such that $C_{\mu, \nu} = 0$ for all but finitely many $(\mu, \nu) \in \mathcal{P} \times \mathcal{P}$.

Since $w \neq 0$, the set S of partitions μ for which there exists some $\nu \in \mathcal{P}$ such that $C_{\mu, \nu} \neq 0$ is nonempty. We may therefore choose $\mu' \in S$ of maximal weight among the elements of S of maximal length. By Lemma 5.5, we have a nonzero vector

$$(5.9) \quad w' = e_{+\mu'}^c w = (-1)^{\ell(\mu')} \sum_{\nu \in \mathcal{P}} C_{\mu', \nu} c_{-\nu} e^{-j+(\lambda+n+\ell(\mu'))c} \neq 0.$$

The set S' consisting of those $\nu \in \mathcal{P}$ for which $C_{\mu', \nu} \neq 0$ is clearly nonempty, hence it has an element ν' of maximal weight. By Lemma 5.5 again, we have

$$(5.10) \quad j_{+\nu'} w' = (-1)^{\ell(\mu')} C_{\mu', \nu'} \prod_{i=1}^{\ell(\nu')} \nu'_i \cdot e^{-j+(\lambda+n+\ell(\mu'))c} \neq 0.$$

As $j_{+\nu'} e_{+\mu'}^c w$ is a nonzero element of $\Pi_{-1}(\lambda)_{\text{top}}$, the proof is complete. \square

Corollary 5.7. $\Pi_{-1}(\lambda)$ is almost irreducible as a U -module.

Remark 5.8. On the other hand, $\Pi_{-1}(\lambda)$ is never irreducible as a U -module. This follows from observing that the generators j and e^c of U have i_0 -eigenvalues 0 and 1, respectively, so U has no elements that reduce the i_0 -eigenvalue. Each $e^{-j+(\lambda+n)c}$, $n \in \mathbb{Z}$, in $\Pi_{-1}(\lambda)_{\text{top}}$ thus generates a distinct U -submodule.

5.2. Almost-irreducibility of $R_M(\lambda)$. Recall from Section 3.3 that each \mathbb{Z}^k -module M yields a BP^k -module $R_M(\lambda) = M \otimes \Pi_{-1}(\lambda)$. If M is irreducible and highest-weight, then the top space of $R_M(\lambda)$ is clearly

$$(5.11) \quad R_M(\lambda)_{\text{top}} = M_{\text{top}} \otimes \Pi_{-1}(\lambda)_{\text{top}}.$$

It shall prove convenient to identify BP^k with its ϕ^k -image in $\mathbb{Z}^k \otimes \Pi$, as per Theorem 3.6. Recalling the explicit form of this realisation, we introduce $\widetilde{W} \in \text{BP}^k$ so that

$$(5.12) \quad \widetilde{W} = G_{-1}^+ G^- = W \otimes \mathbb{1} + \alpha \partial T \otimes \mathbb{1} + \beta T \otimes (i - c) - \mathbb{1} \otimes \omega,$$

where α and β are (k -dependent) constants and ω is an element of Π . Their precise forms will not be needed in what follows.

Theorem 5.9. If M is a weight \mathbb{Z}^k -module that has only top-submodules, then the relaxed highest-weight BP^k -module $R_M(\lambda)$ also has only top-submodules.

Proof. Assume that N is a nonzero BP^k -submodule of $R_M(\lambda)$ and choose a weight vector $w \in N$. Since the vertex subalgebra $U \subset \Pi$ is generated by j and e^c , $\mathbb{1} \otimes U$ is generated by $\mathbb{1} \otimes j = J$ and $\mathbb{1} \otimes e^c = G^+$. Hence, $\mathbb{1} \otimes U \subset \text{BP}^k$. Since $\Pi_{-1}(\lambda)$ has only top-submodules as a U -module (Proposition 5.6), it follows that w may be sent to a nonzero element of $M \otimes \Pi_{-1}(\lambda)_{\text{top}}$ by acting with BP^k . Hence, there exists nonzero $w_0 \in N$ of the form

$$(5.13) \quad w_0 = u_0 \otimes v_{\text{top}},$$

where $u_0 \in M$ and $v_{\text{top}} \in \Pi_{-1}(\lambda)_{\text{top}}$.

Our aim now is to construct a nonzero element in $R_M(\lambda)_{\text{top}}$ from w_0 , by acting with BP^k . We do this by recursively defining weight vectors $w_n = u_n \otimes v_{\text{top}}$, $n = 1, 2, \dots$, in N until we achieve our aim. Here is the definition:

- If there exists $m > 0$ such that $T_m u_n \neq 0$, choose the maximal such m (for definiteness only) and set

$$(5.14) \quad w_{n+1} = L_m w_n = T_m u_n \otimes v_{\text{top}} + u_n \otimes t_m v_{\text{top}} = T_m u_n \otimes v_{\text{top}}.$$

This is then a nonzero element of N and we have $u_{n+1} = T_m u_n$.

- If $T_m u_n = 0$ for all $m > 0$, but there exists $m > 0$ such that $W_m u_n \neq 0$, then choose the maximal such m and set

$$(5.15) \quad \begin{aligned} w_{n+1} &= \widetilde{W}_m w_n \\ &= W_m u_n \otimes v_{\text{top}} - \alpha(m+1) T_m u_n \otimes v_{\text{top}} \\ &\quad + \beta \sum_{r=0}^{\infty} T_{m+r} u_n \otimes (i_{-r} - c_{-r}) v_{\text{top}} - u_n \otimes \omega_m v_{\text{top}} \\ &= W_m u_n \otimes v_{\text{top}}. \end{aligned}$$

This is again a nonzero element of N , this time with $u_{n+1} = W_m u_n$.

- If $T_m u_n = W_m u_n = 0$ for all $m > 0$, then u_n generates a highest-weight submodule of M . As M has only top-submodules, the intersection of this submodule with M_{top} is nonzero, hence we must have $u_n \in M_{\text{top}}$. Thus, $w_n \in R_M(\lambda)_{\text{top}}$ is a nonzero element of N and we are done.

This recursion has to terminate because the conformal weight of u_{n+1} is strictly less than that of u_n and M is positive-energy. There therefore exists n such that $w_n \in R_M(\lambda)_{\text{top}}$ is nonzero and so $R_M(\lambda)$ has only top-submodules. \square

Theorem 5.10. *If M is a top-generated weight \mathbb{Z}^k -module, then the relaxed highest-weight BP^k -module $R_M(\lambda)$ is also top-generated.*

Proof. Consider the submodule of $R_M(\lambda)$ generated by $R_M(\lambda)_{\text{top}} = M_{\text{top}} \otimes \Pi_{-1}(\lambda)_{\text{top}}$, denoting it by N . Clearly, N contains the elements of the form $u_{\text{top}} \otimes v_{\text{top}}$, where $u_{\text{top}} \in M_{\text{top}}$ and $v_{\text{top}} \in \Pi_{-1}(\lambda)_{\text{top}}$. We shall first show that this remains true when u_{top} is replaced by any $u \in M$. As M is top-generated, it is spanned by the elements obtained from the u_{top} by acting iteratively with the T_{-m} and W_{-m} , $m \in \mathbb{Z}_{>0}$. It therefore suffices to assume that $u \otimes v_{\text{top}} \in N$, for some $u \in M$, and then show that upon replacing u by $T_{-m}u$ or $W_{-m}u$, the result is still in N .

Suppose then that $u \otimes v_{\text{top}} \in N$ for some $v_{\text{top}} \in \Pi_{-1}(\lambda)_{\text{top}}$. Acting with L_{-m} , $m \in \mathbb{Z}_{>0}$, then gives $T_{-m}u \otimes v_{\text{top}} + u \otimes t_{-m}v_{\text{top}} \in N$. Since the U -module $\Pi_{-1}(\lambda)$ is top-generated (Proposition 5.4), $u \otimes t_{-m}v_{\text{top}}$ may be obtained from $u \otimes v_{\text{top}} \in N$ by acting with U -modes. But $\mathbb{1} \otimes U \subset \text{BP}^k$, so we have $u \otimes t_{-m}v_{\text{top}} \in N$ and hence $T_{-m}u \otimes v_{\text{top}} \in N$.

Similarly, acting with \widetilde{W}_{-m} , $m \in \mathbb{Z}_{>0}$, results in

$$(5.16) \quad W_{-m}u \otimes v_{\text{top}} + \alpha(m-1)T_{-m}u \otimes v_{\text{top}}$$

$$+ \beta \sum_{r=0}^{\infty} T_{-m+r}u \otimes (i_{-r} - c_{-r})v_{\text{top}} - u \otimes \omega_{-m}v_{\text{top}} \in N.$$

As before, $\Pi_{-1}(\lambda)$ being top-generated implies that $u \otimes \omega_{-m}v_{\text{top}} \in N$ whilst the previous argument gives $T_{-m}u \otimes v_{\text{top}} \in N$. An obvious hybrid of these arguments then shows that $T_{-m+r}u \otimes (i_{-r} - c_{-r})v_{\text{top}} \in N$ and so we conclude that $W_{-m}u \otimes v_{\text{top}} \in N$ as well.

It follows that $M \otimes \Pi_{-1}(\lambda)_{\text{top}} \subset N$. One more appeal to $\Pi_{-1}(\lambda)$ being top-generated then forces $M \otimes \Pi_{-1}(\lambda) \subseteq N$, completing the proof. \square

Corollary 5.11. *If M is an almost-irreducible weight Z^k -module, then the relaxed highest-weight BP^k -module $R_M(\lambda)$ is also almost irreducible.*

5.3. Irreducibility of $R_M(\lambda)$. We now show that when M is an irreducible highest-weight Z^k -module, almost all of the relaxed highest-weight BP^k -modules $R_M(\lambda)$ that we have constructed are irreducible. This gives an *a posteriori* justification for the term “almost irreducible”, see [26].

We recall that a highest-weight vector for BP^k is a J_0 - and L_0 -eigenvector that is annihilated by G_0^+ and all the A_n , $A \in \text{BP}^k$, with $n > 0$. The definition of a *conjugate highest-weight vector* is then obtained by replacing the condition of annihilation by G_0^+ with annihilation by G_0^- . A conjugate highest-weight module is then one which is generated by a single conjugate highest-weight vector.

Theorem 5.12. *Let M be an irreducible highest-weight Z^k -module whose highest-weight vector u has T_0 -eigenvalue Δ and W_0 -eigenvalue w . Then:*

- The relaxed highest-weight BP^k -module $R_M(\lambda)$ is always indecomposable.
- $R_M(\lambda)$ is irreducible if and only if the polynomial

$$(5.17) \quad p_k^{\Delta, w}(x) = w - (k+2)(k+3)\Delta + [(k+3)\Delta - 2(k+2)^2]x + 3(k+2)x^2 - x^3$$

has no roots in the coset $\lambda + \mathbb{Z}$.

- $R_M(\lambda)$ has no highest-weight vectors and its conjugate highest-weight vectors are precisely the $u \otimes e^{-j+\mu c}$ with $\mu \in \lambda + \mathbb{Z}$ satisfying $p_k^{\Delta, w}(\mu) = 0$.

Proof. By Proposition 5.2 and Corollary 5.11, $R_M(\lambda)$ is irreducible if $R_M(\lambda)_{\text{top}}$ is an irreducible $A(\text{BP}^k)$ -module. As the latter has one-dimensional weight spaces, it is irreducible if and only if both G_0^+ and G_0^- act bijectively. From the one-dimensionality of the top space of M , the realisation (3.18), the basis (5.1) of $\Pi_{-1}(\lambda)_{\text{top}}$ and Equation (5.5), we see that G_0^+ always acts bijectively:

$$(5.18) \quad G_0^+(u \otimes e^{-j+(\lambda+n)c}) = u \otimes e_0^c e^{-j+(\lambda+n)c} = u \otimes e^{-j+(\lambda+n+1)c} \neq 0.$$

$R_M(\lambda)_{\text{top}}$ is therefore uniserial, meaning that its submodules S_i form a linear chain under inclusion: $0 \subset S_1 \subset S_2 \subset \dots \subset R_M(\lambda)_{\text{top}}$.

Suppose that $R_M(\lambda)$ was decomposable, hence that $R_M(\lambda) = N \oplus N'$ for nonzero submodules N and N' . Then, $N \cap R_M(\lambda)_{\text{top}}$ and $N' \cap R_M(\lambda)_{\text{top}}$ are both nonzero submodules of $R_M(\lambda)_{\text{top}}$, because $R_M(\lambda)$ has only top-submodules (Theorem 5.9). But, their intersection is clearly zero, in contradiction to $R_M(\lambda)_{\text{top}}$ being uniserial. We therefore conclude that $R_M(\lambda)$ is indecomposable.

Now, any highest-weight vector of $R_M(\lambda)$ must be in $R_M(\lambda)_{\text{top}}$ as otherwise the submodule it generates would contradict Theorem 5.9. But, (5.18) shows that $R_M(\lambda)_{\text{top}}$ has no highest-weight vectors. The situation is similar for conjugate highest-weight vectors except that a calculation somewhat more involved than (5.18) gives

$$(5.19) \quad G_0^-(u \otimes e^{-j+(\lambda+n)c}) = p_k^{\Delta, w}(\lambda+n)u \otimes e^{-j+(\lambda+n-1)c}.$$

Indeed, it follows by combining the following straightforward equalities:

$$\begin{aligned}
(5.20) \quad & (We^{-c})_0v = e_0^{-c}W_0u \otimes e^{-j+(\lambda+n)c} = wu \otimes e^{-j+(\lambda+n-1)c}, \\
& [(\nu_{-1}T)e^{-c}]_0v = e_0^{-c}T_0\nu_0u \otimes e^{-j+(\lambda+n)c} = \Delta(\lambda+n)u \otimes e^{-j+(\lambda+n-1)c}, \\
& [(DT)e^{-c}]_0v = e_0^{-c}(-2T_0)u \otimes e^{-j+(\lambda+n)c} = -2\Delta u \otimes e^{-j+(\lambda+n-1)c}, \\
& [\nu_{-1}^3e^{-c}]_0v = e_0^{-c}\nu_0^3u \otimes e^{-j+(\lambda+n)c} = (\lambda+n)^3u \otimes e^{-j+(\lambda+n-1)c}, \\
& [\nu_{-1}\nu_{-2}e^{-c}]_0v = e_0^{-c}(-\nu_0)\nu_0u \otimes e^{-j+(\lambda+n)c} = -(\lambda+n)^2u \otimes e^{-j+(\lambda+n-1)c}, \\
& [\nu_{-3}e^{-c}]_0v = e_0^{-c}\nu_0u \otimes e^{-j+(\lambda+n)c} = (\lambda+n)u \otimes e^{-j+(\lambda+n-1)c}.
\end{aligned}$$

Here we set $v = u \otimes e^{-j+(\lambda+n)c}$ for brevity.

If $p_k^{\Delta,w}(\lambda+n) \neq 0$ for any $n \in \mathbb{Z}$, then G_0^- acts bijectively and so $R_M(\lambda)$ is irreducible. On the other hand, $p_k^{\Delta,w}(\lambda+n) = 0$ for some $n \in \mathbb{Z}$ implies that $u \otimes e^{-j+(\lambda+n)c}$ is a conjugate highest-weight vector in $R_M(\lambda)$ generating a nonzero proper submodule. \square

5.4. Irreducible submodules of $R_M(\lambda)$. The result of the previous section explicitly realises irreducible relaxed highest-weight BP^k -modules. We now show that we can similarly realise irreducible highest-weight BP^k -modules by analysing submodules of $R_M(\lambda)$, when the latter is reducible.

Proposition 5.13. *Suppose that M is an irreducible highest-weight Z^k -module with highest-weight vector u and that $R_M(\lambda)$ is reducible. Choose $\mu \in \lambda + \mathbb{Z}$ with maximal real part such that $u \otimes e^{-j+\mu c}$ is a conjugate highest-weight vector. Then, $u \otimes e^{-j+\mu c}$ generates an irreducible conjugate highest-weight submodule of $R_M(\lambda)$.*

Proof. By Theorem 5.12, $R_M(\lambda)$ being reducible implies that there exist conjugate highest-weight vectors of the form $u \otimes e^{-j+\mu c}$, so we may indeed choose μ maximal. Then, $u \otimes e^{-j+\mu c}$ is, up to nonzero multiples, the unique conjugate highest-weight vector in the conjugate highest-weight submodule $C = \text{BP}^k \cdot (u \otimes e^{-j+\mu c})$ of $R_M(\lambda)$. Moreover, $A(\text{BP}^k) \cdot (u \otimes e^{-j+\mu c})$ is an irreducible $A(\text{BP}^k)$ -submodule of $R_M(\lambda)_{\text{top}}$.

Choose a nonzero element $v \in C$. Then, $D = \text{BP}^k \cdot v \subseteq C$ is a nonzero submodule of $R_M(\lambda)$ and thus $D \cap R_M(\lambda)_{\text{top}} \neq 0$, by Theorem 5.9. Since

$$(5.21) \quad D \cap R_M(\lambda)_{\text{top}} \subseteq C \cap R_M(\lambda)_{\text{top}} = A(\text{BP}^k) \cdot (u \otimes e^{-j+\mu c})$$

the irreducibility of the latter implies that $u \otimes e^{-j+\mu c} \in D$, whence $D = C$. This shows that every nonzero element of C is cyclic, so this submodule is irreducible. \square

Note that if μ is chosen maximal, as in Proposition 5.13, then the irreducible conjugate highest-weight submodule generated by $u \otimes e^{-j+\mu c}$ has an infinite-dimensional top space.

Proposition 5.14. *Every irreducible conjugate highest-weight BP^k -module whose top space is infinite-dimensional may be explicitly realised as a submodule of $R_M(\lambda)$, for some irreducible highest-weight Z^k -module M and some $\lambda \in \mathbb{C}$.*

Proof. Let C be an arbitrary irreducible conjugate highest-weight BP^k -module with infinite-dimensional top space and let v denote its conjugate highest-weight vector. Then, a basis of C_{top} is given by the $(G_0^+)^n v$ with $n \geq 0$.

If u is the highest-weight vector of some irreducible highest-weight Z^k -module M and Δ and w are its T_0 - and W_0 -eigenvalues, then Proposition 3.4 and Theorem 3.6 give

$$\begin{aligned}
(5.22) \quad & J_0(u \otimes e^{-j+\mu c}) = \left(\mu - \frac{2k+3}{3} \right) u \otimes e^{-j+\mu c} \\
& \text{and } L_0(u \otimes e^{-j+\mu c}) = \left(\Delta + \frac{2k+3}{3} \right) u \otimes e^{-j+\mu c}.
\end{aligned}$$

We may therefore arrange for $u \otimes e^{-j+\mu c}$ to have the same J_0 - and L_0 -eigenvalues as v by choosing μ and Δ in accordance with (5.22). Moreover, if we specialise λ to μ and w to the unique root of the linear (in w) polynomial $p_k^{\Delta,w}(\mu)$, see (5.17), then $u \otimes e^{-j+\mu c}$ is a conjugate highest-weight vector in $R_M(\lambda)$, by Theorem 5.12.

This shows that $u \otimes e^{-j+\mu c}$ generates a conjugate highest-weight submodule V of $R_M(\lambda)$ whose irreducible quotient is isomorphic to C . It moreover generates a $A(\text{BP}^k)$ -module $W = V \cap R_M(\lambda)_{\text{top}}$ with basis $u \otimes e^{-j+(\mu+n)c}$, $n \geq 0$, whose irreducible quotient is isomorphic to the $A(\text{BP}^k)$ -module C_{top} . Comparing bases, we see that $W \cong C_{\text{top}}$ is irreducible and so μ must be the maximal solution in $\lambda + \mathbb{Z}$ of $p_k^{\Delta,w}(\mu) = 0$ (otherwise, V would have another conjugate highest-weight vector in $V \cap R_M(\lambda)_{\text{top}}$ contradicting irreducibility). But now Proposition 5.13 shows that V is irreducible, hence $V \cong C$ as required. \square

Of course, explicit realisations of the irreducible conjugate highest-weight modules lead to explicit realisations of the irreducible highest-weight modules as well, via the conjugation functor of BP^k . Similar to spectral flow functors, this arises from the automorphism of the mode algebra corresponding to

$$(5.23) \quad \begin{aligned} G^+(z) &\mapsto G^-(z), & J(z) &\mapsto -J(z) - \frac{2k+3}{3}z^{-1}, \\ G^-(z) &\mapsto -G^+(z), & L(z) &\mapsto L(z) - \partial J(z) - J(z)z^{-1}. \end{aligned}$$

This then realises all the irreducible highest-weight BP^k -modules with infinite-dimensional top spaces, but as submodules of the conjugates of the $R_M(\lambda)$.

Remark 5.15. Suppose now that $p_k^{\Delta,w}$ has at least two roots in $\lambda + \mathbb{Z}$ and let μ and $\mu' < \mu$ be the maximal and next-to-maximal root, respectively. Then, Proposition 5.13 shows that $u \otimes e^{-j+\mu c}$ generates an irreducible conjugate highest-weight module N of $R_M(\lambda)$ while Theorem 5.12 shows that $u \otimes e^{-j+\mu' c}$ generates a conjugate highest-weight module N' that contains N . Despite the fact that $N'_{\text{top}}/N_{\text{top}}$ is a finite-dimensional irreducible $A(\text{BP}^k)$ -module, it does not necessarily follow that N'/N is an irreducible BP^k -module.

The issue here is that N' , and hence $R_M(\lambda)$, may contain a subsingular vector which, in the language developed here, would generate a submodule that is not top-generated. The upshot is that one can use such quotients to identify (conjugate) highest-weight modules with finite-dimensional top spaces but that this does not amount to a concrete realisation. Instead, one can employ spectral flow.

Corollary 5.16. *Given an irreducible highest-weight BP^k -module N , one of the following possibilities occurs:*

- N may be realised as a submodule of $\sigma^\ell(R_M(\lambda)) \cong M \otimes \Pi_{\ell-1}(\lambda)$, for some $\ell \in \mathbb{Z}_{\geq 1}$, some irreducible highest-weight \mathbb{Z}^k -module and some $\lambda \in \mathbb{C}$.
- The $\sigma^{-\ell}(N)$ have finite-dimensional top spaces for all $\ell \in \mathbb{Z}_{\geq 1}$.

Proof. This follows from the assertion, easily checked using (2.7) and (2.8), that the spectral flow map σ^{-1} takes a highest-weight vector to a conjugate highest-weight vector. For example, $G_n^+v = 0$ for $n \geq 0$ implies that

$$(5.24) \quad G_{n+1}^+ \sigma^{-1}(v) = \sigma^{-1}(\sigma(G_{n+1}^+)v) = \sigma^{-1}(G_n^+v) = 0.$$

$\sigma^{-1}(N)$ is thus an irreducible conjugate highest-weight module. If its top space is finite-dimensional, then it is also a highest-weight module and so we may apply σ^{-1} to its highest-weight vector.

Iterating, we find that either the $\sigma^{-\ell}(N)$, with $\ell \in \mathbb{Z}_{\geq 1}$, all have finite-dimensional top spaces or we arrive at an irreducible conjugate highest-weight module $\sigma^{-\ell}(N)$ with an infinite-dimensional top space. In the latter case, $\sigma^{-\ell}(N)$ embeds into some $R_M(\lambda)$, by Proposition 5.14. Since spectral flow functors are invertible, we conclude that N embeds into $\sigma^\ell(R_M(\lambda))$, as desired. \square

It follows that we can realise any given irreducible highest-weight BP^k -module, as long as its negative spectral flow orbit does not consist exclusively of modules with finite-dimensional top spaces. A generic orbit will not have this property and therefore irreducible highest-weight modules are generically realisable. However, there are some highest-weight modules that cannot be realised in this way. In particular, the irreducible highest-weight BP_k -modules with $2k + 3 \in \mathbb{Z}_{\geq 0}$ are examples because their top spaces are always finite-dimensional (see Remark 6.4 below).

We conclude with an example that illustrates these realisations. As in [51], let us parametrise the highest weights of the irreducible highest-weight Z^k -modules by

$$(5.25) \quad \Delta = \frac{(r-ts)^2 + (r-ts)(r'-ts') + (r'-ts')^2}{3t} - \frac{(t-1)^2}{t}$$

and $w = \frac{(r-ts - (r'-ts'))(2(r-ts) + (r'-ts'))(r-ts + 2(r'-ts'))}{27},$

where $r, r', s, s' \in \mathbb{C}$ and $t = k + 3$. By direct calculation, we see that the polynomial in (5.17) factorises.

Lemma 5.17. *We have*

$$(5.26a) \quad p_k^{\Delta, w}(x) = -(x - x_1)(x - x_2)(x - x_3),$$

where

$$(5.26b) \quad \begin{aligned} x_1 &= t - 1 - \frac{r - ts - (r' - ts')}{3}, & x_2 - x_1 &= r - ts, \\ x_2 &= t - 1 + \frac{2(r - ts) + r' - ts'}{3}, & x_1 - x_3 &= r' - ts', \\ x_3 &= t - 1 - \frac{r - ts + 2(r' - ts')}{3} & x_2 - x_3 &= r + r' - t(s + s'). \end{aligned}$$

Example 5.18. One sees that if $k \notin \frac{1}{2}\mathbb{Z}$, $s = s' = 1$ and $r, r' \in \mathbb{Z}_{>0}$, then the roots x_i , $i = 1, 2, 3$, lie in different cosets of \mathbb{C}/\mathbb{Z} . Each is therefore maximal in the sense of Proposition 5.13, so this proposition and Theorem 5.12 show that, for each $i = 1, 2, 3$,

$$(5.27) \quad C_i = \text{BP}^k \cdot (u \otimes e^{-j+x_i c})$$

is an irreducible conjugate highest-weight BP^k -module with an infinite-dimensional top space. Applying spectral flow, it follows that $\sigma(C_i)$ is an irreducible highest-weight module, for each i . Detailed calculation shows that $\sigma(C_1)$ and $\sigma(C_2)$ always have finite-dimensional top spaces, indeed of dimensions r' and r respectively. The situation for $\sigma(C_3)$ is more subtle: if $3t$ is an integer larger than $r + r'$, then the top space is finite-dimensional (with dimension $3t - r - r'$); otherwise it is infinite-dimensional.

In our forthcoming publications [7], we shall present a detailed study of the structure of $R_M(\lambda)$.

6. REALISATION OF THE VERTEX ALGEBRA BP_k AND ITS RELAXED MODULES

Recall from Theorem 3.6 that we have established an embedding ϕ^k of the universal Bershadsky–Polyakov algebra BP^k , $k \neq -3$, in the tensor product of the universal Zamolodchikov algebra Z^k and the lattice vertex algebra Π . It is natural to ask if this realisation descends to the simple quotients, that is if BP_k embeds in $Z_k \otimes \Pi$. We shall show that the answer is frequently, but not always, yes.

The answer is obviously yes if BP^k is already simple. By [35, Thms. 0.2.1 and 9.1.2], BP^k is not simple if and only if the (noncritical) level k satisfies

$$(6.1) \quad k + 3 = \frac{p'}{p}, \quad \text{for some coprime } p \in \mathbb{Z}_{\geq 1} \text{ and } p' \in \mathbb{Z}_{\geq 2}.$$

For these levels, we consider the projection $\pi_k: Z^k \rightarrow Z_k$ and the composition

$$(6.2) \quad \psi^k: \text{BP}^k \xrightarrow{\phi^k} Z^k \otimes \Pi \xrightarrow{\pi_k \otimes \text{id}} Z_k \otimes \Pi$$

of vertex operator algebra morphisms. Since ψ^k maps the vacuum of BP^k to the vacuum of $Z_k \otimes \Pi$, it is not zero. We shall investigate when $\text{im } \psi^k$ is simple.

The following lemma about singular vectors is a version of Lemma 8.1 from [6]. (We recall that a singular vector is just another name for a highest-weight vector.)

Lemma 6.1. *The vector $(G_{-1}^+)^n \mathbb{1}$, $n > 0$, is singular in BP^k if and only if either $n = k+2$ and $k \in \{-1, 0, 1, 2, \dots\}$ or $n = 2(k+2)$ and $k \in \{-\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, \dots\}$.*

Proof. This is a straightforward computation using the commutation relations (2.4). Since the J_m , L_m , G_{m+1}^- and G_{m-1}^+ , with $m > 0$, clearly annihilate $(G_{-1}^+)^n \mathbb{1}$, we only need to calculate $G_1^-(G_{-1}^+)^n \mathbb{1}$. This is easy (and has already appeared in the proof of [6, Lem. 8.1]):

$$(6.3) \quad G_1^-(G_{-1}^+)^n \mathbb{1} = -n(n-k-2)(n-2k-4)(G_{-1}^+)^{n-1} \mathbb{1}. \quad \square$$

Theorem 6.2. *BP_k embeds into $Z_k \otimes \Pi$ if and only if $2k+3 \notin \mathbb{Z}_{\geq 0}$.*

Proof. Suppose that $\text{im } \psi^k$ has a nonzero proper ideal I . Then, I is a submodule of $Z_k \otimes \Pi = Z_k \otimes \Pi_0(0)$. Here, we may regard I and $Z_k \otimes \Pi$ as BP^k -modules, by (6.2), noting that this makes ψ^k into a BP^k -module homomorphism.

Applying the spectral flow map $\sigma^{-1} = \text{id} \otimes \gamma^{-1}$ of BP^k (γ^{-1} is a spectral flow map of Π , see (3.16)), it follows that $\sigma^{-1}(I)$ is a nonzero submodule of

$$(6.4) \quad (1 \otimes \gamma^{-1})(Z_k \otimes \Pi_0(0)) \cong Z_k \otimes \Pi_{-1}(0) = R_{Z_k}(0).$$

Since Z_k is an irreducible Z^k -module, it has only top-submodules. The BP^k -module $R_{Z_k}(0)$ therefore also has only top-submodules, by Theorem 5.9. Thus,

$$(6.5) \quad R_{Z_k}(0)_{\text{top}} \cap \sigma^{-1}(I) \neq 0.$$

In other words, there exists $n \in \mathbb{Z}$ such that $\mathbb{1} \otimes e^{-j+nc} \in \sigma^{-1}(I)$. Applying $\sigma = \text{id} \otimes \gamma$, we conclude that $\mathbb{1} \otimes e^{nc} \in I$.

As $I \subset \text{im } \psi^k$ and $\text{im } \psi^k$ is a homomorphic image of BP^k , I has nonnegative conformal weights. Thus, $n \in \mathbb{Z}_{\geq 0}$. However, $n = 0$ implies that $I = \text{im } \psi^k$, a contradiction. Therefore, we have

$$(6.6) \quad \psi^k((G_{-1}^+)^n \mathbb{1}) = \mathbb{1} \otimes e^{nc} \in I,$$

for some $n \in \mathbb{Z}_{>0}$, using the explicit realisation of Theorem 3.6.

Now choose $n \in \mathbb{Z}_{>0}$ minimal such that (6.6) holds. Then, $\psi^k((G_{-1}^+)^n \mathbb{1})$ is annihilated by G_1^- , because the result must be proportional to $\psi^k((G_{-1}^+)^{n-1} \mathbb{1})$ which is 0 by minimality. But, it is also annihilated by the J_m , L_m , G_{m+1}^- and G_{m-1}^+ , with $m > 0$. We

conclude that $\psi^k((G_{-1}^+)^n \mathbb{1})$ is a singular vector in I (regarded as a BP^k -module). As $\psi^k((G_{-1}^+)^{n-1} \mathbb{1}) = \mathbb{1} \otimes e^{(n-1)c}$ is nonzero, so is $(G_{-1}^+)^n \mathbb{1} \in \text{BP}^k$.

If $2k + 3 \notin \mathbb{Z}_{\geq 0}$, then this is impossible by Lemma 6.1 and hence $\text{im } \psi^k$ is simple. On the other hand, if k does have this form, then there exists $n \in \mathbb{Z}_{>0}$ such that $\mathbb{1} \otimes e^{nc}$ is a singular vector in $\text{im } \psi^k$ generating a proper nonzero ideal. Thus, $\text{im } \psi^k$ is not simple in this case. \square

A consequence of Theorems 5.12 and 6.2 is that we get families of relaxed highest-weight modules for the simple Bershadsky–Polyakov vertex operator algebra BP_k , at least when $p \neq 1, 2$.

Theorem 6.3. *Assume that $2k + 3 \notin \mathbb{Z}_{\geq 0}$ and that M is an irreducible highest-weight \mathbb{Z}_k -module M whose highest-weight vector u has T_0 -eigenvalue Δ and W_0 -eigenvalue w . Then:*

- $R_M(\lambda)$ is an indecomposable BP_k -module.
- $R_M(\lambda)$ is irreducible if and only if the polynomial $p_k^{\Delta, w}$, defined in (5.17), has no roots in the coset $\lambda + \mathbb{Z}$.
- $R_M(\lambda)$ has no highest-weight vectors and its conjugate highest-weight vectors are precisely the $u \otimes e^{-j+\mu c}$ for which $\mu \in \lambda + \mathbb{Z}$ satisfies $p_k^{\Delta, w}(\mu) = 0$.

Remark 6.4.

- When $k = -\frac{9}{4}$, $\mathbb{Z}_{-9/4} = \mathbb{C}\mathbb{1}$ and so Theorem 6.2 gives one family of relaxed highest-weight $\text{BP}_{-9/4}$ -modules. This family was first constructed in [6, Thm. 7.2].
- When $k = -\frac{5}{3}$, we also have $\mathbb{Z}_{-5/3} = \mathbb{C}\mathbb{1}$, hence one family of relaxed highest-weight $\text{BP}_{-5/3}$ -modules. This family may be constructed by noting that $\text{BP}_{-5/3}$ is a \mathbb{Z}_3 -orbifold of the rank-1 bosonic ghost system [6, Prop. 5.9] and that this ghost system admits a family of relaxed highest-weight modules [47].
- When $p = 2$, so $k \in \{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \dots\}$, BP_k is rational [11]. It therefore has no such families of relaxed highest-weight modules.
- When $k = -1$, BP_{-1} is isomorphic to the rank-1 Heisenberg vertex operator algebra [5]. It therefore also has no such families of relaxed highest-weight modules.
- For $k \in \mathbb{Z}_{\geq 0}$, the relations $(G_{-1}^\pm)^{k+2} \mathbb{1} = 0$ in BP_k ([6] and Lemma 6.1 above) imply that the Zhu algebra $A(\text{BP}_k)$ has only finite-dimensional irreducible modules, more precisely modules of dimension at most $k + 2$. As there are no infinite-dimensional irreducible $A(\text{BP}_k)$ -modules, BP_k likewise has no such families of relaxed highest-weight modules.

Remark 6.5. In the case of the affine vertex algebras $V^k(\mathfrak{sl}_2)$ and $L_k(\mathfrak{sl}_2)$, the irreducibility of the relaxed highest-weight modules was discussed in [2] and [40], using different techniques. As a consequence of our results (with some minor modifications), we can now give a new proof of the irreducibility of these modules.

In particular, [2] showed that all the relaxed highest-weight $L_k(\mathfrak{sl}_2)$ -modules could be realised in the form $\mathcal{E}_{r,s}^\lambda = M_{r,s} \otimes \Pi_{-1}(\lambda)$, where

$$(6.7) \quad k + 2 = \frac{p'}{p}, \quad \text{for some coprime } p, p' \in \mathbb{Z}_{\geq 2},$$

and, for $r = 1, \dots, p - 1$ and $s = 1, \dots, p' - 1$, $M_{r,s}$ is the irreducible highest-weight Virasoro module of central charge and conformal weight

$$(6.8) \quad c_{p,p'} = 1 - \frac{6(p-p')^2}{pp'} \quad \text{and} \quad h_{r,s} = \frac{(p'r - ps)^2 - (p-p')^2}{4pp'}$$

respectively. Completely analogous arguments to those resulting in Theorems 5.12 and 6.3 then prove that $\mathcal{E}_{r,s}^\lambda$ is irreducible if and only if $\lambda \notin \lambda_{r,s}^\pm + \mathbb{Z}$, where $\lambda_{r,s}^\pm$ is as in [2, Sec. 7].

7. CRITICAL-LEVEL RESULTS

A critical level definition of the Bershadsky–Polyakov algebra was investigated in [10, 34].

Definition 7.1. At the critical level $k = -3$, the Bershadsky–Polyakov vertex algebra BP^{-3} is the universal vertex algebra generated by fields S , J , G^+ and G^- subject to the following operator product expansions:

$$(7.1) \quad \begin{aligned} J(z)J(w) &\sim -\frac{1}{(z-w)^2}, & J(z)G^\pm(w) &\sim \pm \frac{G^\pm(w)}{z-w}, & G^\pm(z)G^\pm(w) &\sim 0, \\ S(z)G^\pm(w) &\sim 0, & S(z)J(w) &\sim 0, & S(z)S(w) &\sim 0, \\ G^+(z)G^-(w) &\sim \frac{6}{(z-w)^3} - \frac{6J(w)}{(z-w)^2} + \frac{3 : J(w)J(w) : -3\partial J(w) - S(w)}{z-w}. \end{aligned}$$

Remark 7.2. We can formally obtain the definition of BP^{-3} given above by substituting $S = (k+3)L$ into Definition 2.1 and then setting $k = -3$.

Denote the centre of $V^{-3}(\mathfrak{sl}_3)$ by $\mathcal{Z}(V^{-3}(\mathfrak{sl}_3))$. It is a commutative vertex algebra generated by two fields [29]

$$(7.2) \quad S^2(z) = \sum_{n \in \mathbb{Z}} S_n^2 z^{-n-2} \quad \text{and} \quad S^3(z) = \sum_{n \in \mathbb{Z}} S_n^3 z^{-n-3}.$$

Define the operator $d \in \text{End}(\mathcal{Z}(V^{-3}(\mathfrak{sl}_3)))$ by

$$(7.3) \quad [d, S_n^m] = -n S_n^m, \quad n \in \mathbb{Z}, \quad m = 2, 3.$$

By setting $d\mathbb{1} = 0$, this gives $\mathcal{Z}(V^{-3}(\mathfrak{sl}_3))$ the structure of a $\mathbb{Z}_{\geq 0}$ -graded vertex algebra.

Direct calculation now gives the following critical-level version of Theorem 3.6. The injectivity follows in exactly the same way as in Section 4.

Theorem 7.3. *At the critical level $k = -3$, there is an injective vertex algebra homomorphism $\phi^{-3}: \text{BP}^{-3} \rightarrow \mathcal{Z}(V^{-3}(\mathfrak{sl}_3)) \otimes \Pi$, uniquely determined by*

$$(7.4) \quad \begin{aligned} G^+ &\mapsto \mathbb{1} \otimes e^c, & J &\mapsto \mathbb{1} \otimes j, & S &\mapsto S^2 \otimes \mathbb{1}, \\ G^- &\mapsto \left(S^3 - \frac{1}{2} \partial S^2 \right) \otimes e^{-c} + S^2 \otimes i_{(-1)} e^{-c} \\ &\quad - \mathbb{1} \otimes (i_{(-1)}^3 - 3i_{(-2)}i_{(-1)} + 2i_{(-3)}) e^{-c}. \end{aligned}$$

Remark 7.4. It is proved in [29] that $\mathcal{Z}(V^{-3}(\mathfrak{sl}_3))$ is isomorphic to the critical-level principal W-algebra $W^{-3}(\mathfrak{sl}_3; f_{\text{pr}}) = \mathbb{Z}^{-3}$.

Since $\mathcal{Z}(V^{-3}(\mathfrak{sl}_3))$ is a commutative vertex algebra generated by $S^2(z)$ and $S^3(z)$, its irreducible modules are 1-dimensional and parametrised by $\chi_2, \chi_3 \in \mathbb{C}((z))$ such that

$$(7.5) \quad \chi_m(z) = \sum_{n \in \mathbb{Z}} \chi_m(n) z^{-n-m}, \quad m = 2, 3,$$

and S_n^m acts as multiplication by $\chi_m(n)$ on the irreducible module. We shall therefore denote the irreducible modules by L_{χ_2, χ_3} .

Consider the “relaxed” BP^{-3} -module

$$(7.6) \quad R_{\chi_2, \chi_3}(\lambda) = L_{\chi_2, \chi_3} \otimes \Pi_{-1}(\lambda)$$

(actually, this module is of Wakimoto type [31]). The question of when this module is irreducible may be treated using methods from [1] and we hope to study this in forthcoming publications.

Here, we consider the case in which $R_{\chi_2, \chi_3}(\lambda)$ is $\mathbb{Z}_{\geq 0}$ -gradable. Since Theorem 7.3 gives BP^{-3} the grading defined by $d + t_0$, where t is the conformal vector of Π (see Section 3.2), $R_{\chi_2, \chi_3}(\lambda)$ will be $\mathbb{Z}_{\geq 0}$ -gradable if and only if L_{χ_2, χ_3} is gradable by d . This, in turn, requires that the S_n^m , $m = 2, 3$ and $n \in \mathbb{Z}$, act trivially unless $n = 0$. We therefore conclude that $R_{\chi_2, \chi_3}(\lambda)$ is $\mathbb{Z}_{\geq 0}$ -gradable if and only if

$$(7.7) \quad \chi_2(z) = \frac{\Delta}{z^2} \quad \text{and} \quad \chi_3(z) = \frac{w}{z^3},$$

for some $\Delta, w \in \mathbb{C}$. Moreover, $R_{\chi_2, \chi_3}(\lambda)_{\text{top}} = L_{\chi_2, \chi_3} \otimes \Pi_{-1}(\lambda)_{\text{top}}$ will then be an $A(\text{BP}^{-3})$ -module.

Theorem 7.5. *Assume that $\chi_2(z)$ and $\chi_3(z)$ are given by (7.7) such that*

$$(7.8) \quad g^{\Delta, w}(x) = w + \Delta + (\Delta - 2)x - 3x^2 - x^3$$

has no roots in the coset $\lambda + \mathbb{Z}$. Then, $R_{\chi_2, \chi_3}(\lambda)$ is an irreducible BP^{-3} -module.

Proof. As in the proof of Theorem 5.12, $R_{\chi_2, \chi_3}(\lambda)$ is an irreducible BP^{-3} -module if and only if $R_{\chi_2, \chi_3}(\lambda)_{\text{top}}$ is an irreducible $A(\text{BP}^{-3})$ -module. Moreover, this holds if and only if G_0^+ and G_0^- act bijectively on $R_{\chi_2, \chi_3}(\lambda)_{\text{top}}$. Here, we set $G^+(z) = \sum_n G_n^+ z^{-n-1}$ and $G^-(z) = \sum_n G_n^- z^{-n-2}$. For $u \in L_{\chi_2, \chi_3}$, we have

$$(7.9) \quad \begin{aligned} G_0^+(u \otimes e^{-j+(\lambda+n)c}) &= u \otimes e_0^c e^{-j+(\lambda+n)c} = u \otimes e^{-j+(\lambda+n+1)c} \\ \text{and } G_0^-(u \otimes e^{-j+(\lambda+n)c}) &= g^{\Delta, w}(\lambda + n - 1)u \otimes e^{-j+(\lambda+n-1)c}, \end{aligned}$$

so G_0^+ always acts bijectively and G_0^- acts bijectively if and only if $g^{\Delta, w}(x)$ has no roots in $\lambda + \mathbb{Z}$. The proof follows. \square

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