A Kazhdan–Lusztig correspondence for $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$

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Abstract

The abelian and monoidal structure of the category of smooth weight modules over a non-integrable affine vertex algebra of rank greater than one is an interesting, difficult and essentially wide open problem. Even conjectures are lacking. This work details and tests such a conjecture for $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$ via a logarithmic Kazhdan–Lusztig correspondence.

We first investigate the representation theory of $\overline{\mathcal{U}}_{\mathsf{i}}^{H}(\mathfrak{sl}_3)$, the unrolled restricted quantum group of $\mathfrak{sl}_3$ at fourth root of unity. In particular, we analyse its finite-dimensional weight category, determining Loewy diagrams for all projective indecomposables and decomposing all tensor products of irreducibles. Our motivation is that this category is conjecturally braided tensor equivalent to a category of $W_{A_2}^{0}(2)$-modules. Here, $W_{A_2}^{0}(2)$ is an orbifold of the octuplet vertex algebra $W_{A_2}(2)$ of Semikhatov, the latter being the natural $\mathfrak{sl}_3$-analogue of the well known triplet algebra. Moreover, $W_{A_2}^{0}(2)$ is the parafermionic coset of the affine vertex algebra $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$.

We formulate an explicit conjecture relating the representation theory of $W_{A_2}^{0}(2)$ and $\overline{\mathcal{U}}_{\mathsf{i}}^{H}(\mathfrak{sl}_3)$ and work out the resulting structures of the corresponding $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules. In particular, we obtain conjectural Loewy diagrams for the latter's projective indecomposables and decompositions for the fusion products of its irreducibles. These products coincide with those recently computed via Verlinde's formula. Finally, we give analogous results for $W_{A_2}(2)$.

Contents

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1   Introduction

Two excellent sources of braided tensor categories are the representation categories of modules of quantum groups and of vertex operator algebras. Here, we are mostly interested in categories that are not semisimple. Besides being interesting in their own right, such categories give rise to rich invariants of 3-manifolds [CGP14] and they naturally emerge from quantum field theories [CDGG]. In this work, we study the case of the unrolled restricted quantum group of $\mathfrak{sl}_3$ at fourth root of unity, $\overline{\mathcal{U}}_i^H(\mathfrak{sl}_3)$, and its connection to the simple affine vertex operator algebra of $\mathfrak{sl}_3$ at level $-\frac{3}{2}$, $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$. Let us start by reviewing the state of the art.

1.1   Affine vertex algebras at admissible level

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra. Then, to any $k \in \mathbb{C}$, one associates the simple affine vertex algebra of $\mathfrak{g}$ at level k, which we denote by $L_k(\mathfrak{g})$. If k is a non-negative integer, then $L_k(\mathfrak{g})$ is rational and lisse, hence its module category is a semisimple modular tensor category [Hu08b]. It is a major research goal to thoroughly understand the module categories of $L_k(\mathfrak{g})$ for general k. A relatively accessible, though still extremely challenging, case is when k is an admissible level [KW88].

This goal entails first classifying the irreducible modules, second understanding extensions between them and third establishing a vertex tensor category structure and determining its properties. There are of course different types of categories of modules that one can study. The simplest one is the category of ordinary modules, these being the positive-energy modules whose conformal weight spaces are finite-dimensional. For k admissible, this category is a semisimple vertex tensor category [Ar16, CHY18] for which rigidity is also mostly established [Cr19, CKL22, CGL].

It turns out that when k is not a non-negative integer, $L_k(\mathfrak{g})$ usually has many modules that are not positive-energy or have infinite-dimensional conformal weight spaces. Among other reasons, one must then extend the category of ordinary modules to include some of these in order to obtain a modular-invariant partition function [KW88, CR12, CR13b]. There is therefore significant current effort to understand these extended categories of $L_k(\mathfrak{g})$-modules, particularly those that include the so-called relaxed highest-weight modules [FST98, RW15], see for example [Ad16, RW15a, AFR17, RSW18, Ad19, KR19, CKLR18, AP19, KR21, AW22, FMR20, FK21, AKR21, FKR21, FMK21, KRW21].

In the case $\mathfrak{g} = \mathfrak{sl}_2$, the irreducible positive-energy modules were classified in [AM95]. Reducible positive-energy modules were subsequently considered in many works, including [Ga01, Ri11, CR12, CR13b, KR19]. Moreover, modules that are not positive-energy are often obtained by applying spectral flow functors to those that are. In fact, all irreducible weight modules with finite-dimensional weight spaces can be obtained in this fashion [FT01]. Non-trivial extensions between such spectrally flowed modules were introduced in [Ga01, Ri11] and a general construction of Adamović [Ad19] covers all the indecomposable projective modules [ACK] (in the category of weight modules).

Beyond $\mathfrak{g} = \mathfrak{sl}_2$, the structures of indecomposable projective modules are currently unknown. There are however many conjectural correspondences between categories of modules of certain quantum groups and vertex operator algebras. These are often called logarithmic Kazhdan–Lusztig correspondences and we will review them in a moment. One such correspondence relates the unrolled restricted quantum group $\overline{\mathcal{U}}_i^H(\mathfrak{sl}_3)$, at a fourth root of unity, and a vertex operator algebra related to $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$ [CR22]. (This correspondence is in fact a variant of the conjectural logarithmic Kazhdan–Lusztig correspondence between Feigin–Tipunin algebras and restricted quantum groups [FT10].) Since the representation theory of quantum groups is reasonably accessible, this is a promising direction to deduce otherwise unimaginable structural conjectures for indecomposable projective $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules.

Eventually, one would not only like to understand the linear categories of an affine vertex algebra, but also their monoidal structures. For example, one would like to obtain fusion rules for the irreducible modules. Verlinde's formula [Ve88] is a particularly nice way to obtain (Grothendieck) fusion rules, though it is only established for rational and lisse vertex operator algebras [Hu08a]. There is nevertheless a generalised Verlinde conjecture [CR13a, RW15b] that has been successfully applied to $L_k(\mathfrak{sl}_2)$ at admissible levels [CR12, CR13b]. This conjectural formula has been recently lifted to $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$ in a companion paper [KRW21], so that the tensor product decompositions of $\overline{\mathcal{U}}_i^H(\mathfrak{sl}_3)$-modules and the fusion rules of the corresponding $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules may be compared.

1.2 Logarithmic Kazhdan–Lusztig correspondences

The category of ordinary modules of the affine vertex operator algebra $L_k(\mathfrak{g})$, with $\mathfrak{g}$ simple and $k+h^\vee \notin \mathbb{Q}_{\geq 0}$, is equivalent as a braided tensor category to the category of finite-dimensional modules of Lusztig's quantum group $\mathcal{U}_q(\mathfrak{g})$ at $q = \mathsf{e}^{\pi i/\ell(k+h^\vee)}$, where $h^\vee$ and ℓ are the dual Coxeter and lacing numbers, respectively, of $\mathfrak{g}$. This is the original Kazhdan–Lusztig correspondence [KL93a, KL93b, KL94a, KL94b] established in the 1990s.

Related progress on the mathematics of logarithmic conformal field theory was made in the mid-2000s [FGST06a, FGST06b, FGST06c, FGST07]. These papers conjectured a logarithmic Kazhdan–Lusztig correspondence between the representation categories of the triplet algebras $W_{A_1}(p)$ (parameterised by an integer $p \geq 2$) and the restricted quantum group $\overline{\mathcal{U}}_q(\mathfrak{sl}_2)$ at $q = \mathsf{e}^{\pi i/p}$. It was later noticed that the correspondence is in fact between the triplet algebras and a quasi-Hopf algebra whose underlying algebra is the quantum group. One calls this a quasi-Hopf modification of the quantum group [GR17, CGR20]. In this instance, the quantum group category was quickly understood and it thus provided reliable conjectures about $W_{A_1}(p)$-module categories, in particular their abelian structures and fusion rules. It then took many works [AM08, AM07, KS11, NT11, TW15, CGR20, MY20, CLR21, GN21] to completely settle these conjectures.

A related conjectural logarithmic Kazhdan–Lusztig correspondence involves the unrolled restricted quantum group $\overline{\mathcal{U}}_q^H(\mathfrak{sl}_2)$ at $q = \mathsf{e}^{\pi i/p}$ and the singlet algebra $W_{A_1}^0(p)$ [CGP15], the latter being the $U(1)$-orbifold of $W_{A_1}(p)$. This in turn means that $W_{A_1}(p)$ is a simple current extension of $W_{A_1}^0(p)$ [RW14]. The corresponding relation between the two quantum groups $\overline{\mathcal{U}}_q(\mathfrak{sl}_2)$ and $\overline{\mathcal{U}}_q^H(\mathfrak{sl}_2)$ is sometimes called uprolling and was worked out in [CGR20] (similar quasi-Hopf modifications of the small quantum group also appeared in [AG03]). Moreover, a full understanding of the tensor category of $W_{A_1}^0(p)$ was recently achieved in [CMY20b, CMYc].

This unrolled/singlet correspondence is of significant interest. One reason is that very interesting vertex algebras, called $\mathcal{B}_p$-algebras [CRW14], may be realised as simple current extensions of $W_{A_1}^0(p)$ and a rank-

one Heisenberg vertex algebra. For $p = 2$, one gets the $\beta\gamma$-vertex algebra [Ri10]; for $p = 3$, one gets $L_{-4/3}(\mathfrak{sl}_2)$ [Ad05]; while for larger p, one gets certain subregular W-algebras associated to $\mathfrak{sl}_{p-1}$ [ACGY21]. We remark that the $\mathcal{B}_p$-algebras are important in physics as they arise as chiral algebras of certain Argyres–Douglas theories [BR18, Cr18, ACGY21], these being a type of four-dimensional superconformal quantum field theory. The full ribbon caregory structure of the category of smooth weight modules of the $\mathcal{B}_p$-algebra is obtained in [CMYd].

The quantum group correspondence was then used to give a complete, but as monoidal categories still conjectural, description of the category of smooth weight modules of all the $\mathcal{B}_p$-algebras [ACKR20]. In particular, fusion rules were conjectured via Verlinde's formula and, even better, the normalised entries of the modular S-transformation corresponding to the generalised Verlinde formula were found to coincide with the normalised modified traces of open Hopf links, as was already anticipated in [CMR18, CG17, GR19]. In summary, these logarithmic Kazhdan–Lusztig correspondences between quantum group categories and categories of modules of vertex algebras, both associated to $\mathfrak{sl}_2$, have been invaluable for better understanding the representation theory of the vertex algebras. It is thus reasonable to extend this program to higher ranks.

In this direction, there is a natural generalisation of the triplet algebras called the Feigin–Tipunin algebras [FT10]. They are parameterised by a simply laced simple Lie algebra $\mathfrak{g}$ and an integer $p \geq 2$. The corresponding quantum groups are expected to be quasi-Hopf modifications of the restricted quantum groups $\overline{\mathcal{U}}_q(\mathfrak{g})$ at $q = \mathrm{e}^{\pi \mathrm{i}/p}$. The unrolled version should moreover correspond to the orbifold of the Feigin–Tipunin algebra by the maximal torus of the Lie group of $\mathfrak{g}$. Unfortunately, very few results have thus far been established for these vertex algebras, see [Se14, Se13, CM17, Su21, AMW22, Su22].

As in the case of $\mathfrak{g} = \mathfrak{sl}_2$, there exist extensions of the tensor product of these Feigin–Tipunin orbifolds by Heisenberg vertex algebras of the same rank as $\mathfrak{g}$ [Cr17]. These extensions also arise in physics as chiral algebras of Argyres–Douglas theories [BT17, Cr17]. The very first higher-rank example is $\mathfrak{g} = \mathfrak{sl}_3$ and $p = 2$. In this case, the extension was identified by Adamović as $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$ [Ad16]. It is also worth pointing out that a different extension, obtained by first tensoring with a rank-one Heisenberg vertex algebra, gives the small $N = 4$ superconformal algebra at central charge -9.

The quantum group associated to the orbifold $W_{A_2}^0(2)$ of the Feigin–Tipunin algebra for $\mathfrak{sl}_3$ and $p = 2$ is $\overline{\mathcal{U}}_{\mathrm{i}}^H(\mathfrak{sl}_3)$. General results for these unrolled restricted quantum groups and their uprollings were obtained in [GP13, CGP14, CGP15, Ru22, CR22], but the explicit structures of the projective indecomposables and decompositions of tensor products were not known. We thus decided to study this example in order to compare our findings with the recent results of [KRW21] on the representation theory of $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$, via the conjectural logarithmic Kazhdan–Lusztig correspondence.

The category of finite-dimensional weight modules $\mathrm{Rep}_{wt} \mathcal{U}_q^H(\mathfrak{g})$ over the unrolled quantum group $\mathcal{U}_q^H(\mathfrak{g})$ can be described as modules over the De-Concini - Kac quantum group $\mathcal{U}_q(\mathfrak{g})$ which admit an algebraic action of the torus such that the action of the Cartan part of $\mathcal{U}_q(\mathfrak{g})$ is determined by the torus action. The representation theory of $\mathcal{U}_q^H(\mathfrak{g})$ has been studied in this form before the quantum group itself was introduced (see [AJS94]) and much of the structure of $\mathrm{Rep}_{wt} \mathcal{U}_q^H(\mathfrak{g})$ should be determined by the representation theory of $\mathcal{U}_q(\mathfrak{g})$. For our purposes, we required very explicit knowledge of the structure of indecomposable modules, which we were able to derive directly due to the accessibility of $\mathcal{U}_{\mathrm{i}}^H(\mathfrak{sl}_3)$. To obtain such descriptions of indecomposable modules beyond rank 2, we expect that a thorough understanding of the relationship between the representation theory of $\mathcal{U}_q^H(\mathfrak{g})$ and $\mathcal{U}_q(\mathfrak{g})$ will be invaluable due to the substantial progress which has

already been made for $\mathcal{U}_q(\mathfrak{g})$ [B94, DCK89, DCK91, DCKP93].

1.3 Overview

We begin in Section 2 by recalling the definition of the unrolled restricted quantum group $\overline{\mathcal{U}}_{\mathsf{i}}^{H}(\mathfrak{sl}_3)$ and outlining some of the broader features of its representation theory, following [Ru22]. We also outline the theory of Loewy series and diagrams to fix notation and explain the subtle relation between this theory and duality.

Section 3 is devoted to a detailed study of the category of finite-dimensional weight $\overline{\mathcal{U}}_{\mathsf{i}}^{H}(\mathfrak{sl}_3)$-modules. The simple objects of this category are realised as irreducible quotients L_λ of Verma modules M_λ. It follows from [Ru22, Propositions 4.4 and 4.6] that the irreducibility of M_λ is determined by the "typicality" of the scalars $\lambda_k = \langle \lambda + \rho, \alpha_k \rangle$, $k = 1, 2, 3$, where α_1 and α_2 are the simple roots of $\mathfrak{sl}_3$ and $\alpha_3 = \rho = \alpha_1 + \alpha_2$. We call $\lambda_k \in \mathbb{C}$ typical if $\lambda_k \notin 2\mathbb{Z} + 1$ and a weight $\lambda \in \mathfrak{h}^*$ typical if all λ_j are typical. Weights that are not typical are called atypical and are further subdivided by the degree of atypicality, this being the number of λ_k that are atypical (in $2\mathbb{Z} + 1$).

We find that the structures and invariants of the irreducibles L_λ, Vermas M_λ and projective covers P_λ are captured by four different classes of atypicality for the weight λ:

- λ typical (atypical of degree 0);

- λ atypical of degree 1;

- λ atypical of degree 2 with λ_3 atypical;

- λ atypical of degree 2 with λ_3 typical.

For example, the dimensions of the L_λ corresponding to these four classes are 8, 4, 3 and 1, respectively. This, along with the characters of the L_λ, is the content of Proposition 3.1.

From here, we establish in Theorem 3.2 the Loewy diagrams that summarise the structures of the reducible Verma modules, see Figure 2. The quite non-trivial generalisation for the indecomposable projectives is reported in Theorem 3.5 and Figure 3. Both results again split into cases along the lines of the atypicality classes introduced above. This identification of projectives is moreover extremely helpful for decomposing many of the tensor products of irreducible $\overline{\mathcal{U}}_{\mathsf{i}}^{H}(\mathfrak{sl}_3)$-modules, this being the subject of Section 3.3.

The implications of these results for the representation theories of the vertex operator algebras $W_{A_2}^0(2)$, $W_{A_2}(2)$ and $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$ is discussed in Section 4. The latter here serves as a partial check as some features of the representation theory of $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$ have been recently established in [KRW21]. These results are reviewed in Section 4.1. However, the conjectural logarithmic Kazhdan–Lusztig correspondence involves the Feigin–Tipunin orbifold $W_{A_2}^0(2)$, not $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$, so Section 4.2 details the relationship between these two vertex operator algebras. In particular, we determine the branching rules of the irreducible $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules, that is their decomposition into modules of $W_{A_2}^0(2) \otimes H$ (where H denotes a rank-2 Heisenberg algebra).

The Kazhdan–Lusztig correspondence itself is the topic of Section 4.3, see Conjecture 4.4. This immediately results in conjectural Loewy diagrams for the indecomposable projective $W_{A_2}^0(2)$-modules, see Figure 6. To lift these to $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$, we use the theory of vertex algebra extensions to construct an exact monoidal functor of a suitable subcategory of $W_{A_2}^0(2) \otimes H$-modules to the weight category of $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$. This is detailed in Section 4.4 with the conjectured Loewy diagrams for the indecomposable projective $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules

reported in Figure 7. This gives the first predictions for the structures of these projective covers and will be invaluable guides for their subsequent construction and analysis.

We also test the tensor part of the conjectured equivalence by using the tensor product decompositions for $\overline{\mathcal{U}}_{\mathsf{i}}^H(\mathfrak{sl}_3)$, given in Section 3.3, to predict fusion rules for $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$ in Section 4.5 and checking that they descend to the Grothendieck fusion rules recently calculated in [KRW21]. This is a highly non-trivial check of both our Conjecture 4.4 and the standard Verlinde conjecture for logarithmic models [CR13a, RW15b].

To conclude, we again use vertex algebra extension theory to study the representation theory of $W_{A_2}(2)$. This Feigin–Tipunin algebra was dubbed the octuplet algebra by Semikhatov [Se14, Se13] as it is generated by eight primary fields of conformal weight four. As an $\mathfrak{sl}_3$ generalisation of the triplet algebras, it is expected to be C_2-cofinite. We verify in Section 4.6, again assuming Conjecture 4.4, that there are finitely many irreducibles (up to isomorphism) and that all are ordinary. Predictions for their fusion rules are deduced and Loewy diagrams for the projective indecomposables are proposed in Figure 8.

Acknowledgements

2 Preliminaries

In this section, the unrolled restricted quantum group $\overline{\mathcal{U}}_{\mathsf{i}}^H(\mathfrak{sl}_3)$ is defined and various results are recalled for later use in identifying the Loewy diagrams of its reducible Verma modules and their projective covers.

2.1 The quantum group $\overline{\mathcal{U}}_{\mathsf{i}}^H(\mathfrak{sl}_3)$

Fix a choice of Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}_3$. The simple roots are denoted by $\alpha_i \in \mathfrak{h}^*$, $i = 1, 2$, and the highest root is $\alpha_3 = \alpha_1 + \alpha_2$. The Killing form $\langle -, - \rangle$ is normalised so that $\langle \alpha_i, \alpha_j \rangle = A_{i,j}$, $i = 1, 2$, where $A = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix} \right)$ is the Cartan matrix. The dual form on $\mathfrak{h}^*$ will also be denoted by $\langle -, - \rangle$. Fix also a lattice $L \subset \mathfrak{h}^*$ such that $Q \subseteq L \subseteq P$, where P and Q are the weight and root lattices of $\mathfrak{sl}_3$, respectively.

Here and below, i denotes $\sqrt{-1} \equiv \mathsf{e}^{\pi \mathsf{i}/2}$.

Definition 2.1. The *unrolled restricted quantum group* $\overline{\mathcal{U}}_{\mathsf{i}}^H(\mathfrak{sl}_3)$ is the complex Hopf algebra with generators $X_{\pm j}$, H_j and K_γ, for $j = 1, 2$ and $\gamma \in L$, and relations

$$
K_0 = 1, \qquad\qquad K_{\gamma_1} K_{\gamma_2} = K_{\gamma_1 + \gamma_2}, \quad K_\gamma X_{\pm j} K_{-\gamma} = \mathsf{i}^{\pm \langle \gamma, \alpha_j \rangle} X_{\pm j},
$$

$$
[X_j, X_{-j'}] = \delta_{j,j'} \frac{K_{\alpha_j} - K_{-\alpha_j}}{2\mathsf{i}}, \qquad X_{\pm j}^2 = 0, \qquad (X_{\pm 1} X_{\pm 2})^2 = (X_{\pm 2} X_{\pm 1})^2, \tag{2.1a}
$$

$$
[H_j, X_{\pm j'}] = \pm A_{j,j'} X_{\pm j'}, \qquad [H_j, H_{j'}] = 0, \qquad [H_j, K_\gamma] = 0.
$$

(The notation $K_j = K_{\alpha_j}$ and $K_{-\gamma} = K_\gamma^{-1}$ will also be used.) The coproduct, counit and antipode are given

by

$$
\begin{aligned}
\Delta(K_\gamma) &= K_\gamma \otimes K_\gamma, & \epsilon(K_\gamma) &= 1, & S(K_\gamma) &= K_{-\gamma}, \\
\Delta(X_j) &= X_j \otimes K_j + 1 \otimes X_j, & \epsilon(X_j) &= 0, & S(X_j) &= -X_j K_j^{-1}, \\
\Delta(X_{-j}) &= X_{-j} \otimes 1 + K_j^{-1} \otimes X_{-j}, & \epsilon(X_{-j}) &= 0, & S(X_{-j}) &= -K_j X_{-j} \\
\Delta(H_j) &= 1 \otimes H_j + H_j \otimes 1, & \epsilon(H_j) &= 0, & S(H_j) &= -H_j,
\end{aligned}
\tag{2.1b}
$$

respectively.

Normally, one would augment the relations (2.1a) with Serre relations. However, these become trivial when the quantum group parameter q is specialised to i.

The subalgebra of $\overline{\mathcal{U}}_\mathsf{i}^H(\mathfrak{sl}_3)$ generated by the "simple root vectors" $X_{\pm 1}$ and $X_{\pm 2}$ will be denoted by $\overline{\mathcal{U}}_\mathsf{i}^H(\mathfrak{n}^\pm)$. As in [KS97, Section 6.2.3], the root vectors associated to $\pm\alpha_3$ may be chosen to be

$$
X_3 = -X_1 X_2 - \mathsf{i} X_2 X_1 \qquad \text{and} \qquad X_{-3} = -X_{-2} X_{-1} + \mathsf{i} X_{-1} X_{-2}.
\tag{2.2}
$$

It is easy to check from (2.1a) that $X_{\pm 3}^2 = 0$. The following result is easy to establish.

Proposition 2.2. *The set* $\left\{ X_{\pm 1}^{n_1} X_{\pm 3}^{n_3} X_{\pm 2}^{n_2} \;\middle|\; n_1, n_2, n_3 = 0, 1 \right\}$ *is a vector space basis for* $\overline{\mathcal{U}}_\mathsf{i}^H(\mathfrak{n}^\pm)$.

2.2 Finite-dimensional weight $\overline{\mathcal{U}}_\mathsf{i}^H(\mathfrak{sl}_3)$-modules

To discuss the representation theory of $\overline{\mathcal{U}}_\mathsf{i}^H(\mathfrak{sl}_3)$, it will be convenient to identify the generators H_1 and H_2 with the simple coroots of $\mathfrak{sl}_3$, so that $\mathrm{span}\{H_1, H_2\} = \mathfrak{h}$.

Definition 2.3. Given a $\overline{\mathcal{U}}_\mathsf{i}^H(\mathfrak{sl}_3)$-module V, denote by $V(\lambda)$ the simultaneous eigenspace of H_1 and H_2 with weight $\lambda \in \mathfrak{h}^*$:

$$
V(\lambda) = \left\{ v \in V \;\middle|\; H_j v = \lambda(H_j) v, \; j = 1, 2 \right\}.
\tag{2.3}
$$

A $\overline{\mathcal{U}}_\mathsf{i}^H(\mathfrak{sl}_3)$-module V is called a *weight module* if it is the direct sum of its weight spaces $V(\lambda)$, $\lambda \in \mathfrak{h}^*$, and every K_γ, $\gamma \in L$, acts on each $V(\lambda)$ as multiplication by $\mathsf{i}^{\langle \gamma, \lambda \rangle}$. A non-zero element of $V(\lambda)$ is called a *weight vector* of weight λ. The category of all finite-dimensional weight $\overline{\mathcal{U}}_\mathsf{i}^H(\mathfrak{sl}_3)$-modules will be denoted by $\mathscr{C}$.

Note that the category $\mathscr{C}$ is insensitive to the choice of lattice L made for $\overline{\mathcal{U}}_\mathsf{i}^H(\mathfrak{sl}_3)$ (see Definition 2.1), because $Q \subset L$ and the action of the K_γ, $\gamma \in L$, in $\mathscr{C}$ is completely determined by the action of H_1 and H_2.

Definition 2.4. The *character* of a module V in $\mathscr{C}$ is the following formal series in z:

$$
\mathrm{ch}[V] = \sum_{\lambda \in \mathfrak{h}^*} \dim V(\lambda) \, z^\lambda.
\tag{2.4}
$$

Definition 2.5. A weight vector in a $\overline{\mathcal{U}}_\mathsf{i}^H(\mathfrak{sl}_3)$-module is said to be *maximal* if it is annihilated by X_1 and X_2. Any $\overline{\mathcal{U}}_\mathsf{i}^H(\mathfrak{sl}_3)$-module generated by a single maximal vector is called a *highest-weight module* and the weight of the generating maximal vector is called its *highest weight*.

As usual, one may define the Verma $\overline{\mathcal{U}}_\mathsf{i}^H(\mathfrak{sl}_3)$-module M_λ, $\lambda \in \mathfrak{h}^*$, as the (unique up to isomorphism) universal highest-weight module generated by a maximal vector of weight λ. It has a unique irreducible quotient which
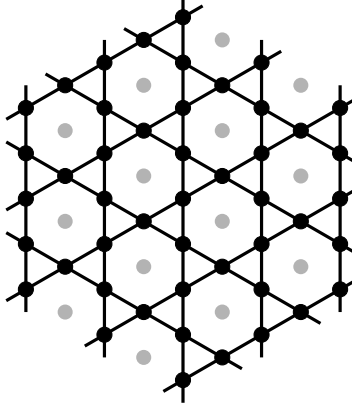
Figure 1: The atypical weights in $\mathfrak{h}^*$, pictured with 0 at the centre. Grey dots indicate typical weights in P, black lines indicate atypical weights of degree 1 and black dots are atypical weights of degree 2. The Weyl vector ρ is the grey dot closest to 0 along the ray $30°$ to the right of the upwards vertical.

we denote by L_λ. Moreover, M_λ is free as a $\overline{\mathcal{U}}_{\mathfrak{i}}^H(\mathfrak{n}^-)$-module so Proposition 2.2 gives the basis

$$\left\{ X_{-1}^{n_1} X_{-3}^{n_3} X_{-2}^{n_2} m_\lambda \mid n_1, n_2, n_3 = 0, 1 \right\}, \tag{2.5}$$

where m_λ is any vector of weight λ. The character of a Verma module is thus given by

$$\mathrm{ch}[M_\lambda] = \sum_{n_1=0}^{1} \sum_{n_2=0}^{1} \sum_{n_3=0}^{1} z^{\lambda - n_1\alpha_1 - n_2\alpha_2 - n_3\alpha_3} = z^\lambda \prod_{i=1}^{3} (1 + z^{-\alpha_i}). \tag{2.6}$$

It moreover follows that both M_λ and L_λ belong to the finite-dimensional weight category $\mathscr{C}$, for all $\lambda \in \mathfrak{h}^*$. As noted in [Ru22, Section 4.2], $\mathscr{C}$ has enough projectives. In particular, L_λ and M_λ have a common projective cover in $\mathscr{C}$, which we denote by P_λ.

The question of when M_λ is irreducible was decided in [Ru22] in terms of the notion of typical weights. Let ρ be the half sum of positive roots, i.e. the Weyl vector of $\mathfrak{sl}_3$.

Definition 2.6. For each $\lambda \in \mathfrak{h}^*$, let

$$\lambda_j = \langle \lambda + \rho, \alpha_j \rangle, \quad j = 1, 2, 3, \tag{2.7}$$

and note that $\lambda_3 = \lambda_1 + \lambda_2$. The index λ_j is said to be *atypical* whenever $\lambda_j \in 2\mathbb{Z} + 1$ and *typical* otherwise. If all of the λ_j, $j = 1, 2, 3$, are typical, then the weight $\lambda \in \mathfrak{h}^*$ is said to be *typical*. For $n = 0, 1, 2$, a weight λ with n atypical indices λ_j is said to be *atypical of degree n*.

For convenience, we illustrate the atypical weights in Figure 1.

Proposition 2.7 ([Ru22, Propositions 4.4 and 4.6]). *M_λ is irreducible if and only if λ is typical.*

We shall see in Sections 3.1 and 3.2 that the subquotient structure of M_λ and P_λ is determined by which of the λ_j, $j = 1, 2, 3$, are atypical.

It is easy to see that two irreducible modules in $\mathscr{C}$ are isomorphic if and only if they have the same character.

One might not expect this to remain true when "irreducible" is replaced by "projective". For example, it is false for the restricted quantum groups $\overline{\mathcal{U}}_q(\mathfrak{g})$ at roots of unity, which appear as subalgebras of $\overline{\mathcal{U}}_q^H(\mathfrak{g})$. However, unrolling allows us to consider additive module weights rather than just multiplicative ones (meaning H_j-eigenvalues rather than K_γ-eigenvalues). This has the following consequence.

Proposition 2.8 ([Ru22, Proposition 4.13]). *If two projective modules in $\mathscr{C}$ have the same character, then they are isomorphic.*

Note that the category $\mathscr{C}$ admits a character-preserving exact contravariant functor $V \mapsto \check{V}$ analogous to the usual duality functor on the category of finite-dimensional modules over a Lie algebra (see [Ru22, Section 4.2] for details).

Definition 2.9. Let $\check{}$ denote the dual defined on $\mathscr{C}$ by the involutive antiautomorphism $S \circ \Omega$, where S is the antipode of $\overline{\mathcal{U}}_i^H(\mathfrak{sl}_3)$ and Ω is the automorphism of $\overline{\mathcal{U}}_i^H(\mathfrak{sl}_3)$ defined by

$$\Omega(X_{\pm j}) = X_{\mp j}, \quad \Omega(K_\gamma) = K_{-\gamma}, \quad \Omega(H_j) = -H_j, \qquad j = 1, 2, \ \gamma \in L. \tag{2.8}$$

For any module V, $\check{V}$ is obtained by twisting the action of $\overline{\mathcal{U}}_q^H(\mathfrak{g})$ on the usual dual $V^* = \mathrm{Hom}(V, \mathbb{C})$ defined by the antipode, by Ω. It is clear that irreducible modules in $\mathscr{C}$ are self-dual with respect to this definition.

Theorem 2.10 ([Ru22, Theorem 4.14]). *With respect to $\check{}$, P_λ is self-dual: $\check{P}_\lambda \cong P_\lambda$.*

This result heavily constrains the Loewy diagram of P_λ and will be used extensively in Section 3.2.

2.3 Filtrations and BGG reciprocity

Recall that a *filtration*, or *series*, of a module V is a strictly increasing family of submodules ordered by inclusion:

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V. \tag{2.9}$$

The integer n is called the *length* of the series. Such a series is called a *composition series* of V if the quotient of each non-zero submodule in the filtration by its predecessor is irreducible: for each $j = 1, \ldots, n$, $V_j/V_{j-1} \cong L_{\lambda_j}$ for some $\lambda_j \in \mathfrak{h}^*$. Similarly, a series is called a *standard filtration* (or *Verma flag*) of V if the successive quotients are Verma modules: for each $j = 1, \ldots, n$, $V_j/V_{j-1} \cong M_{\lambda_j}$ for some $\lambda_j \in \mathfrak{h}^*$.

Every module in $\mathscr{C}$ admits a composition series. While this series is not unique, they all have the same length. Moreover, the set of isomorphism classes of the irreducible quotients (the *composition factors*) is uniquely fixed by the isomorphism class of the module. The set of isomorphism classes of the quotients (the *Verma factors*) of a standard filtration is likewise unique, though not every module in $\mathscr{C}$ admits such a series.

Given a module $V \in \mathscr{C}$ which admits a standard filtration, denote by $(V : M_\lambda)$ the multiplicity with which M_λ appears in the set of Verma factors of any standard filtration of V. Similarly, denote by $[V : L_\lambda]$ the multiplicity with which L_λ appears in the set of irreducible quotients of any composition series of V.

Proposition 2.11 ([Ru22, Proposition 4.12]). *BGG reciprocity holds for the category $\mathscr{C}$. In other words, $(P_\lambda : M_\mu) = [M_\mu : L_\lambda]$ for all $\lambda, \mu \in \mathfrak{h}^*$.*

We shall also be interested in a different type of filtration/series in which the successive quotients are required to be completely reducible. A composition series satisfies this property, but there are usually other examples.

Definition 2.12. A *Loewy series* of $V \in \mathscr{C}$ is one with minimal length amongst all series in which the successive quotients are completely reducible.

There are two standard examples of Loewy series. First, the *radical series*

$$0 = \operatorname{rad}^n V \subset \operatorname{rad}^{n-1} V \subset \cdots \subset \operatorname{rad}^1 V \subset \operatorname{rad}^0 V = V \tag{2.10}$$

of $V \in \mathscr{C}$ is defined inductively by setting $\operatorname{rad}^0 V = V$ and $\operatorname{rad}^j V = \operatorname{rad}(\operatorname{rad}^{j-1} V)$, for $j \geq 1$. Here, the *radical* of $W \in \mathscr{C}$, denoted by $\operatorname{rad} W$, is the smallest submodule such that the quotient is completely reducible. Equivalently, $\operatorname{rad} W$ is the intersection of the maximal proper submodules of W. The successive semisimple quotients are denoted by $\operatorname{rad}_j V = \operatorname{rad}^{j-1} V / \operatorname{rad}^j V$.

The second example is the *socle series*

$$0 = \operatorname{soc}^0 V \subset \operatorname{soc}^1 V \subset \cdots \subset \operatorname{soc}^{n-1} V \subset \operatorname{soc}^n V = V \tag{2.11}$$

of $V \in \mathscr{C}$, this being the series defined inductively by taking $\operatorname{soc}^0 V = 0$ and letting $\operatorname{soc}^j V$ be the unique submodule of V satisfying $\operatorname{soc}(V / \operatorname{soc}^{j-1} V) = \operatorname{soc}_j V$, where $\operatorname{soc}_j V = \operatorname{soc}^j V / \operatorname{soc}^{j-1} V$. Here, the *socle* of $W \in \mathscr{C}$, denoted by $\operatorname{soc} W$, is the largest completely reducible submodule. Equivalently, it is the sum of the irreducible submodules of W.

Note that the radical and socle series of $V \in \mathscr{C}$ are uniquely specified. The dual nature of the definitions suggests that they might actually coincide, however this need not be the case. Indeed, the radical series imposes maximality on quotients, starting from V and working down to 0, while the socle series imposes maximality on submodules, starting from 0 and working up to V. The upshot is that any Loewy series (2.9) must therefore lie between these two extremes:

$$\operatorname{rad}^{n-k} V \subseteq V_k \subseteq \operatorname{soc}^k V, \qquad k = 0, 1, \ldots, n. \tag{2.12}$$

Consequently, V admits a unique Loewy series if and only if the radical and socle series coincide.

In general, it is difficult to know if Loewy series are unique. However, there are easily derived criteria if the length n of the series is sufficiently small. To see this, note that:

- If $\operatorname{soc}_1 V = \operatorname{soc} V$ is irreducible, then $\operatorname{rad}^{n-1} V = \operatorname{soc}^1 V$.

- If the *head* $\operatorname{rad}_1 V = V / \operatorname{rad} V$ of V is irreducible, then $\operatorname{rad}^1 V = \operatorname{soc}^{n-1} V$.

It follows, when $n = 2$, that all Loewy series coincide if either the socle or head of V is irreducible. Similarly, when $n = 3$, all Loewy series coincide if both the socle and head are irreducible.

As our category $\mathscr{C}$ admits a duality functor, one is naturally led to ask for a relationship between the Loewy series of a module V and its dual $\check{V}$. For the radical and socle series, this is the content of the *Landrock lemma* [La83, Lemma 8.4i)] which states that

$$\operatorname{soc}^j \check{V} \cong \left(\frac{V}{\operatorname{rad}^j V} \right)^{\check{}} \qquad \text{and} \qquad \operatorname{rad}^j \check{V} \cong \left(\frac{V}{\operatorname{soc}^j V} \right)^{\check{}}. \tag{2.13}$$

The successive semisimple quotients are then related as follows:

$$\left(\operatorname{soc}_j \check{V}\right)^\vee = \left(\frac{\operatorname{soc}^j \check{V}}{\operatorname{soc}^{j-1} \check{V}}\right)^\vee \cong \left(\left(\frac{V}{\operatorname{rad}^j V}\right)^\vee \Big/ \left(\frac{V}{\operatorname{rad}^{j-1} V}\right)^\vee\right)^\vee$$

$$\cong \ker\left(\frac{V}{\operatorname{rad}^j V} \twoheadrightarrow \frac{V}{\operatorname{rad}^{j-1} V}\right) \cong \frac{\operatorname{rad}^{j-1} V}{\operatorname{rad}^j V} = \operatorname{rad}_j V. \tag{2.14}$$

Since the irreducible $\overline{\mathcal{U}}_{\mathsf{i}}^H(\mathfrak{sl}_3)$-modules in $\mathscr{C}$ are self-dual, we therefore have $\operatorname{rad}_j V \cong \operatorname{soc}_j \check{V}$ for all $j = 1, \ldots, n$.

Definition 2.13. Given a Loewy series (2.9) of $V \in \mathscr{C}$, the associated *Loewy diagram* is a picture of the composition factors of V arranged in horizontal layers that reflect the successive quotients of the series. In particular, the j-th layer from the bottom consists of the irreducible direct summands of the completely reducible module V_j/V_{j-1}.

We shall generally annotate a Loewy diagram with arrows pointing from a composition factor in the j-th layer to one in the $(j-1)$-th layer whenever these factors correspond to a non-split subquotient of V_j/V_{j-2}. This additional information helps to describe the submodule structure of V, though it need not fix it completely.

Since the irreducibles L_λ are self-dual, (2.14) implies that the Loewy diagram for the radical (socle) series of $\check{V}$ (*sans* arrows) is obtained from that of the socle (radical) series of $V \in \mathscr{C}$ by turning it upside-down. In particular, the radical and socle series of the projectives P_λ are related in this way, by Theorem 2.10.

3 Representation theory of $\overline{\mathcal{U}}_{\mathsf{i}}^H(\mathfrak{sl}_3)$

The Verma modules of $\overline{\mathcal{U}}_{\mathsf{i}}^H(\mathfrak{sl}_3)$ are only 8-dimensional, implying that their submodule structures may be readily computed. This is the topic of the present section, along with the more intricate determination of the structures of the indecomposable projective modules and the decomposition of all tensor products of irreducibles in $\mathscr{C}$.

3.1 Verma modules and irreducible quotients

Recall from Proposition 2.7 that the irreducibility of the Verma module M_λ is determined by the typicality of the scalars $\lambda_j = \langle \lambda + \rho, \alpha_j \rangle$, $j = 1, 2, 3$. Here, the structures of the reducible Verma modules are determined and the characters of their irreducible quotients computed. Note that at most two of the λ_j can be atypical (the third index must then lie in $2\mathbb{Z}$).

Proposition 3.1.

(i) *If λ is typical, so $\lambda_1, \lambda_2, \lambda_3 \notin 2\mathbb{Z} + 1$, then $\dim L_\lambda = 8$ and*

$$\operatorname{ch}[L_\lambda] = z^\lambda + z^{\lambda - \alpha_1} + z^{\lambda - \alpha_2} + 2z^{\lambda - \alpha_3} + z^{\lambda - \alpha_1 - \alpha_3} + z^{\lambda - \alpha_2 - \alpha_3} + z^{\lambda - 2\alpha_3}. \tag{3.1}$$

(ii) *If λ is atypical of degree 1, then $\dim L_\lambda = 4$ and*

$$\operatorname{ch}[L_\lambda] = z^\lambda + z^{\lambda - \alpha_j} + z^{\lambda - \alpha_k} + z^{\lambda - \alpha_j - \alpha_k}, \tag{3.2}$$

where $j, k \in \{1, 2, 3\}$ are chosen so that λ_j and λ_k are typical.

(iii) *If λ is atypical of degree 2 and λ_3 is atypical, then* $\dim L_\lambda = 3$ *and*

$$\mathrm{ch}[L_\lambda] = z^\lambda + z^{\lambda - \alpha_j} + z^{\lambda - \alpha_3}, \tag{3.3}$$

where $j \in \{1, 2\}$ is chosen so that λ_j is typical.

(iv) *Otherwise, λ is atypical of degree 2 and λ_3 is typical. In this case,* $\dim L_\lambda = 1$ *and*

$$\mathrm{ch}[L_\lambda] = z^\lambda. \tag{3.4}$$

Proof. Case (i) follows immediately from Proposition 2.7 and the Verma character (2.6). We therefore start with (iv), which assumes that $\lambda_1, \lambda_2 \in 2\mathbb{Z} + 1$. If v is a vector of weight λ and $j, j' = 1, 2$, then

$$[X_j, X_{-j'}]\, v = \delta_{j,j'} \frac{K_j - K_j^{-1}}{2\mathrm{i}} v = \delta_{j,j'} \frac{\mathrm{i}^{\langle \lambda, \alpha_j \rangle} - \mathrm{i}^{-\langle \lambda, \alpha_j \rangle}}{2\mathrm{i}} v = -\frac{1}{2}\delta_{j,j'}\left(\mathrm{i}^{\lambda_j} + \mathrm{i}^{-\lambda_j}\right) v. \tag{3.5}$$

Taking $v = m_\lambda$, a generating maximal vector in M_λ, it follows that $X_j X_{-j'} m_\lambda$ is always 0, for $j \neq j'$, and $X_j X_{-j} m_\lambda = 0$ if and only if $\lambda_j \in 2\mathbb{Z} + 1$. Thus, $X_{-1} m_\lambda$ and $X_{-2} m_\lambda$ belong to the maximal proper submodule J_λ of M_λ, when λ is typical, hence so does $X_{-3} m_\lambda$. L_λ is therefore one-dimensional, spanned by the equivalence class of m_λ, as claimed.

In case (iii), take (without loss of generality) $\lambda_1 \in 2\mathbb{Z}$ and $\lambda_2, \lambda_3 \in 2\mathbb{Z}+1$. Then, $X_{-2} m_\lambda \in M_\lambda$ is maximal, as above, and so it and its descendants lie in J_λ. We claim that $X_{-1} X_{-3} m_\lambda \in J_\lambda$ as well. Since $X_2 X_{-1} X_{-3} m_\lambda$ vanishes for weight reasons, this follows from

$$\begin{aligned}
X_1 X_{-1} X_{-3} m_\lambda &= -X_1 X_{-1} X_{-2} X_{-1} m_\lambda = -[X_1, X_{-1}] X_{-2} X_{-1} m_\lambda - X_{-1} X_{-2} [X_1, X_{-1}] m_\lambda \\
&= \frac{1}{2}\left(\mathrm{i}^{\lambda_1 - 1} + \mathrm{i}^{-\lambda_1 + 1}\right) X_{-2} X_{-1} m_\lambda + \frac{1}{2}\left(\mathrm{i}^{\lambda_1} + \mathrm{i}^{-\lambda_1}\right) X_{-1} X_{-2} m_\lambda \\
&= \frac{1}{2}\left(\mathrm{i}^{\lambda_1} + \mathrm{i}^{-\lambda_1}\right) X_{-1} X_{-2} m_\lambda,
\end{aligned} \tag{3.6}$$

which is non-zero since $\lambda_1 \in 2\mathbb{Z}$. Similar calculations establish that X_2 sends $X_{-3} m_\lambda$ to a non-zero multiple of $X_{-1} m_\lambda$, which is in turn sent to a non-zero multiple of m_λ by X_1. These states are therefore not in J_λ, hence their images span L_λ and $\dim L_\lambda = 3$.

It remains to prove (ii). Suppose first that $\lambda_2 \in 2\mathbb{Z} + 1$, so that $\lambda_1, \lambda_3 \in \mathbb{C} \setminus \mathbb{Z}$. The previous case shows that $X_{-2} m_\lambda$ and its descendants are in J_λ. This time, $X_{-1} X_{-3} m_\lambda$ is not because

$$X_1 X_{-1} X_{-3} m_\lambda = \frac{1}{2}\left(\mathrm{i}^{\lambda_1 - 1} + \mathrm{i}^{-(\lambda_1 - 1)}\right) X_{-2} X_{-1} m_\lambda \qquad (\mathrm{mod}\ J_\lambda), \tag{3.7}$$

which is non-zero as $\lambda_1 \notin 2\mathbb{Z}$. Moreover, $X_{-2} X_{-1} m_\lambda \notin J_\lambda$ as X_2 sends it to a non-zero multiple of $X_{-1} m_\lambda$ and X_1 sends the latter to m_λ. $\dim L_\lambda$ is thus 4.

As $\lambda_1 \in 2\mathbb{Z} + 1$ obviously follows in the same way, the last case to consider is $\lambda_3 \in 2\mathbb{Z} + 1$ and $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{Z}$. As above, it easily follows that $X_{-1} m_\lambda, X_{-2} m_\lambda \notin J_\lambda$. However there is a vector of weight $\lambda - \alpha_3$, unique up to scalar multiples, that is maximal hence in J_λ:

$$v = \left(\mathrm{i}^{\lambda_1} + \mathrm{i}^{-\lambda_1}\right) X_{-1} X_{-2} m_\lambda - \left(\mathrm{i}^{\lambda_1 + 1} + \mathrm{i}^{-\lambda_1 - 1}\right) X_{-2} X_{-1} m_\lambda. \tag{3.8}$$

λ atypical of degree 1, $\lambda_i \in 2\mathbb{Z}+1$.

λ atypical of degree 2, $\lambda_i, \lambda_3 \in 2\mathbb{Z}+1$.

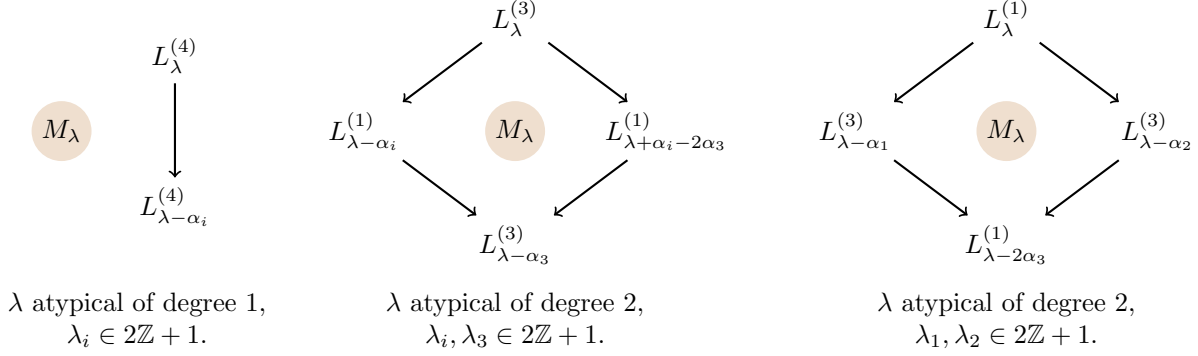λ atypical of degree 2, $\lambda_1, \lambda_2 \in 2\mathbb{Z}+1$.

Figure 2: The Loewy diagrams of the reducible Verma modules M_λ of $\overline{\mathcal{U}}_{\mathsf{i}}^H(\mathfrak{sl}_3)$. Here, the superscript on a composition factor indicates its dimension.

Checking $X_1 v = 0$ is straightforward and an easy computation gives

$$X_2 v = -\left(\mathsf{i}^{\lambda_3} + \mathsf{i}^{-\lambda_3}\right) X_{-1} m_\lambda, \tag{3.9}$$

which vanishes because $\lambda_3 \in 2\mathbb{Z}+1$. Now one checks that v generates a four-dimensional submodule of M_λ (which is therefore J_λ). It follows that $\dim L_\lambda = 4$, completing the proof. $\qquad\square$

It is very convenient that the dimension of L_λ completely determines the four different cases in Proposition 3.1. This dimension also completely determines the Loewy diagrams of the Verma modules of $\overline{\mathcal{U}}_{\mathsf{i}}^H(\mathfrak{sl}_3)$.

Theorem 3.2. *The Loewy series of any given Verma module M_λ of $\overline{\mathcal{U}}_{\mathsf{i}}^H(\mathfrak{sl}_3)$ all coincide. When M_λ is reducible, its (unique) Loewy diagram is given in Figure 2.*

Proof. Suppose first that $\dim L_\lambda = 4$, so that there is a unique $i = 1, 2, 3$ satisfying $\lambda_i \in 2\mathbb{Z}+1$ and thus $\lambda_j \in \mathbb{C} \setminus \mathbb{Z}$ for all $j \neq i$. The proof of Proposition 3.1 established that the maximal proper submodule $J_\lambda = \operatorname{rad} M_\lambda$ is generated by a maximal vector of weight $\mu = \lambda - \alpha_i$. Because $\mu_i = \lambda_i - 2 \in 2\mathbb{Z}+1$ and $\mu_j = \lambda_j \pm 1 \in \mathbb{C} \setminus \mathbb{Z}$, for $j \neq i$, $\dim L_\mu$ is also 4. This implies that $\operatorname{rad} M_\lambda \cong L_\mu$, fixing the Loewy diagram of the radical series. Because this series has length 2 and the head is irreducible (true for all Verma modules), all Loewy series coincide and so the Loewy diagram is unique.

Next, take $\dim L_\lambda = 1$, so $\lambda_1, \lambda_2 \in 2\mathbb{Z}+1$. In this case, the proof of Proposition 3.1 established that $\operatorname{rad} M_\lambda$ is generated by the maximal vectors $X_{-i} m_\lambda$ of weights $\lambda - \alpha_i$, $i = 1, 2$. Obviously, the submodule V^i generated by $X_{-i} m_\lambda$ does not contain the other. Moreover, $\dim L_{\lambda-\alpha_i} = 3$, so the dimensions of the composition factors found thus far sum to 7. There is therefore one remaining and its dimension must be 1. It is easy to check that it is $L_{\lambda-2\alpha_3}$, generated by the maximal vector $(X_{-1} X_{-2})^2 m_\lambda = (X_{-2} X_{-1})^2 m_\lambda$, which is clearly contained in $V^1 \cap V^2$. This forces $\operatorname{rad}^2 M_\lambda \cong L_{\lambda-2\alpha_3}$ and $\operatorname{rad}^1 M_\lambda \cong V^1 + V^2$, hence $\operatorname{rad}_2 M_\lambda \cong L_{\lambda-\alpha_1} \oplus L_{\lambda-\alpha_2}$. The length of the radical series is thus 3 and it has an irreducible head. The socle is obviously the irreducible $L_{\lambda-2\alpha_3}$, so again there is a unique Loewy diagram.

It remains to consider $\dim L_\lambda = 3$. Without loss of generality, take $\lambda_1 \in 2\mathbb{Z}$ and $\lambda_2 \in 2\mathbb{Z}+1$. Then, the proof of Proposition 3.1 shows that $\operatorname{rad} M_\lambda$ is the 5-dimensional submodule generated by the maximal vector $X_{-2} m_\lambda$ and the non-maximal vector $X_{-1} X_{-3} m_\lambda$. Under the action of X_1, the latter is mapped to a non-zero

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multiple of $X_{-1}X_{-2}m_\lambda$ which is easily checked to be maximal. We have thus identified three composition factors of rad M_λ: $L_{\lambda-\alpha_2}$, $L_{\lambda-\alpha_1-\alpha_3}$ and $L_{\lambda-\alpha_3}$. Their dimensions are 1, 1 and 3 by Proposition 3.1, so there are no more factors. This proves that $\mathrm{rad}^2 M_\lambda \cong L_{\lambda-\alpha_3}$ and $\mathrm{rad}_2 M_\lambda \cong L_{\lambda-\alpha_2} \oplus L_{\lambda-\alpha_1-\alpha_3}$. The length of the radical series is thus 3 and soc $M_\lambda \cong L_{\lambda-\alpha_3}$, so M_λ once more has a unique Loewy diagram. $\qquad\square$

A useful consequence of Theorem 3.2 is the classification of indecomposable extensions of irreducibles in $\mathscr{C}$. Self-extensions are ruled out as they are necessarily non-weight, hence not in $\mathscr{C}$. Such an extension is therefore a highest-weight module, hence a Verma quotient, or the dual of a highest-weight module. Either way, the extension is unique up to isomorphism.

Corollary 3.3. *A non-split extension between L_λ and L_μ exists, and is then unique up to isomorphism, if and only if one of the following conditions is satisfied:*

- *λ is atypical of degree 1 and $\mu = \lambda \pm \alpha_i$, where $i = 1, 2, 3$ satisfies $\lambda_i \in 2\mathbb{Z}+1$.*

- *λ is atypical of degree 2 with $\lambda_1, \lambda_2 \in 2\mathbb{Z}+1$ and $\mu = \lambda \pm \alpha_1, \lambda \pm \alpha_2, \lambda - \alpha_1 + 2\alpha_3, \lambda - \alpha_2 + 2\alpha_3$.*

- *λ is atypical of degree 2 with $\lambda_3 \in 2\mathbb{Z}+1$ and $\mu = \lambda \pm \alpha_i, \lambda + \alpha_i - 2\alpha_3$, where $i = 1, 2$ satisfies $\lambda_i \in 2\mathbb{Z}+1$.*

The blocks of $\overline{\mathcal{U}}_i^H(\mathfrak{sl}_3)$ are thus as follows. If λ is typical, then L_λ generates a semisimple block in which it is the only irreducible. There are three one-parameter families of blocks corresponding to λ atypical of degree 1, one for each positive root α_i. The parameter may be taken as either of the typical $\lambda_i \in \mathbb{C} \setminus \mathbb{Z}$. Finally, there is a unique block for λ atypical of degree 2. This is the principal block as it includes L_0.

3.2 Projective modules

We turn to the Loewy diagrams of the projective indecomposables P_λ. These are almost completely determined by combining the Loewy diagrams of the Verma modules with their standard (and costandard) filtrations. The following lemma will help to complete the picture.

Lemma 3.4. *Given $\lambda_i, \lambda_3 \in 2\mathbb{Z}+1$, for some $i = 1, 2$, there does not exist a $\overline{\mathcal{U}}_i^H(\mathfrak{sl}_3)$-module whose Loewy diagram is the form*

$$
\begin{array}{ccc}
 & L_\lambda^{(3)} & \\
\swarrow & & \searrow \\
L_{\lambda+\alpha_i}^{(1)} & & L_{\lambda+\alpha_i-2\alpha_3}^{(1)} \\
\searrow & & \swarrow \\
 & L_\lambda^{(3)} &
\end{array}
\qquad (3.10)
$$

Proof. Suppose that V is such a module, with $i = 2$ say (the argument for $i = 1$ is identical). Then,

$$
\mathrm{ch}[V] = z^{\lambda+\alpha_2} + 2\left(z^\lambda + z^{\lambda-\alpha_1} + z^{\lambda-\alpha_3}\right) + z^{\lambda+\alpha_2-2\alpha_3}, \qquad (3.11)
$$

by Proposition 3.1. Let $v \in V$ be cyclic of weight λ. Then, the arrow from L_λ to $L_{\lambda+\alpha_2}$ in (3.10) means that $X_2 v \neq 0$ and that from $L_{\lambda+\alpha_2}$ to L_λ means that $w = X_{-2}X_2 v$ is linearly independent of v (it generates the

socle of V). However, $X_{-2}v = 0$ since $\lambda - \alpha_2$ is not a weight of V, hence

$$w = [X_{-2}, X_2]v = -\frac{K_2 - K_2^{-1}}{2\mathrm{i}}v \tag{3.12}$$

is proportional to v, a contradiction. $\qquad\square$

Theorem 3.5. *If λ is typical, then $P_\lambda = M_\lambda = L_\lambda$. Otherwise, the projective P_λ has a unique Loewy diagram which is presented, along with its character as a sum of Verma characters, in Figure 3.*

Proof. The fact that L_λ is projective, for λ typical, follows immediately from Corollary 3.3. Otherwise, the multiplicities of the Verma factors may be obtained from Proposition 2.11 and Theorem 3.2.

Suppose first that $\dim L_\lambda = 4$ and $i = 1, 2, 3$ is the unique index satisfying $\lambda_i \in 2\mathbb{Z} + 1$. Then, L_λ can only appear as the head or the socle of another Verma of the same type. These two possibilities give $(P_\lambda : M_\mu) = [M_\mu : L_\lambda] = \delta^{\mu,\lambda} + \delta^{\mu,\lambda+\alpha_i}$. In the Grothendieck group of $\mathscr{C}$, one therefore has

$$[P_\lambda] = [M_\lambda] + [M_{\lambda+\alpha_i}]. \tag{3.13}$$

Obviously, only M_λ has the same head as P_λ. Moreover, Theorem 3.2 shows that only $M_{\lambda+\alpha_i}$ has the same socle as P_λ:

$$\operatorname{soc} P_\lambda = \operatorname{soc}_1 P_\lambda \cong \operatorname{rad}_1 P_\lambda = \frac{P_\lambda}{\operatorname{rad} P_\lambda} \cong L_\lambda, \tag{3.14}$$

P_λ thus admits a single standard filtration: $0 \subset M_{\lambda+\alpha_i} \subset P_\lambda$.

It follows that $L_{\lambda-\alpha_i}$ appears in $\operatorname{rad}_2 P_\lambda$ and $L_{\lambda+\alpha_i}$ appears in $\operatorname{soc}_2 P_\lambda$. Since projectives are self-dual (Theorem 2.10), (2.14) leads to $\operatorname{rad}_2 P_\lambda = \operatorname{soc}_2 P_\lambda \cong L_{\lambda-\alpha_i} \oplus L_{\lambda+\alpha_i}$. The Loewy length of P_λ is thus 3 and the Loewy diagram (with some arrows missing) is
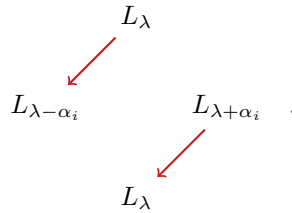

This diagram obviously needs two more arrows. While it is clear where they go, it is instructive for the other cases to see them arise from a *costandard filtration* of P_λ, that is one whose successive quotients are *dual Verma modules*. This series is easily deduced by applying the duality functor to the short exact sequence

$$0 \longrightarrow M_{\lambda+\alpha_i} \longrightarrow P_\lambda \longrightarrow M_\lambda \longrightarrow 0, \tag{3.15}$$

the result being $0 \subset \widetilde{M}_\lambda \subset P_\lambda$ with $P_\lambda/\widetilde{M}_\lambda \cong \widetilde{M}_{\lambda+\alpha_i}$. Since Verma modules have unique Loewy diagrams, the same is true for their duals and the latter's diagrams are obtained by turning those of Figure 2 upside-down. Drawing in blue the arrows corresponding to these dual Verma modules on the partial Loewy diagram above (to distinguish them from the Verma arrows drawn in red), we arrive at the Loewy diagram of Figure 3. As the socle and head are irreducible, it follows that this is the unique Loewy series of P_λ.
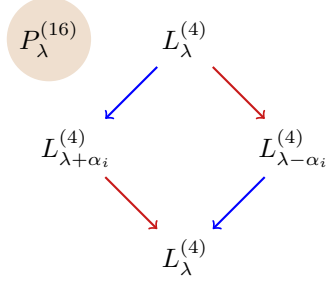
λ atypical of degree 1, $\lambda_i \in 2\mathbb{Z}+1$,
$\mathrm{ch}[P_\lambda] = \mathrm{ch}[M_{\lambda+\alpha_i}] + \mathrm{ch}[M_\lambda]$.
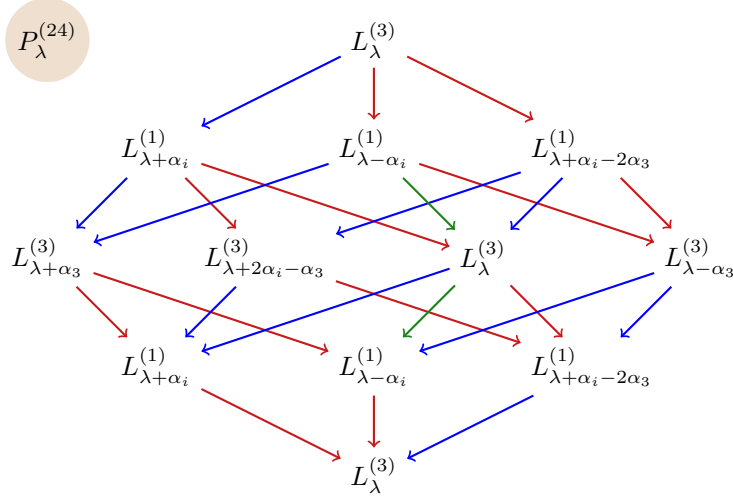
λ atypical of degree 2, $\lambda_i, \lambda_3 \in 2\mathbb{Z}+1$,
$\mathrm{ch}[P_\lambda] = \mathrm{ch}[M_{\lambda+\alpha_3}] + \mathrm{ch}[M_{\lambda+\alpha_i}] + \mathrm{ch}[M_\lambda]$.
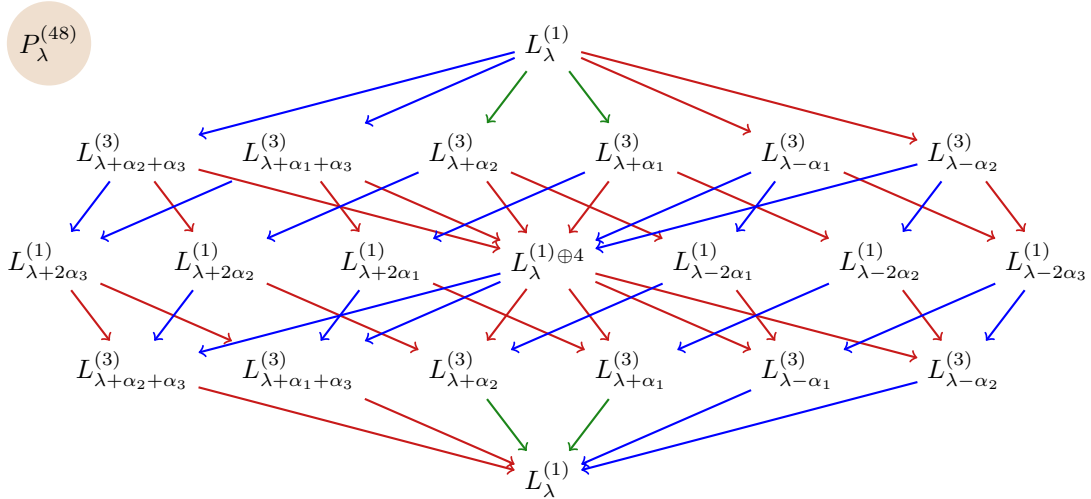
λ atypical of degree 2, $\lambda_1, \lambda_2 \in 2\mathbb{Z}+1$,
$\mathrm{ch}[P_\lambda] = \mathrm{ch}[M_{\lambda+2\alpha_3}] + \mathrm{ch}[M_{\lambda+2\alpha_3-\alpha_1}] + \mathrm{ch}[M_{\lambda+2\alpha_3-\alpha_2}] + \mathrm{ch}[M_{\lambda+\alpha_1}] + \mathrm{ch}[M_{\lambda+\alpha_2}] + \mathrm{ch}[M_\lambda]$.

Figure 3: Loewy diagrams and characters of the projectives $P_\lambda^{(d)}$ with λ atypical. Here, d is the dimension. Red arrows are associated with standard filtrations, blue with costandard filtrations and all other arrows are green. The bottom diagram is not quite complete, because we make no attempt to resolve the four composition factors isomorphic to $L_\lambda^{(1)}$ in the middle row.

Suppose next that $\dim L_\lambda = 3$, so that $\lambda_i, \lambda_3 \in 2\mathbb{Z}+1$ for some $i = 1, 2$. Consulting Figure 2, BGG reciprocity implies this time that

$$[P_\lambda] = [M_\lambda] + [M_{\lambda+\alpha_i}] + [M_{\lambda+\alpha_3}]. \tag{3.16}$$

Again, the only Verma factor whose head matches that of P_λ is M_λ and the only Verma whose socle matches that of P_λ is $M_{\lambda+\alpha_3}$. We conclude that in this case, there is also a unique standard series for P_λ and it has the form

$$0 \subset M_{\lambda+\alpha_3} \subset N \subset P_\lambda, \qquad \text{with } \frac{N}{M_{\lambda+\alpha_3}} \cong M_{\lambda+\alpha_i} \text{ and } \frac{P_\lambda}{N} \cong M_\lambda. \tag{3.17}$$

Combining this with (2.14) and Theorem 3.2 identifies some of the composition factors in the second Loewy layers of the radical and socle series:

$$L_{\lambda-\alpha_i} \oplus L_{\lambda+\alpha_i-2\alpha_3} \subseteq \mathrm{rad}_2\, P_\lambda \cong \mathrm{soc}_2\, P_\lambda \supseteq L_{\lambda+\alpha_i} \oplus L_{\lambda-\alpha_i}. \tag{3.18}$$

It follows that $L_{\lambda+\alpha_i}$ appears in $\mathrm{rad}_2\, P_\lambda$ and $L_{\lambda+\alpha_i-2\alpha_3}$ appears in $\mathrm{soc}_2\, P_\lambda$. These extra composition factors are the head and socle of $M_{\lambda+\alpha_i}$, hence the length of the radical and socle series of P_λ is 5. In fact, these series coincide and the Loewy diagram (with some arrows missing) is as follows:
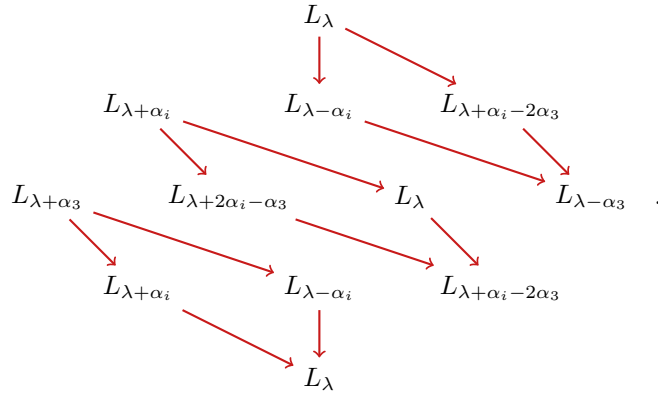
To complement these red Verma arrows, one next computes the corresponding costandard filtration of P_λ by dualising (3.17):

$$\widetilde{M}_\lambda \hookrightarrow P_\lambda \xrightarrow{\pi} \check{N} \qquad \text{and} \qquad \widetilde{M}_{\lambda+\alpha_i} \hookrightarrow \check{N} \xrightarrow{\pi'} \widetilde{M}_{\lambda+\alpha_3}. \tag{3.19}$$

The first says that $\widetilde{M}_\lambda = \ker \pi$ is a submodule of P_λ and the second says that $\pi^{-1}(\widetilde{M}_{\lambda+\alpha_i})$ is too. Of course, $\pi^{-1}(\widetilde{M}_{\lambda+\alpha_i})/\widetilde{M}_\lambda = \pi^{-1}(\widetilde{M}_{\lambda+\alpha_i})/\ker \pi \cong \widetilde{M}_{\lambda+\alpha_i}$. Similarly, $\pi^{-1}(\widetilde{M}_{\lambda+\alpha_i}) = \ker(\pi' \circ \pi)$ and so $P_\lambda/\pi^{-1}(\widetilde{M}_{\lambda+\alpha_i}) \cong \widetilde{M}_{\lambda+\alpha_3}$. The required costandard series is thus

$$0 \subset \widetilde{M}_\lambda \subset \pi^{-1}(\widetilde{M}_{\lambda+\alpha_i}) \subset P_\lambda \tag{3.20}$$

and the dual Verma quotients are $\widetilde{M}_\lambda$, $\widetilde{M}_{\lambda+\alpha_i}$ and $\widetilde{M}_{\lambda+\alpha_3}$. Drawing these dual Verma arrows on the Loewy diagram of P_λ in blue (and not repeating if the arrow already exists) gives

$$
\begin{array}{c}
L_\lambda \\[4pt]
L_{\lambda+\alpha_i} \qquad L_{\lambda-\alpha_i} \qquad L_{\lambda+\alpha_i-2\alpha_3} \\[4pt]
L_{\lambda+\alpha_3} \qquad L_{\lambda+2\alpha_i-\alpha_3} \qquad L_\lambda \qquad L_{\lambda-\alpha_3} \\[4pt]
L_{\lambda+\alpha_i} \qquad L_{\lambda-\alpha_i} \qquad L_{\lambda+\alpha_i-2\alpha_3} \\[4pt]
L_\lambda
\end{array} \qquad .
$$
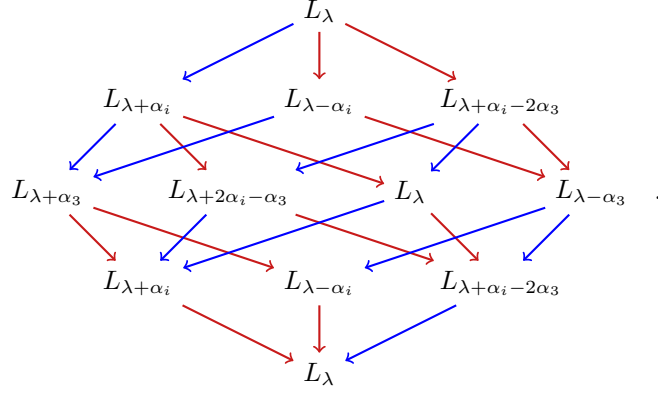
It remains to check for any additional arrows beyond those already established. There are only a few possibilities and most are ruled out by Corollary 3.3. The only possible additions are an arrow from the composition factor $L_{\lambda-\alpha_i}$ in the second-top layer to L_λ in the middle layer and from the latter factor to $L_{\lambda-\alpha_i}$ in the second-bottom layer. In both cases, the absence of the arrow would imply that P_λ has a quotient or submodule whose Loewy diagram is

$$
\begin{array}{c}
L_\lambda \\[4pt]
L_{\lambda+\alpha_i} \qquad L_{\lambda+\alpha_i-2\alpha_3} \\[4pt]
L_\lambda
\end{array} \qquad .
$$
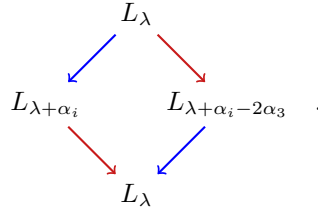
But, such a $\overline{\mathcal{U}}_{\mathsf{i}}^{H}(\mathfrak{sl}_3)$-module does not exist, by Lemma 3.4. Both arrows must therefore be present and the Loewy diagram is thus as in Figure 3.

Finally, we consider the case $\dim L_\lambda = 1$, hence $\lambda_1, \lambda_2 \in 2\mathbb{Z}+1$. Now, BGG reciprocity gives six Verma modules for the projective:

$$
[P_\lambda] = [M_\lambda] + [M_{\lambda+\alpha_1}] + [M_{\lambda+\alpha_2}] + [M_{\lambda+2\alpha_3-\alpha_1}] + [M_{\lambda+2\alpha_3-\alpha_2}] + [M_{\lambda+2\alpha_3}]. \tag{3.21}
$$

Following the same procedure as in the previous case, we find that the radical and socle series again coincide and have length 5. The Loewy diagram (with arrows corresponding to any of the standard filtrations) is

$$L_\lambda$$

$$L_{\lambda-\alpha_1+2\alpha_3} \quad L_{\lambda-\alpha_2+2\alpha_3} \quad L_{\lambda+\alpha_2} \quad L_{\lambda+\alpha_1} \quad L_{\lambda-\alpha_1} \quad L_{\lambda-\alpha_2}$$

$$L_{\lambda+2\alpha_3} \quad L_{\lambda+2\alpha_2} \quad L_\lambda \quad L_{\lambda+2\alpha_1} \quad L_\lambda \quad L_\lambda \quad L_{\lambda-2\alpha_1} \quad L_\lambda \quad L_{\lambda-2\alpha_2} \quad L_{\lambda-2\alpha_3}$$

$$L_{\lambda-\alpha_1+2\alpha_3} \quad L_{\lambda-\alpha_2+2\alpha_3} \quad L_{\lambda+\alpha_2} \quad L_{\lambda+\alpha_1} \quad L_{\lambda-\alpha_1} \quad L_{\lambda-\alpha_2}$$

$$L_\lambda$$
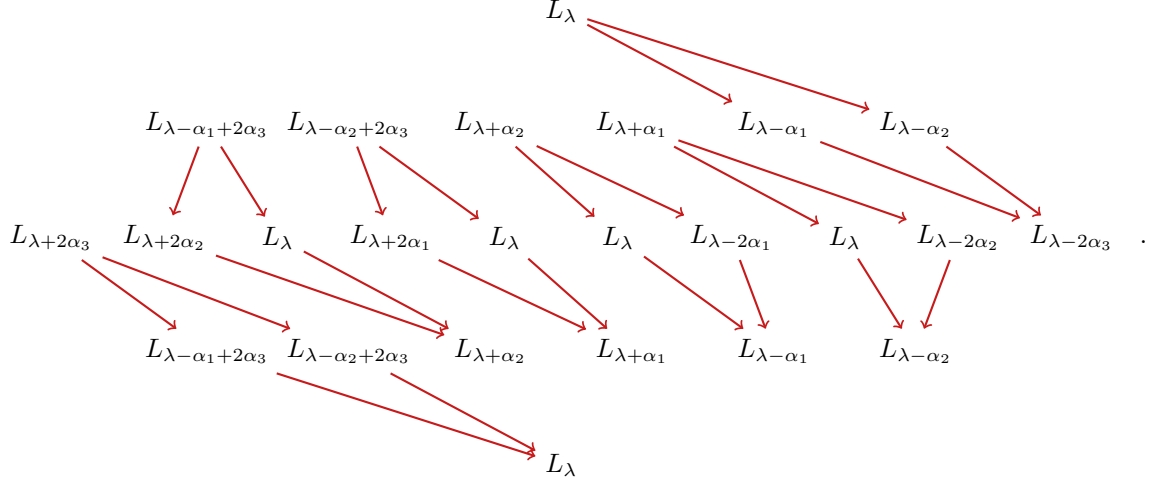
Unfortunately, it is not straightforward to add the arrows corresponding to (any of) the costandard filtrations because L_λ occurs in the middle row with multiplicity 4 — roughly speaking, it is difficult to identify which copy (more precisely, which linear combination of copies) of L_λ corresponds to the head or tail of the arrows in a costandard filtration. Rather than deal with this, we shall be content with a partial Loewy diagram in which these 4 copies of L_λ are collapsed to $L_\lambda^{\oplus 4}$. With this, adding the costandard arrows is easy. Finally, there are obvious arrows to add between the first and second, and fourth and fifth, rows. Corollary 3.3 then rules out any more. $\qquad\square$

3.3 Tensor product decompositions

Recall that there are four types of irreducible $\overline{\mathcal{U}}_i^H(\mathfrak{sl}_3)$-modules in $\mathscr{C}$, distinguished by their dimensions. Including this dimension as a superscript (as in Figures 2 and 3), these types are characterised by the following conditions on their highest weights:

$$
\begin{aligned}
L_\lambda^{(1)}: &\quad \lambda_1, \lambda_2 \in 2\mathbb{Z}+1,\ \lambda_3 \in 2\mathbb{Z}; \\
L_\lambda^{(3)}: &\quad \lambda_i, \lambda_3 \in 2\mathbb{Z}+1,\ \lambda_j \in 2\mathbb{Z} &&(\{i,j\}=\{1,2\}); \\
L_\lambda^{(4)}: &\quad \lambda_i \in 2\mathbb{Z}+1,\ \lambda_j, \lambda_k \notin \mathbb{Z} &&(\{i,j,k\}=\{1,2,3\}); \\
L_\lambda^{(8)}: &\quad \lambda_1, \lambda_2, \lambda_3 \notin 2\mathbb{Z}+1.
\end{aligned}
\tag{3.22}
$$

Recall also that λ_i is said to be atypical if $\lambda_i \in 2\mathbb{Z}+1$ and typical otherwise. In this section, the tensor product of (almost) every pair of irreducible $\overline{\mathcal{U}}_i^H(\mathfrak{sl}_3)$-modules in $\mathscr{C}$ is explicitly decomposed (up to isomorphism) as a direct sum of indecomposables.

It is worth mentioning that these computations are simplified by the fact that $\mathscr{C}$ is braided. This follows immediately from the fact that $\overline{\mathcal{U}}_i^H(\mathfrak{sl}_3)$ admits a quasitriangular structure [GP13, Theorem 41], see also [Ru22, Equations (4.8–9)].

The easiest tensor products to identify are those of the form $L_\lambda^{(1)} \otimes L_\mu$. These have dimension $\dim L_\mu$ and possess a maximal vector of weight $\lambda + \mu$. Since

$$(\lambda + \mu)_i = \lambda_i + \mu_i - \langle \rho, \alpha_i \rangle, \quad i = 1, 2, 3, \tag{3.23}$$

and $\lambda_i - \langle \rho, \alpha_i \rangle \in 2\mathbb{Z}$ for each i, it follows that $\lambda + \mu$ satisfies the same set of conditions in (3.22) as μ. This

leads to the following conclusion.

Proposition 3.6. *Given $\lambda, \mu \in \mathfrak{h}^*$ satisfying $\lambda_1, \lambda_2 \in 2\mathbb{Z} + 1$, one has*

$$L_\lambda^{(1)} \otimes L_\mu \cong L_{\lambda+\mu}. \tag{3.24}$$

Tensor products involving a typical irreducible $L_\mu^{(8)}$ are likewise relatively easy to identify because of the following facts:

- $\mathscr{C}$ is pivotal (in particular, rigid) [GPV13, Lemma 17], so the projectives form a tensor ideal.
- Every projective is self-dual (Theorem 2.10), thus injective, hence their extensions split.
- Projectives are uniquely determined (up to isomorphism) by their characters (Proposition 2.8).

The tensor product of any module of $\mathscr{C}$ with an $L_\mu^{(8)}$ thus decomposes as a direct sum of projectives which may be identified from its character.

To illustrate how one identifies such a tensor product, consider $L_\lambda^{(3)} \otimes L_\mu^{(8)}$, so that the μ_i, $i = 1, 2, 3$, are typical (note that $L_\lambda^{(1)} \otimes L_\mu^{(8)}$ was already identified in Proposition 3.6). Without loss of generality, assume that $\lambda_1 \in 2\mathbb{Z} + 1$ and $\lambda_2 \in 2\mathbb{Z}$. Then, the character of the tensor product satisfies

$$\mathrm{ch}[L_\lambda^{(3)} \otimes L_\mu^{(8)}] = \mathrm{ch}[L_\lambda^{(3)}] \, \mathrm{ch}[L_\mu^{(8)}] = (z^\lambda + z^{\lambda - \alpha_2} + z^{\lambda - \alpha_3}) \, \mathrm{ch}[M_\mu]$$
$$= \mathrm{ch}[M_{\lambda+\mu}] + \mathrm{ch}[M_{\lambda+\mu-\alpha_2}] + \mathrm{ch}[M_{\lambda+\mu-\alpha_3}]. \tag{3.25}$$

It therefore remains to identify which projective has this character. This splits into several cases.

When $\lambda + \mu$, $\lambda + \mu - \alpha_2$ and $\lambda + \mu - \alpha_3$ are all typical, the projective is $P_{\lambda+\mu}^{(8)} \oplus P_{\lambda+\mu-\alpha_2}^{(8)} \oplus P_{\lambda+\mu-\alpha_3}^{(8)}$, where the dimension of each indecomposable summand is likewise indicated in the superscript. By (3.23), the typicality of all three weights is equivalent to $\mu_i \notin \mathbb{Z}$, $i = 1, 2, 3$.

Suppose now that $\mu_1 \in 2\mathbb{Z}$, but $\mu_2, \mu_3 \notin \mathbb{Z}$. Then, $\lambda + \mu$ is typical while $\lambda + \mu - \alpha_2$ and $\lambda + \mu - \alpha_3$ are atypical of degree 1: $(\lambda + \mu - \alpha_2)_1 = \lambda_1 + \mu_1 \in 2\mathbb{Z} + 1$ and $(\lambda + \mu - \alpha_3)_1 = \lambda_1 + \mu_1 - 2 \in 2\mathbb{Z} + 1$. It follows that $P_{\lambda+\mu}^{(8)}$ splits off as a direct summand of the tensor product and the remaining two Verma modules combine to form a 16-dimensional indecomposable projective. Comparing characters with those in Figure 3, the latter projective is identified as $P_{\lambda+\mu-\alpha_3}^{(16)}$.

The story is similar when μ_2 or μ_3 is the only integral label (which then lies in $2\mathbb{Z}$). The last remaining possibility is $\mu_1, \mu_2, \mu_3 \in 2\mathbb{Z}$. Then, $\lambda + \mu$, $\lambda + \mu - \alpha_2$ and $\lambda + \mu - \alpha_3$ are atypical of degree 2, corresponding to irreducibles of dimension 3, 1 and 3, respectively. The only projective consistent with this is $P_{\lambda+\mu-\alpha_3}^{(24)}$, again by Figure 3.

Proposition 3.7. *Given $\lambda \in \mathfrak{h}^*$ satisfying $\lambda_i, \lambda_3 \in 2\mathbb{Z} + 1$ and $\lambda_j \in 2\mathbb{Z}$, where $\{i, j\} = \{1, 2\}$, and $\mu \in \mathfrak{h}^*$ typical, $L_\lambda^{(3)} \otimes L_\mu^{(8)}$ is isomorphic to*

$$
\begin{aligned}
\mu_1, \mu_2, \mu_3 \in 2\mathbb{Z}: && P_{\lambda+\mu-\alpha_3}^{(24)}, \\
\mu_i \in 2\mathbb{Z}, \ \mu_j, \mu_3 \notin \mathbb{Z}: && P_{\lambda+\mu}^{(8)} \oplus P_{\lambda+\mu-\alpha_3}^{(16)}, \\
\mu_j \in 2\mathbb{Z}, \ \mu_i, \mu_3 \notin \mathbb{Z}: && P_{\lambda+\mu-\alpha_j}^{(16)} \oplus P_{\lambda+\mu-\alpha_3}^{(8)},
\end{aligned}
\tag{3.26}
$$

$$\mu_3 \in 2\mathbb{Z}, \ \mu_1, \mu_2 \notin \mathbb{Z}: \qquad P^{(8)}_{\lambda+\mu-\alpha_j} \oplus P^{(16)}_{\lambda+\mu-\alpha_3},$$

$$\mu_1, \mu_2, \mu_3 \notin \mathbb{Z}: \qquad P^{(8)}_{\lambda+\mu} \oplus P^{(8)}_{\lambda+\mu-\alpha_j} \oplus P^{(8)}_{\lambda+\mu-\alpha_3}.$$

Explicit decompositions for $L^{(4)}_\lambda \otimes L^{(8)}_\mu$ and $L^{(8)}_\lambda \otimes L^{(8)}_\mu$ may be readily found using the same methods. We omit the details.

Proposition 3.8. *Given* $\lambda \in \mathfrak{h}^*$ *satisfying* $\lambda_i \in 2\mathbb{Z}+1$ *and* $\lambda_j, \lambda_3 \notin \mathbb{Z}$, *where* $\{i,j\} = \{1,2\}$, *and* $\mu \in \mathfrak{h}^*$ *typical,* $L^{(4)}_\lambda \otimes L^{(8)}_\mu$ *is isomorphic to*

$$
\begin{aligned}
\mu_i, \lambda_j + \mu_j \in 2\mathbb{Z}, \ \lambda_3 + \mu_3 \in 2\mathbb{Z}+1: &\qquad P^{(24)}_{\lambda+\mu-\alpha_3} \oplus P^{(8)}_{\lambda+\mu-\alpha_j-\alpha_3}, \\
\mu_i, \lambda_3 + \mu_3 \in 2\mathbb{Z}, \ \lambda_j + \mu_j \in 2\mathbb{Z}+1: &\qquad P^{(8)}_{\lambda+\mu} \oplus P^{(24)}_{\lambda+\mu-\alpha_j-\alpha_3}, \\
\mu_i \in 2\mathbb{Z}, \ \lambda_j + \mu_j, \lambda_3 + \mu_3 \notin \mathbb{Z}: &\qquad P^{(8)}_{\lambda+\mu} \oplus P^{(16)}_{\lambda+\mu-\alpha_3} \oplus P^{(8)}_{\lambda+\mu-\alpha_j-\alpha_3}, \\
\lambda_j + \mu_j \in 2\mathbb{Z}, \ \mu_i, \lambda_3 + \mu_3 \notin \mathbb{Z}: &\qquad P^{(16)}_{\lambda+\mu-\alpha_j} \oplus P^{(8)}_{\lambda+\mu-\alpha_3} \oplus P^{(8)}_{\lambda+\mu-\alpha_j-\alpha_3}, \\
\lambda_j + \mu_j \in 2\mathbb{Z}+1, \ \mu_i, \lambda_3 + \mu_3 \notin \mathbb{Z}: &\qquad P^{(8)}_{\lambda+\mu} \oplus P^{(8)}_{\lambda+\mu-\alpha_3} \oplus P^{(16)}_{\lambda+\mu-\alpha_j-\alpha_3}, \\
\lambda_3 + \mu_3 \in 2\mathbb{Z}, \ \mu_i, \lambda_j + \mu_j \notin \mathbb{Z}: &\qquad P^{(8)}_{\lambda+\mu} \oplus P^{(8)}_{\lambda+\mu-\alpha_j} \oplus P^{(16)}_{\lambda+\mu-\alpha_j-\alpha_3}, \\
\lambda_3 + \mu_3 \in 2\mathbb{Z}+1, \ \mu_i, \lambda_j + \mu_j \notin \mathbb{Z}: &\qquad P^{(8)}_{\lambda+\mu-\alpha_j} \oplus P^{(16)}_{\lambda+\mu-\alpha_3} \oplus P^{(8)}_{\lambda+\mu-\alpha_j-\alpha_3}, \\
\mu_1, \lambda_2 + \mu_2, \lambda_3 + \mu_3 \notin \mathbb{Z}; &\qquad P^{(8)}_{\lambda+\mu} \oplus P^{(8)}_{\lambda+\mu-\alpha_j} \oplus P^{(8)}_{\lambda+\mu-\alpha_3} \oplus P^{(8)}_{\lambda+\mu-\alpha_j-\alpha_3}.
\end{aligned}
\tag{3.27}
$$

Proposition 3.9. *Given* $\lambda \in \mathfrak{h}^*$ *satisfying* $\lambda_1, \lambda_2 \notin \mathbb{Z}$ *and* $\lambda_3 \in 2\mathbb{Z}+1$, *and* $\mu \in \mathfrak{h}^*$ *typical,* $L^{(4)}_\lambda \otimes L^{(8)}_\mu$ *is isomorphic to*

$$
\begin{aligned}
\lambda_i + \mu_i, \mu_3 \in 2\mathbb{Z}, \ \lambda_j + \mu_j \in 2\mathbb{Z}+1: &\qquad P^{(8)}_{\lambda+\mu-\alpha_j} \oplus P^{(24)}_{\lambda+\mu-\alpha_3}, \\
\mu_3 \in 2\mathbb{Z}, \ , \lambda_1 + \mu_1, \lambda_2 + \mu_2 \notin \mathbb{Z}: &\qquad P^{(8)}_{\lambda+\mu-\alpha_1} \oplus P^{(8)}_{\lambda+\mu-\alpha_2} \oplus P^{(16)}_{\lambda+\mu-\alpha_3}, \\
\lambda_i + \mu_i \in 2\mathbb{Z}, \ \lambda_j + \mu_j, \mu_3 \notin \mathbb{Z}: &\qquad P^{(16)}_{\lambda+\mu-\alpha_i} \oplus P^{(8)}_{\lambda+\mu-\alpha_j} \oplus P^{(8)}_{\lambda+\mu-\alpha_3}, \\
\lambda_i + \mu_i \in 2\mathbb{Z}+1, \ \lambda_j + \mu_j, \mu_3 \notin \mathbb{Z}: &\qquad P^{(8)}_{\lambda+\mu} \oplus P^{(8)}_{\lambda+\mu-\alpha_i} \oplus P^{(16)}_{\lambda+\mu-\alpha_3}, \\
\lambda_1 + \mu_1, \lambda_2 + \mu_2, \mu_3 \notin \mathbb{Z}; &\qquad P^{(8)}_{\lambda+\mu} \oplus P^{(8)}_{\lambda+\mu-\alpha_1} \oplus P^{(8)}_{\lambda+\mu-\alpha_2} \oplus P^{(8)}_{\lambda+\mu-\alpha_3},
\end{aligned}
\tag{3.28}
$$

where $\{i,j\} = \{1,2\}$.

Proposition 3.10. *Given typical* $\lambda, \mu \in \mathfrak{h}^*$, $L^{(8)}_\lambda \otimes L^{(8)}_\mu$ *is isomorphic to*

$$
\begin{aligned}
\lambda_1 + \mu_1, \lambda_2 + \mu_2, \lambda_3 + \mu_3 \in 2\mathbb{Z}: &\quad 2P^{(8)}_{\lambda+\mu-\alpha_3} \oplus P^{(48)}_{\lambda+\mu-2\alpha_3}, \\
\lambda_i + \mu_i \in 2\mathbb{Z}, \ \lambda_j + \mu_j, \lambda_3 + \mu_3 \in 2\mathbb{Z}+1: &\quad P^{(8)}_{\lambda+\mu-\alpha_j} \oplus P^{(8)}_{\lambda+\mu-\alpha_i-\alpha_3} \oplus P^{(24)}_{\lambda+\mu-\alpha_3} \oplus P^{(24)}_{\lambda+\mu-2\alpha_3}, \\
\lambda_1 + \mu_1, \lambda_2 + \mu_2 \in 2\mathbb{Z}+1, \ \lambda_3 + \mu_3 \in 2\mathbb{Z}: &\quad P^{(8)}_{\lambda+\mu} \oplus P^{(24)}_{\lambda+\mu-\alpha_1-\alpha_3} \oplus P^{(24)}_{\lambda+\mu-\alpha_2-\alpha_3} \oplus P^{(8)}_{\lambda+\mu-2\alpha_3}, \\
\lambda_i + \mu_i \in 2\mathbb{Z}, \ \lambda_j + \mu_j, \lambda_3 + \mu_3 \notin \mathbb{Z}: &\quad P^{(16)}_{\lambda+\mu-\alpha_i} \oplus P^{(8)}_{\lambda+\mu-\alpha_j} \oplus 2P^{(8)}_{\lambda+\mu-\alpha_3} \\
&\qquad \oplus P^{(8)}_{\lambda+\mu-\alpha_i-\alpha_3} \oplus P^{(16)}_{\lambda+\mu-2\alpha_3}, \\
\lambda_i + \mu_i \in 2\mathbb{Z}+1, \ \lambda_j + \mu_j, \lambda_3 + \mu_3 \notin \mathbb{Z}: &\quad P^{(8)}_{\lambda+\mu} \oplus P^{(8)}_{\lambda+\mu-\alpha_1} \oplus P^{(16)}_{\lambda+\mu-\alpha_3} \\
&\qquad \oplus P^{(16)}_{\lambda+\mu-\alpha_i-\alpha_3} \oplus P^{(8)}_{\lambda+\mu-\alpha_j-\alpha_3} \oplus P^{(8)}_{\lambda+\mu-2\alpha_3}, \\
\lambda_3 + \mu_3 \in 2\mathbb{Z}, \ \lambda_1 + \mu_1, \lambda_2 + \mu_2 \notin \mathbb{Z}: &\quad P^{(8)}_{\lambda+\mu} \oplus 2P^{(8)}_{\lambda+\mu-\alpha_3} \oplus P^{(16)}_{\lambda+\mu-\alpha_1-\alpha_3}
\end{aligned}
\tag{3.29}
$$

$$\oplus\, P^{(16)}_{\lambda+\mu-\alpha_2-\alpha_3} \oplus P^{(8)}_{\lambda+\mu-2\alpha_3},$$

$$\lambda_3+\mu_3\in 2\mathbb{Z}+1,\ \lambda_1+\mu_1,\lambda_2+\mu_2\notin\mathbb{Z}:\quad P^{(8)}_{\lambda+\mu-\alpha_1}\oplus P^{(8)}_{\lambda+\mu-\alpha_2}\oplus P^{(16)}_{\lambda+\mu-\alpha_3}$$

$$\oplus\, P^{(8)}_{\lambda+\mu-\alpha_1-\alpha_3}\oplus P^{(8)}_{\lambda+\mu-\alpha_2-\alpha_3}\oplus P^{(16)}_{\lambda+\mu-2\alpha_3},$$

$$\lambda_1+\mu_1,\lambda_2+\mu_2,\lambda_3+\mu_3\notin\mathbb{Z}:\quad P^{(8)}_{\lambda+\mu}\oplus P^{(8)}_{\lambda+\mu-\alpha_1}\oplus P^{(8)}_{\lambda+\mu-\alpha_2}\oplus 2P^{(8)}_{\lambda+\mu-\alpha_3}$$

$$\oplus\, P^{(8)}_{\lambda+\mu-\alpha_1-\alpha_3}\oplus P^{(8)}_{\lambda+\mu-\alpha_2-\alpha_3}\oplus P^{(8)}_{\lambda+\mu-2\alpha_3},$$

where $\{i,j\}=\{1,2\}$ and $2V$ means $V^{\oplus 2}$.

It only remains to decompose the tensor products of the 3- and 4-dimensional irreducibles. Here, identifying projective summands is not always sufficient and one must sometimes resort to brute force computation. However, there are still many cases for which projectivity is very helpful.

As an illustration, consider the decomposition of $L^{(4)}_\lambda \otimes L^{(4)}_\mu$, where $\lambda_1,\mu_1\in 2\mathbb{Z}+1$ and $\lambda_2,\mu_2\notin\mathbb{Z}$. This tensor product obviously has a maximal vector v of weight $\lambda+\mu$ that satisfies $(\lambda+\mu)_1\in 2\mathbb{Z}+1$. Assume that $\lambda+\mu$ is atypical of degree 1, so $(\lambda+\mu)_2,(\lambda+\mu)_3\notin\mathbb{Z}$. Then, v generates a submodule of dimension at least 4 in which $X_{-2}v$ is not maximal.

Now, the weight $\lambda+\mu-\alpha_2$ has multiplicity 2 in the tensor product while $\lambda+\mu$ has multiplicity 1. X_2 must therefore annihilate some vector w of weight $\lambda+\mu-\alpha_2$. As $\lambda+\mu+\alpha_1-\alpha_2$ is not a weight of $L^{(4)}_\lambda\otimes L^{(4)}_\mu$, w is maximal. But, $\lambda+\mu-\alpha_2$ is typical, hence w generates a copy of the irreducible projective $P^{(8)}_{\lambda+\mu-\alpha_2}$ which is therefore a direct summand of the tensor product.

Of the 8 weights (counted with multiplicity) not accounted for by the copy of $P^{(8)}_{\lambda+\mu-\alpha_2}$, at least 4 belong to the submodule, isomorphic to $L^{(4)}_{\lambda+\mu}$, generated by v. That leaves 4, the highest of which is easily checked to be $\lambda+\mu-\alpha_2-\alpha_3$. However, this is also atypical of degree 1, hence the corresponding composition factor is $L^{(4)}_{\lambda+\mu-\alpha_2-\alpha_3}$ which accounts for all the remaining weights.

The tensor product therefore has three composition factors: $L^{(4)}_{\lambda+\mu}$, $P^{(8)}_{\lambda+\mu-\alpha_2}$ and $L^{(4)}_{\lambda+\mu-\alpha_2-\alpha_3}$. The only unknown left is whether the 4-dimensional irreducibles are direct summands or parts of an 8-dimensional indecomposable. However, the latter is impossible, by Corollary 3.3.

Similar arguments result in the following decompositions.

Proposition 3.11.

- *Given $\lambda,\mu,\lambda+\mu\in\mathfrak{h}^*$ atypical of degree 1, $L^{(4)}_\lambda\otimes L^{(4)}_\mu$ is isomorphic to*

$$
\begin{array}{lll}
\lambda_i,\mu_j\in 2\mathbb{Z}+1: & P^{(8)}_{\lambda+\mu}\oplus P^{(8)}_{\lambda+\mu-\alpha_k} & (\{i,j,k\}=\{1,2,3\}),\\[4pt]
\lambda_i,\mu_i\in 2\mathbb{Z}+1: & L^{(4)}_{\lambda+\mu}\oplus P^{(8)}_{\lambda+\mu-\alpha_j}\oplus L^{(4)}_{\lambda+\mu-\alpha_j-\alpha_3} & (\{i,j\}=\{1,2\}),\qquad (3.30)\\[4pt]
\lambda_3,\mu_3\in 2\mathbb{Z}+1: & P^{(8)}_{\lambda+\mu}\oplus L^{(4)}_{\lambda+\mu-\alpha_1}\oplus L^{(4)}_{\lambda+\mu-\alpha_2}.
\end{array}
$$

- *Given $\lambda\in\mathfrak{h}^*$ atypical of degree 2 with $\lambda_3\in 2\mathbb{Z}+1$ and μ atypical of degree 1, $L^{(3)}_\lambda\otimes L^{(4)}_\mu$ is isomorphic to*

$$
\begin{array}{lll}
\lambda_i,\mu_j\in 2\mathbb{Z}+1: & P^{(8)}_{\lambda+\mu}\oplus L^{(4)}_{\lambda+\mu-\alpha_k} & (i\in\{1,2\},\ \{i,j,k\}=\{1,2,3\}),\\[4pt]
\lambda_i,\mu_i\in 2\mathbb{Z}+1: & L^{(4)}_{\lambda+\mu}\oplus P^{(8)}_{\lambda+\mu-\alpha_j} & (\{i,j\}=\{1,2\}).
\end{array}\qquad (3.31)
$$

$K_\lambda^{(16)}$ $L_\lambda^{(3)}$ $L_{\lambda-\alpha_3}^{(3)}$ $R_\lambda^{(8)}$ $L_\lambda^{(1)}$

$L_{\lambda+\alpha_i}^{(1)}$ $L_{\lambda-\alpha_i}^{(1)}$ $L_{\lambda+\alpha_i-2\alpha_3}^{(1)}$ $L_{\lambda-\alpha_i-2\alpha_3}^{(1)}$ $L_{\lambda+\alpha_j+\alpha_3}^{(3)}$ $L_{\lambda-\alpha_j}^{(3)}$

$L_\lambda^{(3)}$ $L_{\lambda-\alpha_3}^{(3)}$ $L_\lambda^{(1)}$

$S_\lambda^{(15)}$ $L_{\lambda-\alpha_1}^{(3)}$ $L_{\lambda-\alpha_2}^{(3)}$ $N_\lambda^{(9)}$ $L_\lambda^{(3)}$

$L_{\lambda-2\alpha_1}^{(1)}$ $L_\lambda^{(1)}$ $L_{\lambda-2\alpha_3}^{(1)}$ $L_{\lambda-2\alpha_2}^{(1)}$ $L_{\lambda+\alpha_i}^{(1)}$ $L_{\lambda-\alpha_i}^{(1)}$ $L_{\lambda+\alpha_i-2\alpha_3}^{(1)}$

$L_{\lambda-\alpha_1}^{(3)}$ $L_{\lambda-\alpha_2}^{(3)}$ $L_\lambda^{(3)}$
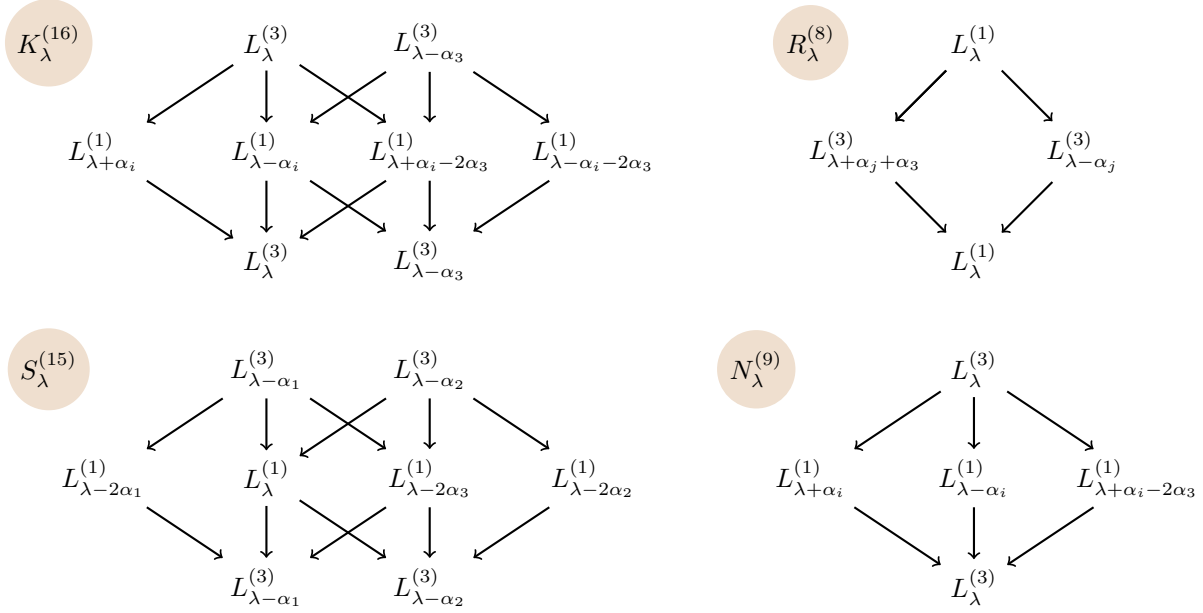
Figure 4: The Loewy diagrams of the indecomposable $\overline{\mathcal{U}}_i^H(\mathfrak{sl}_3)$-modules appearing in Proposition 3.12.

- *Given $\lambda, \mu \in \mathfrak{h}^*$ atypical of degree 2 with $\lambda_i, \lambda_3, \mu_j, \mu_3 \in 2\mathbb{Z}+1$, where $\{i,j\} = \{1,2\}$, one has*

$$L_\lambda^{(3)} \otimes L_\mu^{(3)} \cong P_{\lambda+\mu}^{(8)} \oplus L_{\lambda+\mu-\alpha_3}^{(1)}. \tag{3.32}$$

The remaining tensor products are $L_\lambda^{(4)} \otimes L_\mu^{(4)}$, with $\lambda + \mu$ atypical of degree 2, and $L_\lambda^{(3)} \otimes L_\mu^{(3)}$, with $\lambda_i, \lambda_3, \mu_i, \mu_3 \in 2\mathbb{Z}+1$. It seems that there is no easy way to identify these decompositions except by brute force computation. We outline the results below.

Proposition 3.12.

- *Given $\lambda, \mu \in \mathfrak{h}^*$ atypical of degree 1 and $\lambda + \mu$ atypical of degree of 2, $L_\lambda^{(4)} \otimes L_\mu^{(4)}$ is isomorphic to*

$$\begin{array}{lll}
\lambda_i, \mu_i, (\lambda+\mu)_i, (\lambda+\mu)_j \in 2\mathbb{Z}+1: & K_{\lambda+\mu-\alpha_j}^{(16)} & (i,j \in \{1,2\}), \\
\lambda_i, \mu_i, (\lambda+\mu)_i, (\lambda+\mu)_3 \in 2\mathbb{Z}+1: & R_{\lambda+\mu-\alpha_j-\alpha_3}^{(8)} \oplus P_{\lambda+\mu-\alpha_j}^{(8)} & (i,j \in \{1,2\}), \\
\lambda_3, \mu_3, (\lambda+\mu)_1, (\lambda+\mu)_2 \in 2\mathbb{Z}+1: & S_{\lambda+\mu}^{(16)},
\end{array} \tag{3.33}$$

where $K_{\lambda+\mu-\alpha_j}^{(16)}$ is an indecomposable quotient of $P_{\lambda+\mu-\alpha_j}^{(24)} \oplus P_{\lambda+\mu-\alpha_j-\alpha_3}^{(24)}$, $R_{\lambda+\mu-\alpha_j-\alpha_3}^{(8)}$ is an indecomposable quotient of $P_{\lambda+\mu-\alpha_j-\alpha_3}^{(48)}$ and $S_{\lambda+\mu}^{(16)}$ is an indecomposable quotient of $P_{\lambda+\mu-\alpha_1}^{(24)} \oplus P_{\lambda+\mu-\alpha_2}^{(24)}$. Their (unique) Loewy diagrams are pictured in Figure 4.

- *Given $\lambda, \mu \in \mathfrak{h}^*$ atypical of degree 2 with $\lambda_i, \lambda_3, \mu_i, \mu_3 \in 2\mathbb{Z}+1$, for some $i \in \{1,2\}$, one has*

$$L_\lambda^{(3)} \otimes L_\mu^{(3)} \cong N_{\lambda+\mu-\alpha_j}^{(9)}, \tag{3.34}$$

where $N_{\lambda+\mu-\alpha_j}^{(9)}$ is an indecomposable quotient of $P_{\lambda+\mu-\alpha_j}^{(24)}$, $\{i,j\} = \{1,2\}$. Its (unique) Loewy diagram is again pictured in Figure 4.

Proof. We outline the argument resulting in (3.34), the other tensor products being similar though slightly more intricate. So let $v \in L_\lambda^{(3)} \otimes L_\mu^{(3)}$ have weight $\lambda + \mu$. Then, v is maximal and $(\lambda + \mu)_1, (\lambda + \mu)_2 \in 2\mathbb{Z} + 1$, hence there is a composition factor $L_{\lambda+\mu}^{(1)}$. Using the explicit coproduct of (2.1b) and the structure of the irreducibles worked out in the proof of Proposition 3.1, one computes that $X_{-1}v = 0$ but $X_{-2}v \neq 0$. Indeed, $X_{-2}v$ is maximal and so there is a composition factor $L_{\lambda+\mu-\alpha_2}^{(3)}$. From the structure of this factor, it follows that $X_{-2}X_{-1}X_{-2}v \neq 0$. However, $(X_{-1}X_{-2})^2 v = (X_{-2}X_{-1})^2 v = 0$ and so v generates a 4-dimensional submodule V of $L_\lambda^{(3)} \otimes L_\mu^{(3)}$.

Next, note that $\lambda + \mu - \alpha_2$ has multiplicity 2 in $L_\lambda^{(3)} \otimes L_\mu^{(3)}$. Explicit computation shows that X_2 does not act as 0 on the corresponding weight space. In other words, there exists w of this weight with $X_2 w = v$. Since $X_1 w = 0$ by weight considerations, $[w]$ is maximal in $(L_\lambda^{(3)} \otimes L_\mu^{(3)})/V$ and so gives a second composition factor $L_{\lambda+\mu-\alpha_2}^{(3)}$. It follows that $X_{-1}[w], X_{-2}X_{-1}[w] \neq 0$. However, explicit calculation shows that $X_{-2}[w]$ and $X_{-1}X_{-2}X_{-1}[w]$ are also non-zero, indicating that $[w]$ does not generate an irreducible submodule of $(L_\lambda^{(3)} \otimes L_\mu^{(3)})/V$. Indeed, these calculations imply that $(L_\lambda^{(3)} \otimes L_\mu^{(3)})/V$, and hence $L_\lambda^{(3)} \otimes L_\mu^{(3)}$, has $L_{\lambda+\mu-2\alpha_2}^{(1)}$ and $L_{\lambda+\mu-2\alpha_3}^{(1)}$ as composition factors.

By dimensional considerations, these complete the set of composition factors of $L_\lambda^{(3)} \otimes L_\mu^{(3)}$. To complete the description of the structure of this module, one checks that $X_{-1}X_{-2}w$ and $X_1 X_{-1} X_{-2} X_{-1} w$ are both non-zero multiples of $X_{-2}X_{-1}X_{-2}v$. With this, it is easy to compute the radical and socle series of $L_\lambda^{(3)} \otimes L_\mu^{(3)}$ (which again coincide) and draw the Loewy diagram. Finally, the fact that $L_\lambda^{(3)} \otimes L_\mu^{(3)}$ may be realised as a quotient of $P_{\lambda+\mu-\alpha_2}^{(24)}$ follows from the latter's role as the projective cover of the head of the former. $\square$

4 Categories related to $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$

In this section, a Kazhdan–Lusztig correspondence is conjectured between the category of finite-dimensional weight $\overline{\mathcal{U}}_i^H(\mathfrak{sl}_3)$-modules and a module category of a vertex operator algebra. More precisely, the conjecture is that this correspondence takes the form of an equivalence of braided tensor categories. The vertex operator algebra appearing in this correspondence is not $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$, but its parafermionic coset. This is of course the commutant of the rank-2 Heisenberg subalgebra of $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$ generated by the fields associated with the Cartan subalgebra of $\mathfrak{sl}_3$.

The reason why one might expect a Kazhdan–Lusztig correspondence involving this parafermionic vertex operator algebra is that it has recently been shown [Ad16, AMW22] to be isomorphic to $W_{A_2}^0(2)$, a rank-2 analogue of the singlet algebras $W_{A_1}^0(p)$, $p \in \mathbb{Z}_{\geq 2}$ [Ka91]. This isomorphism extends the known coincidences between the $L_k(\mathfrak{sl}_2)$-parafermions, with $k = -\frac{1}{2}$ and $-\frac{4}{3}$, and the $W_{A_1}^0(p)$, with $p = 2, 3$ respectively [Ad05, Ri11] (as well as between $k = -\frac{2}{3}$ and the bosonic orbifold of an $N = 1$ "supersinglet" algebra [Ad16, ACR18]).

$W_{A_2}^0(2)$ is one of a family, parametrised by $p \in \mathbb{Z}_{\geq 2}$ and an ADE Dynkin diagram, of vertex algebras introduced by Feigin and Tipunin [FT10], see also [CM17]. Aside from the A_1 case (the singlet algebras), the representation theories of these vertex operators algebras have not yet received much attention. However, that of the singlet algebras is very well understood, see [AM07, AM09, TW15, RW14, CMY20b]. In this case, a conjectural braided tensor equivalence has been proposed [CGP14, CM14] and studied [CMR18, CGR20, GN21, CLR21]. It connects a category of modules over $W_{A_1}^0(p)$ with the category of finite-dimensional weight modules of the unrolled restricted quantum group $\overline{\mathcal{U}}_q^H(\mathfrak{sl}_2)$ with $q = e^{\pi i/p}$.

It is thus natural to conjecture [CR22] that this equivalence generalises to higher ranks and, in particular,

to a braided tensor equivalence between the category of finite-dimensional weight $\overline{\mathcal{U}}_i^H(\mathfrak{sl}_3)$-modules and an appropriate category of $W_{A_2}^0(2)$-modules. Here, this conjecture is made explicit by comparing with the representation theory of the simple affine vertex operator algebra $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$. Consequences for other related vertex operator algebra categories are also explored.

4.1 Representation theory of $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$

The simple affine vertex operator algebra $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$ has central charge -8. A list of its irreducible modules includes the following:

- Four highest-weight modules $\widehat{\mathcal{L}}_0$, $\widehat{\mathcal{L}}_{-\frac{3}{2}\omega_1}$, $\widehat{\mathcal{L}}_{-\frac{3}{2}\omega_2}$ and $\widehat{\mathcal{L}}_{-\frac{1}{2}\rho}$. These are highest-weight with respect to the standard Borel of $\widehat{\mathfrak{sl}}_3$; the subscript indicates the projection of the highest weight onto $\mathfrak{h}$. Their ground states have conformal weight 0, $-\frac{1}{2}$, $-\frac{1}{2}$ and $-\frac{1}{2}$, respectively.

- A single family of semirelaxed highest-weight modules $\widehat{\mathcal{S}}_{[\mu]}$, for all $[\mu] \in (-\frac{3}{2}\omega_1 + \mathbb{C}\alpha_1)/\mathbb{Z}\alpha_1$ except for $[\mu] = [-\frac{3}{2}\omega_1]$ and $[\mu] = [-\frac{1}{2}\rho]$. The root vectors of $\mathfrak{sl}_3$ corresponding to $\pm\alpha_1$, $-\alpha_2$ and $-\alpha_3$ act injectively while those for α_2 and α_3 act locally nilpotently. The $\mathfrak{sl}_3$-weights of the ground states constitute the set $\mu + \mathbb{Z}\alpha_1 - \mathbb{Z}_{\geq 0}\alpha_2 - \mathbb{Z}_{\geq 0}\alpha_3$; these always have conformal weight $-\frac{1}{2}$.

- A single family of relaxed highest-weight modules $\widehat{\mathcal{R}}_{[\mu]}$, for all $[\mu] \in \mathfrak{h}^*/Q$ except

$$[\mu] \in [-\tfrac{3}{2}\omega_1 + \mathbb{C}\alpha_1] \cup [-\tfrac{1}{2}\rho + \mathbb{C}\alpha_2] \cup [-\tfrac{3}{2}\omega_1 + \mathbb{C}\alpha_3]. \tag{4.1}$$

All root vectors of $\mathfrak{sl}_3$ act injectively and the set of $\mathfrak{sl}_3$-weights of the ground states is $\mu + Q$; these also always have conformal weight $-\frac{1}{2}$.

The highest-weight modules were originally classified in [Pe08], while the relaxed highest-weight modules were constructed, but not proved to be exhaustive, in [Ad16]. A complete classification was first obtained in [AFR17] using Gelfand-Tsetlin methods, see also [KR19] for an alternative classification based on Mathieu's coherent families [Ma00].

A complete set of irreducible positive-energy weight $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules, with finite-dimensional weight spaces, is obtained from the irreducibles listed above by twisting by the automorphisms of $\mathfrak{sl}_3$ that preserve the chosen Cartan subalgebra. These automorphisms form a group isomorphic to the dihedral group D_6. Similarly, a complete set of irreducible weight $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules, with finite-dimensional weight spaces, is obtained by also twisting by the spectral flow automorphisms σ^ω, $\omega \in P$, of $\widehat{\mathfrak{sl}}_3$. Recall that the latter are defined by

$$\sigma^\omega(e_n^\alpha) = e_{n-\langle\alpha,\omega\rangle}^\alpha, \quad \sigma^\omega(h_n) = h_n - \langle\omega,h\rangle\delta_{n,0}K, \quad \sigma^\omega(K) = K. \tag{4.2}$$

Occasionally, a given D_6- or spectral flow twist will relate two of the irreducibles listed above, for example $\sigma^{\omega_i}(\widehat{\mathcal{L}}_0) \cong \widehat{\mathcal{L}}_{-3\omega_i/2}$ for $i = 1, 2$. However, most twists result in new irreducibles. A complete list of the exceptions may be found in [KRW21, Equation (5.9)].

Denote the category of irreducible positive-energy weight $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules, with finite-dimensional weight spaces and *real* weights, by $\mathscr{D}$. The list [KRW21, Equation (5.9)] of exceptions shows that the irreducibles in $\mathscr{D}$ may all be realised as spectral flows of the following irreducibles:

$$\widehat{\mathcal{L}}_0, \quad \widehat{\mathcal{L}}_{-\frac{1}{2}\rho}, \quad c(\widehat{\mathcal{L}}_{-\frac{1}{2}\rho}), \quad \widehat{\mathcal{S}}_{[\mu]}, \quad w_2(\widehat{\mathcal{S}}_{[\mu]}), \quad w_1 w_2(\widehat{\mathcal{S}}_{[\mu]}), \quad \widehat{\mathcal{R}}_{[\mu]}. \tag{4.3}$$

Here, $\mathrm{w}_i \in D_6$, $i = 1, 2, 3$, is the Weyl reflection of $\mathfrak{sl}_3$ corresponding to α_i and $\mathrm{c} \in D_6$ is the conjugation automorphism (which acts on the Cartan subalgebra of $\mathfrak{sl}_3$ as multiplication by -1).

The relaxed family $\widehat{\mathcal{R}}_{[\mu]}$ also includes reducible $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules. Writing $\omega_3 = \omega_2 - \omega_1$, their composition factors were determined in [KRW21, Equations (5.11–12)] and are as follows:

$$
\begin{aligned}
\mu \in -\tfrac{3}{2}\omega_1 + \mathbb{C}\alpha_1, \quad [\mu] \neq [-\tfrac{3}{2}\omega_1], [-\tfrac{1}{2}\rho]: \quad & \widehat{\mathcal{S}}_{[\mu]}, \ \sigma^{-\omega_2}(\widehat{\mathcal{S}}_{[\mu - \frac{1}{2}\alpha_1]}), \\
\mu \in -\tfrac{1}{2}\rho + \mathbb{C}\alpha_2, \quad [\mu] \neq [-\tfrac{1}{2}\rho], [-\tfrac{3}{2}\omega_2]: \quad & \mathrm{w}_1\mathrm{w}_2(\widehat{\mathcal{S}}_{[\mathrm{w}_2 \cdot \mathrm{w}_1(\mu)]}), \ \sigma^{\omega_1}\mathrm{w}_1\mathrm{w}_2(\widehat{\mathcal{S}}_{[\mathrm{w}_2\mathrm{w}_1 \cdot \mu + \frac{3}{2}\omega_2]}), \\
\mu \in -\tfrac{3}{2}\omega_1 + \mathbb{C}\alpha_3, \quad [\mu] \neq [-\tfrac{3}{2}\omega_1], [-\tfrac{3}{2}\omega_2]: \quad & \mathrm{w}_2(\widehat{\mathcal{S}}_{[\mathrm{w}_2(\mu)]}), \ \sigma^{\omega_3}\mathrm{w}_2(\widehat{\mathcal{S}}_{[\mathrm{w}_2 \cdot \mu + \frac{3}{2}\omega_2]}),
\end{aligned} \tag{4.4a}
$$

$$
\begin{aligned}
\mu = [-\tfrac{3}{2}\omega_1]: \quad & \sigma^{\omega_1}(\widehat{\mathcal{L}}_0), \ \sigma^{-\omega_1}(\widehat{\mathcal{L}}_0), \ \sigma^{\omega_2}\mathrm{c}(\widehat{\mathcal{L}}_{-\frac{1}{2}\rho}), \ \sigma^{-\omega_2}(\widehat{\mathcal{L}}_{-\frac{1}{2}\rho}), \\
\mu = [-\tfrac{1}{2}\rho]: \quad & \widehat{\mathcal{L}}_{-\frac{1}{2}\rho}, \ \mathrm{c}(\widehat{\mathcal{L}}_{-\frac{1}{2}\rho}), \ \sigma^{\omega_3}(\widehat{\mathcal{L}}_0), \ \sigma^{-\omega_3}(\widehat{\mathcal{L}}_0), \\
\mu = [-\tfrac{3}{2}\omega_2]: \quad & \sigma^{\omega_2}(\widehat{\mathcal{L}}_0), \ \sigma^{-\omega_2}(\widehat{\mathcal{L}}_0), \ \sigma^{\omega_1}\mathrm{c}(\widehat{\mathcal{L}}_{-\frac{1}{2}\rho}), \ \sigma^{-\omega_1}(\widehat{\mathcal{L}}_{-\frac{1}{2}\rho}).
\end{aligned} \tag{4.4b}
$$

Here, every composition factor has been expressed as a spectral flow of an irreducible in (4.3).

Grothendieck fusion rules for all irreducibles in $\mathscr{D}$ were computed in [KRW21] using a conjectural Verlinde formula [CR13a, RW15b]. As Grothendieck fusion respects D_6-twists and spectral flow,

$$
\begin{aligned}
\mathrm{w}(M) \,\widehat{\boxtimes}\, \mathrm{w}(N) &= \mathrm{w}(M \,\widehat{\boxtimes}\, N), & \mathrm{w} &\in D_6, \\
\sigma^\omega(M) \,\widehat{\boxtimes}\, \sigma^{\omega'}(N) &= \sigma^{\omega+\omega'}(M \,\widehat{\boxtimes}\, N), & \omega, \omega' &\in P,
\end{aligned} \tag{4.5}
$$

it suffices to list the following rules [KRW21, Equations (5.54, 56, 58, 61, 62, 64, 65 and 68)]:

$$
\widehat{\mathcal{L}}_{-\frac{1}{2}\rho} \,\widehat{\boxtimes}\, \widehat{\mathcal{L}}_{-\frac{1}{2}\rho} = \widehat{\mathcal{L}}_0 + \sigma^{2\omega_1}(\widehat{\mathcal{L}}_0) + \sigma^{2\omega_2}(\widehat{\mathcal{L}}_0) + 2\,\sigma^\rho\mathrm{c}(\widehat{\mathcal{L}}_{-\frac{1}{2}\rho}), \tag{4.6a}
$$

$$
\widehat{\mathcal{L}}_{-\frac{1}{2}\rho} \,\widehat{\boxtimes}\, \mathrm{c}(\widehat{\mathcal{L}}_{-\frac{1}{2}\rho}) = \widehat{\mathcal{L}}_0 + \widehat{\mathcal{R}}_{[0]}, \tag{4.6b}
$$

$$
\widehat{\mathcal{L}}_{-\frac{1}{2}\rho} \,\widehat{\boxtimes}\, \widehat{\mathcal{S}}_{[\mu]} = \sigma^{\omega_1}(\widehat{\mathcal{S}}_{[\mu + \frac{1}{2}\alpha_1]}) + \sigma^{\omega_2}(\widehat{\mathcal{R}}_{[\mu + \frac{1}{2}\alpha_2]}), \tag{4.6c}
$$

$$
\widehat{\mathcal{L}}_{-\frac{1}{2}\rho} \,\widehat{\boxtimes}\, \widehat{\mathcal{R}}_{[\mu]} = \widehat{\mathcal{R}}_{[\mu - \frac{1}{2}\rho]} + \sigma^{\omega_1}(\widehat{\mathcal{R}}_{[\mu + \frac{1}{2}\alpha_1]}) + \sigma^{\omega_2}(\widehat{\mathcal{R}}_{[\mu + \frac{1}{2}\alpha_2]}), \tag{4.6d}
$$

$$
\widehat{\mathcal{S}}_{[\lambda]} \,\widehat{\boxtimes}\, \widehat{\mathcal{S}}_{[\mu]} = \sigma^{\omega_1}(\widehat{\mathcal{S}}_{[\lambda + \mu + \frac{3}{2}\omega_1]}) + \sigma^{\omega_2}(\widehat{\mathcal{R}}_{[\lambda + \mu + \frac{3}{2}\omega_2]}) + \sigma^{\omega_3}(\widehat{\mathcal{S}}_{[\lambda + \mu + \frac{3}{2}\omega_3]}), \tag{4.6e}
$$

$$
\widehat{\mathcal{S}}_{[\lambda]} \,\widehat{\boxtimes}\, \mathrm{w}_2(\widehat{\mathcal{S}}_{[\mu]}) = \widehat{\mathcal{R}}_{[\lambda + \mathrm{w}_2(\mu)]} + \sigma^{\omega_1}(\widehat{\mathcal{R}}_{[\lambda + \mathrm{w}_2(\mu) + \frac{3}{2}\omega_1]}), \tag{4.6f}
$$

$$
\widehat{\mathcal{S}}_{[\lambda]} \,\widehat{\boxtimes}\, \mathrm{w}_1\mathrm{w}_2(\widehat{\mathcal{S}}_{[\mu]}) = \widehat{\mathcal{R}}_{[\lambda + \mathrm{w}_1\mathrm{w}_2(\mu)]} + \sigma^{\omega_3}(\widehat{\mathcal{R}}_{[\lambda + \mathrm{w}_1\mathrm{w}_2(\mu) + \frac{3}{2}\omega_3]}), \tag{4.6g}
$$

$$
\widehat{\mathcal{S}}_{[\lambda]} \,\widehat{\boxtimes}\, \widehat{\mathcal{R}}_{[\mu]} = \widehat{\mathcal{R}}_{[\lambda + \mu]} + \sum_{i=1}^3 \sigma^{\omega_i}(\widehat{\mathcal{R}}_{[\lambda + \mu + \frac{3}{2}\omega_i]}), \tag{4.6h}
$$

$$
\widehat{\mathcal{R}}_{[\lambda]} \,\widehat{\boxtimes}\, \widehat{\mathcal{R}}_{[\mu]} = 2\,\widehat{\mathcal{R}}_{[\lambda + \mu]} + \sum_{i=1}^3 \left(\sigma^{\omega_i}(\widehat{\mathcal{R}}_{[\lambda + \mu + \frac{3}{2}\omega_i]}) + \sigma^{-\omega_i}(\widehat{\mathcal{R}}_{[\lambda + \mu + \frac{3}{2}\omega_i]}) \right). \tag{4.6i}
$$

The restriction made above, that modules in $\mathscr{D}$ have real weights, is purely technical in nature. However, it is worth noting that the decompositions (4.6) were also deduced in [KRW21] under the assumption that all weights are real.

4.2 A parafermionic coset

The vertex operator algebra $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$ has a rank-2 Heisenberg subalgebra H generated by the fields associated with the Cartan subalgebra $\mathfrak{h}$. As noted above, the commutant of this subalgebra in $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$ is known [Ad16, AMW22] to be isomorphic to the Feigin–Tipunin algebra $W^0_{A_2}(2)$. The consequent conformal embedding

$W^0_{A_2}(2) \otimes H \hookrightarrow L_{-\frac{3}{2}}(\mathfrak{sl}_3)$ then leads to decompositions of $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules as $W^0_{A_2}(2) \otimes H$-modules.

To analyse these decompositions, the following conjecture is useful. It is understood to be in force for all that follows.

Conjecture 4.1. *The category $\mathscr{D}$ of weight $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules with finite-dimensional weight spaces and real weights admits the structure of a vertex tensor category. Furthermore, the tensor (fusion) product $\boxtimes$ of $\mathscr{D}$ descends to the Grothendieck group and the resulting Grothendieck fusion rules are as in (4.5) and (4.6).*

Assuming this conjecture, the results of [CKLR19] for general Heisenberg commutants imply that the submodule structure of a given module in $\mathscr{D}$ is shared by each of the $W^0_{A_2}(2)$-modules appearing in its decomposition. Moreover, they also imply that $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$ decomposes into simple currents:

$$L_{-\frac{3}{2}}(\mathfrak{sl}_3) \cong \bigoplus_{\lambda \in Q} E^{(1)}_\lambda \otimes F_\lambda, \qquad E^{(1)}_\lambda \boxtimes E^{(1)}_\mu \cong E^{(1)}_{\lambda+\mu}. \tag{4.7}$$

Here, F_λ denotes the Fock module of H of highest weight λ and $E^{(1)}_\lambda$, $\lambda \in Q$, is an irreducible $W^0_{A_2}(2)$-module.

The natural $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules with which to start this decomposition game are the relaxed highest-weight $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules $\widehat{\mathcal{R}}_{[\mu]} \in \mathscr{D}$, because these (along with their spectral flows) are the *standard modules* in the sense of [CR13a, RW15b]. In particular, every other irreducible weight module appears as a composition factor of a standard one, see (4.4). As the $\mathfrak{sl}_3$-weights of $\widehat{\mathcal{R}}_{[\mu]}$ are precisely the elements of $[\mu] = \mu + Q$, it decomposes into modules over $W^0_{A_2}(2) \otimes H$ as follows:

$$\widehat{\mathcal{R}}_{[\mu]} \cong \bigoplus_{\lambda \in \mu+Q} E^{(8)}_\lambda \otimes F_\lambda. \tag{4.8}$$

Here, the $E^{(8)}_\lambda$ are $W^0_{A_2}(2)$-modules satisfying $E^{(8)}_\lambda \boxtimes E^{(1)}_\mu \cong E^{(8)}_{\lambda+\mu}$ [CKLR19].

The character of $E^{(8)}_\lambda$ is now easily deduced from [KRW21, Equation (5.14)]:

$$\mathrm{ch}[\widehat{\mathcal{R}}_{[\mu]}] = \frac{1}{\eta(q)^4} \sum_{\lambda \in \mu+Q} z^\lambda = \sum_{\lambda \in \mu+Q} \frac{q^{\frac{1}{3}\|\lambda\|^2}}{\eta(q)^2} \frac{z^\lambda q^{-\frac{1}{3}\|\lambda\|^2}}{\eta(q)^2} \quad \Rightarrow \quad \mathrm{ch}[E^{(8)}_\lambda] = \frac{q^{\frac{1}{3}\|\lambda\|^2}}{\eta(q)^2}. \tag{4.9}$$

This suggests that $E^{(8)}_\lambda$ may be constructed as another rank-2 Fock module, albeit one whose central charge is -10 (this free field approach to $W^0_{A_2}(2)$ is that originally used in [FT10]). Applying [KRW21, Equation (5.4)] now gives the character decomposition of a general standard module:

$$\mathrm{ch}[\sigma^\omega(\widehat{\mathcal{R}}_{[\mu]})] = \sum_{\lambda \in \mu+Q} \frac{q^{\frac{1}{3}\|\lambda\|^2}}{\eta(q)^2} \frac{z^{\lambda-\frac{3}{2}\omega} q^{-\frac{1}{3}\|\lambda-\frac{3}{2}\omega\|^2}}{\eta(q)^2} = \sum_{\lambda \in \mu+Q} \mathrm{ch}[E^{(8)}_\lambda] \, \mathrm{ch}[F_{\lambda-\frac{3}{2}\omega}]. \tag{4.10}$$

When $\widehat{\mathcal{R}}_{[\mu]}$ is irreducible (typical), this shows that

$$\sigma^\omega(\widehat{\mathcal{R}}_{[\mu]}) \cong \bigoplus_{\lambda \in \mu+Q} E^{(8)}_\lambda \otimes F_{\lambda-\frac{3}{2}\omega}, \tag{4.11}$$

hence that spectral flow only changes the Fock module labels. The same must then be true for the atypical $\widehat{\mathcal{R}}_{[\mu]}$ as well.
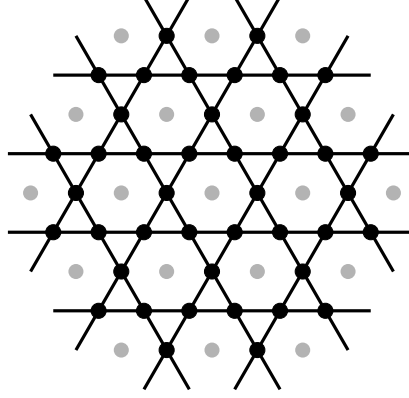
Figure 5: The atypical weights $\lambda \in \mathfrak{h}^*$ of the $W^0_{A_2}(2)$-modules $E^{(8)}_\lambda$, pictured with 0 at the centre. Grey dots indicate typical weights in Q, black lines indicate atypical weights of degree 1 and black dots are atypical weights of degree 2. The highest root α_3 is the grey dot closest to 0 along the ray $30°$ to the right of the upwards vertical.

4.3   A logarithmic Kazhdan–Lusztig correspondence

Recall that $\widehat{\mathcal{R}}_{[\mu]}$ is typical unless $[\mu] \in \mathfrak{h}^*/Q$ belongs to the set in (4.1). The $E^{(8)}_\lambda$ appearing in (4.8) are then irreducible (typical) unless

$$\lambda \in \bigcup_{\substack{i,j=1 \\ i \neq j}}^{3} \left((\mathbb{Z} + \tfrac{1}{2})\alpha_i + \mathbb{R}\alpha_j \right) \tag{4.12}$$

(recalling the restriction to real weights in the definition of $\mathscr{D}$). These typicality conditions are illustrated in Figure 5. Crucially, the conditions for $E^{(8)}_\lambda$ match those given in Figure 1 for the Verma modules $M_{\lambda'}$ of $\overline{\mathcal{U}}^H_{\mathsf{i}}(\mathfrak{sl}_3)$, if λ and λ' are related by the affine transformation $\lambda' = \varphi(\lambda) + \rho$ of $\mathfrak{h}^*$. Here, φ is the linear map satisfying

$$\varphi(\alpha_1) = 2\omega_1, \quad \varphi(\alpha_2) = 2\omega_3 \quad \text{and} \quad \varphi(\alpha_3) = 2\omega_2. \tag{4.13}$$

This suggests the following (partial) Kazhdan–Lusztig correspondence.

Conjecture 4.2. *Any $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-module decomposes as a direct sum of $W^0_{A_2}(2) \otimes H$ modules. Let $\mathscr{E}$ denote that category of $W^0_{A_2}(2)$-modules which appear in such a decomposition. Then $\mathscr{E}$ is a vertex tensor category and the identification of $E^{(8)}_\lambda$ with $M_{\varphi(\lambda)+\rho}$ partially defines a braided tensor equivalence between $\mathscr{E}$ and the full subcategory $\mathscr{C}_{\mathbb{R}}$ of $\mathscr{C}$ containing the finite-dimensional weight $\overline{\mathcal{U}}^H_{\mathsf{i}}(\mathfrak{sl}_3)$-modules with* real *weights, that is, weights whose image lies in $\mathbb{R}$.*

Of course, if λ is typical for $W^0_{A_2}(2)$, then $\varphi(\lambda) + \rho$ is typical for $\overline{\mathcal{U}}^H_{\mathsf{i}}(\mathfrak{sl}_3)$, hence $M_{\varphi(\lambda)+\rho}$ is irreducible and the identification in Conjecture 4.2 becomes

$$E^{(8)}_\lambda \longleftrightarrow L^{(8)}_{\varphi(\lambda)+\rho}. \tag{4.14}$$

The complete (conjectural) Kazhdan–Lusztig correspondence is obtained by extending this identification to the remaining irreducible $\overline{\mathcal{U}}^H_{\mathsf{i}}(\mathfrak{sl}_3)$-modules in $\mathscr{C}_{\mathbb{R}}$.

Remark 4.3. The category $\mathscr{C}_{\mathbb{R}}$ is ribbon and of course we expect an equivalence of ribbon categories. In a

vertex tensor category the ribbon twist is expected to be given by $e^{2\pi i L_0}$ with L_0 the Virasoro zero-mode, or equivalently by $e^{2\pi i \Delta_M}$ with Δ_M the conformal weight of the top level of the $\mathbb{Z}$-graded module M. Let $\lambda = \lambda_1 \alpha_1 + \lambda_3 \alpha_3$ be generic. The conformal weight of the top level of $E_\lambda^{(8)}$ is

$$\Delta(E_\lambda^{(8)}) = \frac{\lambda^2}{3} - \frac{1}{2} = \frac{2}{3}\left(\lambda_1^2 + \lambda_3^2 + \lambda_1 \lambda_3\right) - \frac{1}{2}.$$

On the other hand the ribbon twist of $L_{\varphi(\lambda)+\rho}^{(8)}$ is given by [Ru22, Eq. (4.10)] and is

$$\theta_{L_{\varphi(\lambda)+\rho}^{(8)}} = e^{\frac{\pi i}{2}\left(\varphi(\lambda)^2 - \rho^2\right)} = -e^{\frac{4\pi i}{3}\left(\lambda_1^2 + \lambda_3^2 + \lambda_1 \lambda_3\right)}$$

and so

$$\theta_{L_{\varphi(\lambda)+\rho}^{(8)}} = e^{2\pi i \Delta(E_\lambda^{(8)})}.$$

To extend the identification in Equation (4.14) to the remaining irreducibles, first note that degree-1 atypicality in $\mathscr{C}$ corresponds to the $\widehat{\mathcal{S}}_{[\mu]}$ and their D_6-twists in $\mathscr{D}$. Decomposing defines (generically irreducible) $W_{A_2}^0(2)$-modules $E_\lambda^{(4)}$:

$$\widehat{\mathcal{S}}_{[\mu]} \cong \bigoplus_{\lambda \in \mu + Q} E_\lambda^{(4)} \otimes F_\lambda, \qquad\qquad \mu \in -\tfrac{3}{2}\omega_1 + \mathbb{R}\alpha_1. \qquad (4.15a)$$

This defines the $E_\lambda^{(4)}$ when $\lambda \in -\tfrac{3}{2}\omega_1 + \mathbb{R}\alpha_1 + Q$. However, twisting by w_2 and $w_1 w_2$ extends this to $\lambda \in -\tfrac{3}{2}\omega_1 + \mathbb{R}\alpha_3 + Q$ and $\lambda \in -\tfrac{1}{2}\rho + \mathbb{R}\alpha_2 + Q$, respectively:

$$w_2(\widehat{\mathcal{S}}_{[w_2(\mu)]}) \cong \bigoplus_{\lambda \in \mu + Q} E_\lambda^{(4)} \otimes F_\lambda, \qquad\qquad \mu \in -\tfrac{3}{2}\omega_1 + \mathbb{R}\alpha_3, \qquad (4.15b)$$

$$w_1 w_2(\widehat{\mathcal{S}}_{[w_2 w_1(\mu)]}) \cong \bigoplus_{\lambda \in \mu + Q} E_\lambda^{(4)} \otimes F_\lambda, \qquad\qquad \mu \in -\tfrac{1}{2}\rho + \mathbb{R}\alpha_2. \qquad (4.15c)$$

For degree-2 atypicality, the decompositions are instead

$$\widehat{\mathcal{L}}_{-\frac{1}{2}\rho} \cong \bigoplus_{\lambda \in -\frac{1}{2}\rho + Q} E_\lambda^{(3)} \otimes F_\lambda, \qquad c(\widehat{\mathcal{L}}_{-\frac{1}{2}\rho}) \cong \bigoplus_{\lambda \in -\frac{1}{2}\rho + Q} E_\lambda^{(\overline{3})} \otimes F_\lambda. \qquad (4.16)$$

Recall also the decomposition (4.7) defining the $E_\lambda^{(1)}$, $\lambda \in Q$. In all cases, [CKLR19] gives

$$E_\lambda^{(1)} \boxtimes E_\mu^{(d)} \cong E_{\lambda+\mu}^{(d)}, \quad d = 1, 3, 4, 8. \qquad (4.17)$$

The full Kazhdan–Lusztig correspondence is as follows.

Conjecture 4.4. *There is a braided tensor equivalence between $\mathscr{E}$ and $\mathscr{C}_\mathbb{R}$ defined by*

$$E_\lambda^{(8)} \longleftrightarrow L_{\varphi(\lambda)+\rho}^{(8)} \qquad \text{for generic } \lambda, \qquad (4.18a)$$

$$E_\lambda^{(4)} \longleftrightarrow \begin{cases} L_{\varphi(\lambda)+\rho}^{(4)} & \text{for generic } \lambda \text{ in } -\tfrac{3}{2}\omega_1 + \mathbb{R}\alpha_1 + Q \text{ or } -\tfrac{3}{2}\omega_1 + \mathbb{R}\alpha_3 + Q, \\ L_{\varphi(\lambda)}^{(4)} & \text{for generic } \lambda \text{ in } -\tfrac{1}{2}\rho + \mathbb{R}\alpha_2 + Q, \end{cases} \qquad (4.18b)$$

$$E_\lambda^{(3)} \longleftrightarrow L_{\varphi(\lambda)+\rho}^{(3)}, \quad E_\lambda^{(\bar{3})} \longleftrightarrow L_{\varphi(\lambda)}^{(3)} \qquad \text{for } \lambda \in -\frac{1}{2}\rho + Q, \tag{4.18c}$$

$$E_\lambda^{(1)} \longleftrightarrow L_{\varphi(\lambda)}^{(1)} \qquad \text{for } \lambda \in Q. \tag{4.18d}$$

This conjecture will be tested in what follows by comparing the tensor products of $\mathscr{C}_\mathbb{R}$, computed in Section 3.3, with the Grothendieck fusion rules of $\mathscr{D}$, computed in [KRW21] and listed in (4.6).

This correspondence predicts Loewy diagrams for the projective covers of the irreducible $W_{A_2}^0(2)$-modules in $\mathscr{E}$. More precisely, the prediction is that the irreducible $E_\mu^{(8)}$ are already projective and for the projective covers of the other irreducibles, Loewy diagrams are obtained by replacing each composition factor in Figure 3 by its image in $\mathscr{E}$ under the equivalence (4.18), see Figure 6. For this, it is convenient to invert (4.18):

$$L_\mu^{(8)} \longleftrightarrow E_{\varphi^{-1}(\mu-\rho)}^{(8)} \qquad \text{for generic } \mu, \tag{4.19a}$$

$$L_\mu^{(4)} \longleftrightarrow \begin{cases} E_{\varphi^{-1}(\mu-\rho)}^{(8)} & \text{for generic } \mu \text{ in } \mathbb{R}\omega_1 + 2P \text{ or } \mathbb{R}\omega_2 + 2P, \\ E_{\varphi^{-1}(\mu)}^{(8)} & \text{for generic } \mu \text{ in } \omega_1 + \mathbb{R}\omega_3 + 2P = \omega_2 + \mathbb{R}\omega_3 + 2P, \end{cases} \tag{4.19b}$$

$$L_\mu^{(3)} \longleftrightarrow E_{\varphi^{-1}(\mu-\rho)}^{(3)} \quad \text{for } \mu \in \omega_1 + 2P, \qquad L_\mu^{(3)} \longleftrightarrow E_{\varphi^{-1}(\mu)}^{(\bar{3})} \quad \text{for } \mu \in \omega_2 + 2P, \tag{4.19c}$$

$$L_\mu^{(1)} \longleftrightarrow E_{\varphi^{-1}(\mu)}^{(1)} \quad \text{for } \mu \in 2P. \tag{4.19d}$$

To illustrate this replacement process, let $Q_\lambda^{(24)}$ denote the projective cover of $E_\lambda^{(3)}$, $\lambda \in -\frac{1}{2}\rho + Q$. Under the equivalence (4.18), $E_\lambda^{(3)} \longleftrightarrow L_\mu^{(3)}$ so $Q_\lambda^{(24)} \longleftrightarrow P_\mu^{(24)}$, where $\mu = \varphi(\lambda) + \rho \in \omega_1 + 2P$. The composition factors of $P_\mu^{(24)}$ are given in Figure 3. Ignoring repetitions, there are only 7:

$$L_\mu^{(3)}, \quad L_{\mu+\alpha_2}^{(1)}, \quad L_{\mu-\alpha_2}^{(1)}, \quad L_{\mu+\alpha_2-2\alpha_3}^{(1)}, \quad L_{\mu+\alpha_3}^{(3)}, \quad L_{\mu+2\alpha_2-\alpha_3}^{(3)} \quad \text{and} \quad L_{\mu-\alpha_3}^{(3)}. \tag{4.20}$$

Noting that $\mu \pm \alpha_3, \mu + 2\alpha_2 - \alpha_3 \in \omega_2 + 2P$, the corresponding $W_{A_2}^0(2)$-modules under (4.19) are

$$E_\lambda^{(3)}, \quad E_{\lambda+\frac{3}{2}\rho}^{(1)}, \quad E_{\lambda-\frac{3}{2}\omega_3}^{(1)}, \quad E_{\lambda+\frac{3}{2}\omega_3}^{(1)}, \quad E_{\lambda+3\omega_1}^{(\bar{3})}, \quad E_\lambda^{(\bar{3})} \quad \text{and} \quad E_{\lambda+3\omega_2}^{(\bar{3})}, \tag{4.21}$$

respectively. For this, it is convenient to record the following useful facts:

$$\varphi^{-1}(\alpha_1) = -\tfrac{3}{2}\omega_3, \quad \varphi^{-1}(\alpha_2) = \tfrac{3}{2}\omega_2 \quad \text{and} \quad \varphi^{-1}(\alpha_3) = \tfrac{3}{2}\omega_1. \tag{4.22}$$

Making these replacements in the Loewy diagram of $P_\mu^{(24)}$ now gives that of $Q_\lambda^{(24)}$.

Note that the typical, atypical of degree 1 and atypical of degree 2 irreducibles $E_\lambda^{(d)}$ appear in 1, 2 and 3 distinct Loewy layers of its projective cover, suggesting that the Virasoro zero-mode acts on the projectives with Jordan blocks of rank 1, 2 and 3, respectively.

4.4 Reconstructing $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules

Just as $W_{A_2}^0(2)$-modules, tensored with Fock modules, may be obtained by restricting $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules, so too may $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules be obtained by *inducing* $W_{A_2}^0(2)$-modules, again after tensoring with appropriate Fock modules. For the situation at hand, the setup is as follows:

• Restriction defines a functor from the category $\mathscr{D}$ of weight $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules with finite multiplicities and
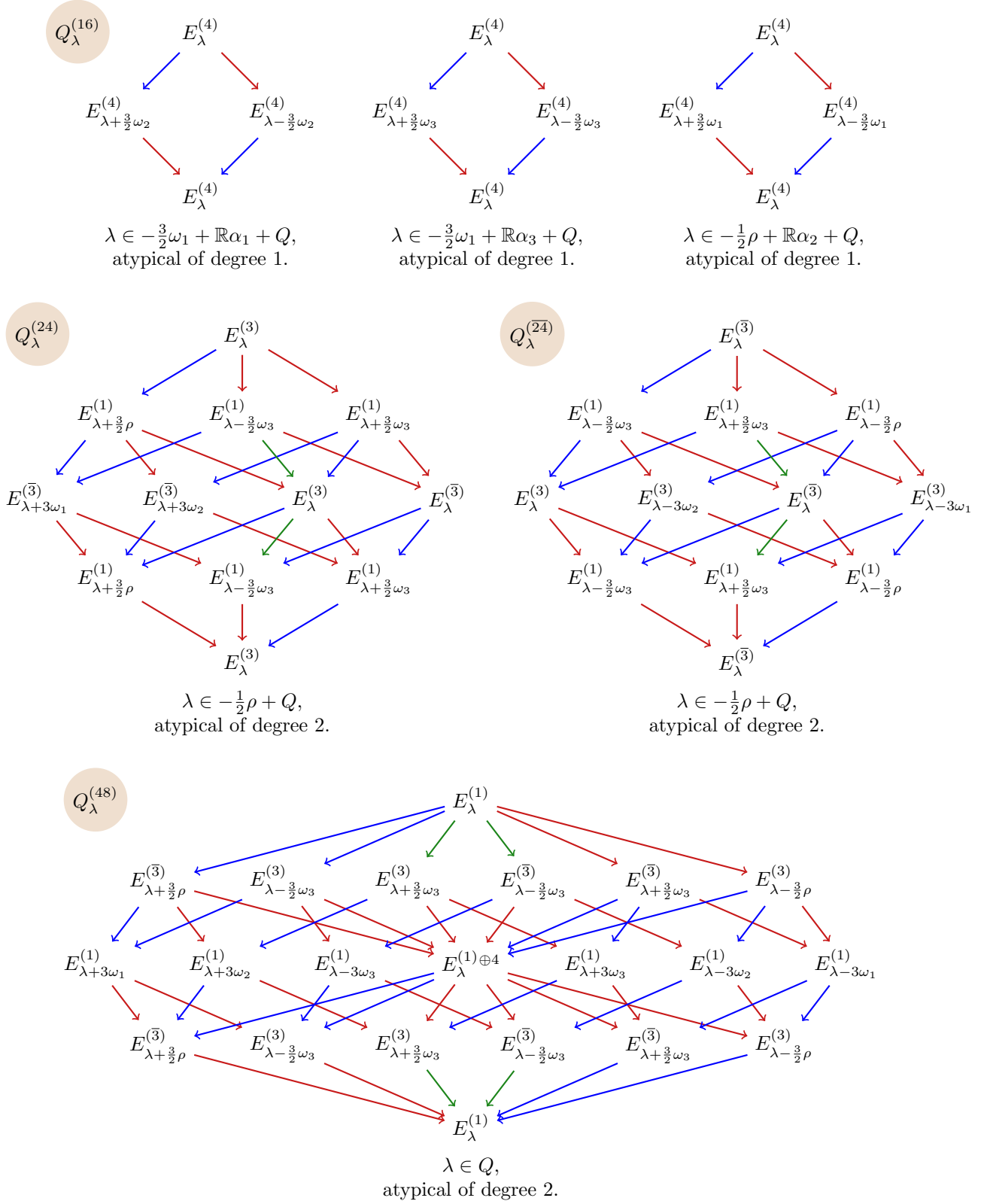
Figure 6: Conjectural Loewy diagrams for the projective covers of the atypical irreducible $W_{A_2}^0(2)$-modules $E_\lambda^{(4)}$, $E_\lambda^{(3)}$, $E_\lambda^{(\overline{3})}$ and $E_\lambda^{(1)}$.

real weights to a direct limit completion [CMY20a] of the Deligne product of the category $\mathscr{E}$ of $W_{A_2}^0(2)$-modules and the category $\mathscr{F}$ of Fock modules over H.

- The Fock modules in $\mathscr{F}$ will be restricted to having *real* highest weights so that $\mathscr{F}$ is a vertex tensor category [CKLR19, Theorem 2.3]. This is the reason for restricting $\mathscr{D}$ to have real weights.

- Conjecture 4.2 assumes that $\mathscr{E}$ is also a vertex tensor category.

- As $\mathscr{F}$ is semisimple, the Deligne product of $\mathscr{E}$ and $\mathscr{F}$ is then a vertex tensor category [CKM19, Theorem 5.5] as is its direct limit completion [CMY20a].

As $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$ is a simple current extension in this direct limit completion [CMY20a, AR18], it now follows from [CKL20a, CKM17] that $\mathscr{D}$ is the category of local modules for $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$, viewed as a commutative algebra object in the direct limit completion. There is therefore an induction functor that maps any module V in the completion that centralises the algebra object $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$ to an object $\mathrm{Ind}(V)$ in $\mathscr{D}$. We use freely here the fact that the induction restriction adjunction guarantees that the induction of a projective object is projective [ACKR20, Lemma 17]. Moreover, this functor is a vertex tensor functor [CKM17], meaning that it respects the fusion products of the completion and $\mathscr{D}$:

$$\mathrm{Ind}(V \boxtimes V') \cong \mathrm{Ind}(V) \,\widehat{\boxtimes}\, \mathrm{Ind}(V'). \tag{4.23}$$

Practically, one starts with a $W_{A_2}^0(2)$-module such as $E_\lambda^{(8)}$, tensors it with some Fock module F_μ and induces to a $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-module by fusing with the decomposition (4.7):

$$\mathrm{Ind}(E_\lambda^{(8)} \otimes F_\mu) = \bigoplus_{\nu \in Q} \left(E_\lambda^{(8)} \otimes F_\mu \right) \boxtimes \left(E_\nu^{(1)} \otimes F_\nu \right) = \bigoplus_{\nu \in Q} E_{\lambda+\nu}^{(8)} \otimes F_{\mu+\nu}. \tag{4.24}$$

The resulting object in the direct limit completion defines a local $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-module if and only if the conformal weights of the summands differ by integers. By (4.9), the conformal weights of $E_\lambda^{(8)} \otimes F_\mu$ are given (mod 1) by

$$\Delta(\lambda, \mu) = \tfrac{1}{3}\|\lambda\|^2 - \tfrac{1}{2} - \tfrac{1}{3}\|\mu\|^2 = \tfrac{1}{3}\langle \lambda - \mu, \lambda + \mu \rangle - \tfrac{1}{2}. \tag{4.25}$$

The difference between the conformal weights of summands whose ν-labels differ by $\beta \in Q$ is then

$$\Delta(\lambda + \nu + \beta, \mu + \nu + \beta) - \Delta(\lambda + \nu, \mu + \nu) = \tfrac{2}{3}\langle \lambda - \mu, \beta \rangle. \tag{4.26}$$

This proves that inducing $E_\lambda^{(8)} \otimes F_\mu$ gives an $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-module if and only if $\lambda - \mu \in \frac{3}{2}P$. In this case, (4.10) identifies the result:

$$\mathrm{Ind}(E_\lambda^{(8)} \otimes F_\mu) \cong \sigma^{\frac{2}{3}(\lambda-\mu)}(\widehat{\mathcal{R}}_{[\lambda]}). \tag{4.27a}$$

Similarly, the decompositions (4.7), (4.15) and (4.16) (and their spectral flows) lead to the following identifications, again assuming that $\lambda - \mu \in \frac{3}{2}P$:

$$\mathrm{Ind}(E_\lambda^{(4)} \otimes F_\mu) \cong \begin{cases} \sigma^{\frac{2}{3}(\lambda-\mu)}(\widehat{\mathcal{S}}_{[\lambda']}), & \lambda \in -\frac{3}{2}\omega_1 + \mathbb{R}\alpha_1 + Q, \\ \sigma^{\frac{2}{3}(\lambda-\mu)}\mathrm{w}_2(\widehat{\mathcal{S}}_{[\mathrm{w}_2(\lambda')]}), & \lambda \in -\frac{3}{2}\omega_1 + \mathbb{R}\alpha_3 + Q, \\ \sigma^{\frac{2}{3}(\lambda-\mu)}\mathrm{w}_1\mathrm{w}_2(\widehat{\mathcal{S}}_{[\mathrm{w}_2\mathrm{w}_1(\lambda')]}), & \lambda \in -\frac{1}{2}\rho + \mathbb{R}\alpha_2 + Q, \end{cases} \tag{4.27b}$$

(here, $\lambda' \in \lambda + Q$ is chosen so that $\lambda' \in -\frac{3}{2}\omega_1 + \mathbb{R}\alpha_1$, $-\frac{3}{2}\omega_1 + \mathbb{R}\alpha_3$ and $-\frac{3}{2}\omega_3 + \mathbb{R}\alpha_2$, respectively)

$$\begin{aligned}
\operatorname{Ind}\big(E_\lambda^{(3)} \otimes F_\mu\big) &\cong \sigma^{\frac{2}{3}(\lambda-\mu)}(\widehat{\mathcal{L}}_{-\frac{1}{2}\rho}), \\
\operatorname{Ind}\big(E_\lambda^{(\overline{3})} \otimes F_\mu\big) &\cong \sigma^{\frac{2}{3}(\lambda-\mu)}\mathrm{c}(\widehat{\mathcal{L}}_{-\frac{1}{2}\rho}),
\end{aligned} \qquad \lambda \in -\tfrac{1}{2}\rho + Q, \tag{4.27c}$$

$$\operatorname{Ind}\big(E_\lambda^{(1)} \otimes F_\mu\big) \cong \sigma^{\frac{2}{3}(\lambda-\mu)}(\widehat{\mathcal{L}}_0), \qquad \lambda \in Q. \tag{4.27d}$$

With the inductions (4.27) identified, it is now straightforward to predict Loewy diagrams for the projective covers of the irreducible $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules: the irreducible relaxed modules $\widehat{\mathcal{R}}_{[\lambda]}$ are already projective and Loewy diagrams for the remaining projective covers are obtained by tensoring each composition factor in Figure 6 by F_μ and inducing. The results are pictured in Figure 7.

To verify the (predicted) diagram of the projective cover $\widehat{\mathcal{Q}}_{[\lambda]}$ of $\widehat{\mathcal{S}}_{[\lambda]}$, $[\lambda] \in (-\frac{3}{2}\omega_1 + \mathbb{R}\alpha_1)/\mathbb{Z}\alpha_1$, start with the (predicted) diagram in Figure 6 for the projective cover $Q_\lambda^{(16)}$ of the irreducible $W_{A_2}^0(2)$-module $E_\lambda^{(4)}$. The composition factors of $Q_\lambda^{(16)}$ induce, after tensoring with F_λ, to give

$$\widehat{\mathcal{S}}_{[\lambda']}, \quad \sigma^{\omega_2}(\widehat{\mathcal{S}}_{[(\lambda+\frac{3}{2}\omega_2)']}) \quad \text{and} \quad \sigma^{-\omega_2}(\widehat{\mathcal{S}}_{[(\lambda-\frac{3}{2}\omega_2)']}) \tag{4.28}$$

(the first with multiplicity 2), by (4.27b). Here, λ' may be taken to be λ because the latter already belongs to $-\frac{3}{2}\omega_1 + \mathbb{R}\alpha_1$ by assumption. However, $\lambda \pm \frac{3}{2}\omega_2$ does not. To determine the corresponding primed weight, note that $\frac{3}{2}\omega_2 = \frac{1}{2}\alpha_1 + \alpha_2$ so we may take

$$(\lambda \pm \tfrac{3}{2}\omega_2)' = \lambda \pm (\tfrac{3}{2}\omega_2 - \alpha_2) = \lambda \pm \tfrac{1}{2}\alpha_1 \in \lambda \pm \tfrac{3}{2}\omega_2 + Q, \tag{4.29}$$

which does lie in $-\frac{3}{2}\omega_1 + \mathbb{R}\alpha_1$. The Loewy diagram in Figure 7 now follows.

Note again that the multiplicity with which the head of each projective appears suggests that the Virasoro zero-mode acts with rank-2 Jordan blocks on the $\widehat{\mathcal{Q}}_{[\lambda]}$ and with rank-3 Jordan blocks on $\widehat{\mathcal{P}}_0$ and $\widehat{\mathcal{P}}_{-\frac{1}{2}\rho}$ (and their spectral flows). This is consistent with the recent work [ACG21] who have constructed indecomposable $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules with rank-2 and 3 blocks using inverse quantum hamiltonian reduction [Ad19, AKR21]. It will be interesting to try to identify their constructions in the setting presented here.

4.5 Fusion rules

Another important check of this Kazhdan–Lusztig correspondence is that it is indeed a tensor equivalence: it maps tensor products in $\mathscr{C}_\mathbb{R}$ to fusion rules in $\mathscr{E}$. As the latter are not known, the former will be used to deduce fusion rules in $\mathscr{D}$ (again using [CKM17]). These in turn will be compared with the Grothendieck fusion rules (4.6) computed for $\mathscr{D}$ in [KRW21]. Note that a given Grothendieck fusion rule completely characterises the corresponding fusion rule if the fusion product in question happens to be completely reducible.

The first case to analyse is the generic fusion product of two relaxed $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules (in which the result is completely reducible). In the decomposition (4.8), the $W_{A_2}^0(2)$-modules are the irreducible $E_\lambda^{(8)}$. Their fusion product is computed using Conjecture 4.4 and Proposition 3.10 as follows:

$$E_\lambda^{(8)} \boxtimes E_\mu^{(8)} \longleftrightarrow L_{\varphi(\lambda)+\rho}^{(8)} \otimes L_{\varphi(\mu)+\rho}^{(8)} \cong \bigoplus_{i,j=1}^3 L_{\varphi(\lambda+\mu)+2\rho-\alpha_i-\alpha_j}^{(8)} = \bigoplus_{i,j=1}^3 L_{\varphi(\lambda+\mu+\varphi^{-1}(\alpha_3-\alpha_i-\alpha_j))+\rho}^{(8)}$$
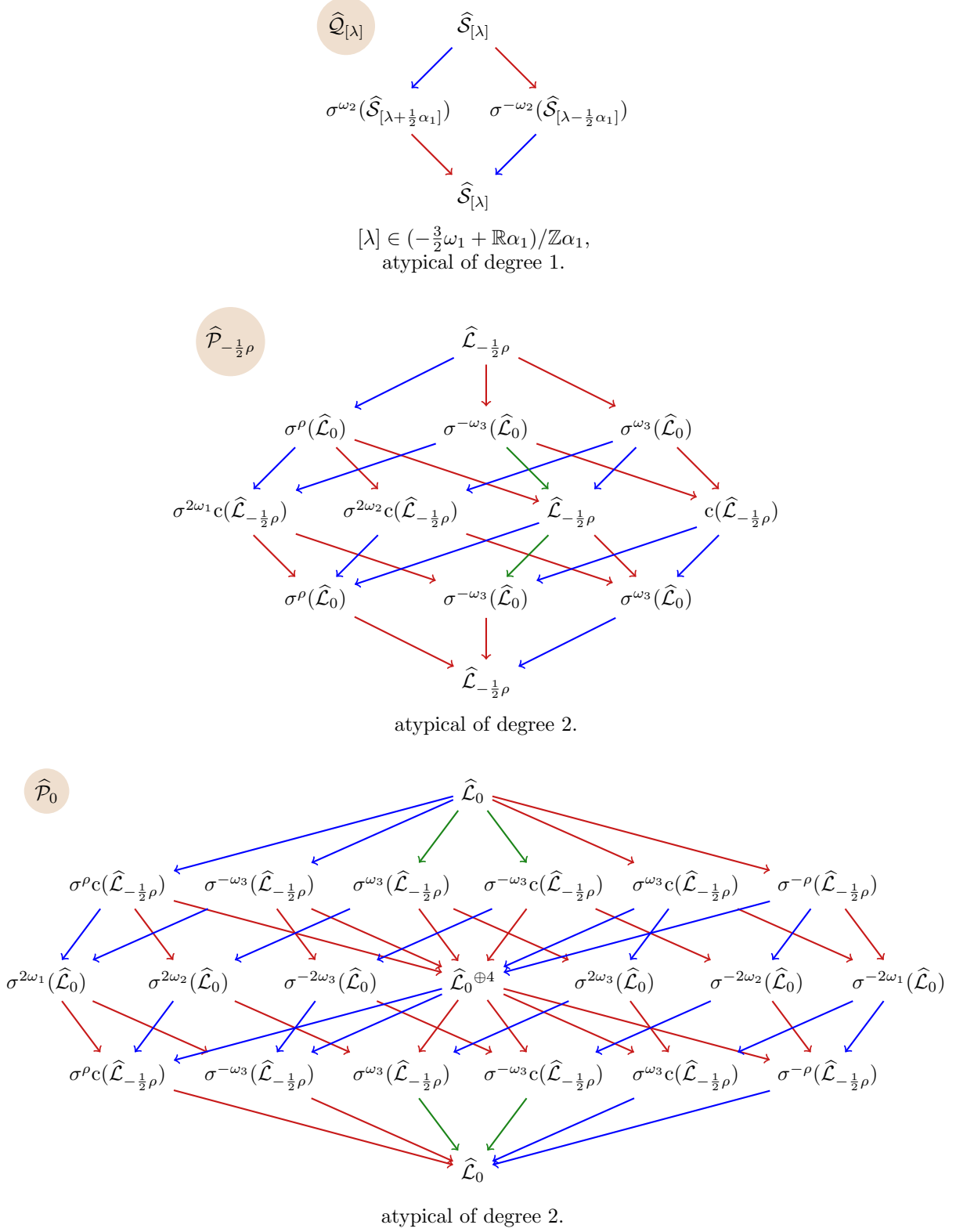
Figure 7: Conjectural Loewy diagrams for the projective covers of the atypical irreducible $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules $\widehat{\mathcal{S}}_{[\lambda]}$, $\widehat{\mathcal{L}}_{-\frac{1}{2}\rho}$ and $\widehat{\mathcal{L}}_0$. Projective covers for their D_6 and spectral flow twists are obtained by applying the twist to each automorphism.

$$\longrightarrow 2\,E^{(8)}_{\lambda+\mu} \oplus \bigoplus_{i=1}^{3}\Big(E^{(8)}_{\lambda+\mu+\frac{3}{2}\omega_i} \oplus E^{(8)}_{\lambda+\mu-\frac{3}{2}\omega_i}\Big). \tag{4.30}$$

Tensoring with $F_\lambda \boxtimes F_\mu \cong F_{\lambda+\mu}$ and inducing now gives

$$\widehat{\mathcal{R}}_{[\lambda]} \,\widehat{\boxtimes}\, \widehat{\mathcal{R}}_{[\mu]} \cong \mathrm{Ind}\big(E^{(8)}_\lambda \otimes F_\lambda\big) \,\widehat{\boxtimes}\, \mathrm{Ind}\big(E^{(8)}_\mu \otimes F_\mu\big) \cong \mathrm{Ind}\big((E^{(8)}_\lambda \boxtimes E^{(8)}_\mu) \otimes F_{\lambda+\mu}\big)$$

$$\cong 2\,\mathrm{Ind}\big(E^{(8)}_{\lambda+\mu} \otimes F_{\lambda+\mu}\big) \oplus \bigoplus_{i=1}^{3}\Big(\mathrm{Ind}\big(E^{(8)}_{\lambda+\mu+\frac{3}{2}\omega_i} \otimes F_{\lambda+\mu}\big) \oplus \mathrm{Ind}\big(E^{(8)}_{\lambda+\mu-\frac{3}{2}\omega_i} \otimes F_{\lambda+\mu}\big)\Big)$$

$$\cong 2\,\widehat{\mathcal{R}}_{[\lambda+\mu]} \oplus \bigoplus_{i=1}^{3}\Big(\sigma^{\omega_i}(\widehat{\mathcal{R}}_{[\lambda+\mu+\frac{3}{2}\omega_i]}) \oplus \sigma^{-\omega_i}(\widehat{\mathcal{R}}_{[\lambda+\mu-\frac{3}{2}\omega_i]})\Big), \tag{4.31}$$

courtesy of (4.23) and (4.27a). This correctly reproduces the generic fusion rule given in (4.6i).

Similar calculations recover the other generic fusion rules in (4.6) and thereby provide strong evidence for the proposed Kazhdan–Lusztig correspondence (Conjecture 4.4). Here, the generic fusion rule involving $\widehat{\mathcal{S}}_{[\lambda]}$ and $\mathrm{w}_1\mathrm{w}_2(\widehat{\mathcal{S}}_{[\mathrm{w}_2\mathrm{w}_1(\mu)]})$, where $\lambda \in -\frac{3}{2}\omega_1 + \mathbb{R}\alpha_1$ and $\mu \in -\frac{1}{2}\rho + \mathbb{R}\alpha_2$, is computed to illustrate the different formulae in (4.18).

First, note that $E^{(4)}_\lambda \boxtimes E^{(4)}_\mu$ corresponds to

$$L^{(4)}_{\varphi(\lambda)+\rho} \otimes L^{(4)}_{\varphi(\mu)} \cong L^{(8)}_{\varphi(\lambda+\mu)+\rho} \oplus L^{(8)}_{\varphi(\lambda+\mu)+\rho-\alpha_1}, \tag{4.32}$$

by Proposition 3.11, because $\big(\varphi(\lambda)+\rho\big)_2, \varphi(\mu)_3 \in 2\mathbb{Z}+1$. The conjectural fusion rule in $\mathscr{E}$ is thus

$$E^{(4)}_\lambda \boxtimes E^{(4)}_\mu \cong E^{(8)}_{\lambda+\mu} \oplus E^{(8)}_{\lambda+\mu+\frac{3}{2}\omega_3}, \tag{4.33}$$

by (4.22). Inducing then yields the generic fusion rule

$$\widehat{\mathcal{S}}_{[\lambda]} \,\widehat{\boxtimes}\, \mathrm{w}_1\mathrm{w}_2(\widehat{\mathcal{S}}_{[\mathrm{w}_2\mathrm{w}_1(\mu)]}) \cong \widehat{\mathcal{R}}_{[\lambda+\mu]} \oplus \sigma^{\omega_3}(\widehat{\mathcal{R}}_{[\lambda+\mu+\frac{3}{2}\omega_3]}) \tag{4.34}$$

in $\mathscr{D}$, which is equivalent to (4.6g).

More interesting are the non-generic fusion rules that result in reducible but indecomposable direct summands. The structures of these indecomposables are not visible in the Grothendieck fusion rules of (4.6), but they can now be predicted from the Loewy diagrams of the indecomposable projectives of $\mathscr{D}$ depicted in Figure 7.

As an example, consider the tensor product rule

$$L^{(3)}_\lambda \otimes L^{(8)}_\mu \cong P^{(24)}_{\lambda+\mu-\alpha_3} \tag{4.35}$$

from Proposition 3.7, which holds if $\lambda_i, \lambda_3 \in 2\mathbb{Z}+1$ and $\mu_1, \mu_2, \mu_3 \in 2\mathbb{Z}$. Take $i = 2$, so that these conditions are equivalent to $\lambda \in \omega_1 + 2P$ and $\mu \in \rho + 2P$. As $\lambda+\mu-\alpha_3 \in \omega_1 + 2P$, the correspondence (4.18) converts this tensor product rule into the fusion rule

$$E^{(3)}_{\lambda'} \boxtimes E^{(8)}_{\mu'} \cong Q^{(24)}_{\lambda'+\mu'}, \tag{4.36}$$

where $\lambda' = \varphi^{-1}(\lambda - \rho) \in -\frac{1}{2}\rho + Q$ and $\mu' = \varphi^{-1}(\mu - \rho) \in Q$. Tensoring with $F_{\lambda'+\mu'}$ and inducing now

predicts the following fusion rule of $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-modules:

$$\widehat{\mathcal{L}}_{-\frac{1}{2}\rho} \,\widehat{\boxtimes}\, \widehat{\mathcal{R}}_{[0]} \cong \widehat{\mathcal{P}}_{-\frac{1}{2}\rho}. \tag{4.37}$$

One can use (4.4b) to check that the composition factors of this fusion product match those given in the corresponding Grothendieck fusion rule (4.6d).

As a second example, suppose that one wished to identify the fusion product of the irreducible $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$-module $\widehat{\mathcal{R}}_{[\lambda]}$ and its conjugate $\mathrm{c}(\widehat{\mathcal{R}}_{[\lambda]}) \cong \widehat{\mathcal{R}}_{[-\lambda]}$. This can be computed from the $W_{A_2}^0(2)$-module fusion product $E_\lambda^{(8)} \boxtimes E_{-\lambda}^{(8)}$ which corresponds, under the Kazhdan–Lusztig correspondence (4.18) to the $\overline{\mathcal{U}}_{\mathrm{i}}^H(\mathfrak{sl}_3)$ tensor product $L_{\lambda'}^{(8)} \otimes L_{\mu'}^{(8)}$, where $\lambda' = \varphi(\lambda) + \rho$ and $\mu' = -\varphi(\lambda) + \rho$. It is easy to verify that $\lambda'_i + \mu'_i \in 2\mathbb{Z}$, for $i = 1, 2, 3$, hence that the required tensor product rule is

$$L_{\varphi(\lambda)+\rho}^{(8)} \otimes L_{-\varphi(\lambda)+\rho}^{(8)} \cong 2L_\rho^{(8)} \oplus P_0^{(48)}, \tag{4.38}$$

by Proposition 3.10. Using the correspondence once again, specifically (4.19), the $W_{A_2}^0(2)$ fusion product is

$$E_\lambda^{(8)} \boxtimes E_{-\lambda}^{(8)} \cong 2E_0^{(8)} \oplus Q_0^{(48)}. \tag{4.39}$$

Tensoring with Fock spaces and inducing then gives the desired fusion rule for $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$:

$$\widehat{\mathcal{R}}_{[\lambda]} \,\widehat{\boxtimes}\, \mathrm{c}(\widehat{\mathcal{R}}_{[\lambda]}) \cong 2\widehat{\mathcal{R}}_{[0]} \oplus \widehat{\mathcal{P}}_0. \tag{4.40}$$

Many other affine fusion rules follow similarly from the tensor product computations of Section 3.3.

4.6 An octuplet algebra

In [Se14, Se13], Semikhatov introduced $\mathfrak{sl}_3$ analogues of the well known triplet algebras $W_{A_1}(p)$ of Kausch [Ka91] (which correspond to $\mathfrak{sl}_2$). As these analogues are generated by 8 W_3 primary fields, rather than 3 Virasoro ones, he dubbed these the *octuplet* algebras. Semikhatov realised the octuplet algebras inside free field algebras and worked out the operator product algebra. The representation theory of this algebra is mostly unexplored except that some modules and their simplicity are known [AMW22, Su21]. The first nontrivial octuplet algebra $W_{A_2}(2)$ has central charge $c = -10$ and turns out to be a simple current extension of $W_{A_2}^0(2)$, just as $W_{A_1}(p)$ is a simple current extension of $W_{A_1}^0(p)$.

Each of the $E_\lambda^{(1)}$, $\lambda \in Q$, is a simple current, by (4.7). The conformal weight Δ_λ of its highest-weight vector is given by the difference between the minimal conformal weight of a vector of weight λ in $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$ and the conformal weight $\frac{1}{3}\|\lambda\|^2$ of the highest-weight vector of F_λ. Obviously, $\Delta_0 = 0$. The next-smallest conformal weights occur when $\lambda = \pm\alpha_i$, $i = 1, 2, 3$, in which case $\Delta_\lambda = \frac{5}{3}$, and when $\lambda = \pm 3\omega_i$, $i = 1, 2, 3$, giving $\Delta_\lambda = 4$. It is this latter set of six simple currents that define $W_{A_2}(2)$:

$$W_{A_2}(2) = \bigoplus_{\lambda \in 3P} E_\lambda^{(1)}. \tag{4.41}$$

The remaining two octuplet fields have weight 0. Their existence is easily checked by using the decomposition

$W^0_{A_2}(2)$-mod.	$W_{A_2}(2)$-mod.	top-space dim.	$\mathfrak{sl}_3$-weights	conf. weight
	$\mathbb{O}^{(1)}_{[0]}$	1	0	0
$E^{(1)}_\lambda$	$\mathbb{O}^{(1)}_{[\rho]}$	3	$\rho,\ \rho-3\omega_1,\ \rho-3\omega_2$	$\frac{5}{3}$
	$\mathbb{O}^{(1)}_{[-\rho]}$	3	$-\rho,\ -\rho+3\omega_1,\ -\rho+3\omega_2$	$\frac{5}{3}$
	$\mathbb{O}^{(3)}_{[-\frac12\rho]}$	1	$-\frac12\rho$	$-\frac13$
$E^{(3)}_\lambda$	$\mathbb{O}^{(3)}_{[\frac12\rho]}$	3	$\frac12\rho,\ \frac12\rho-3\omega_1,\ \frac12\rho-3\omega_2$	$\frac23$
	$\mathbb{O}^{(3)}_{[\frac32\rho]}$	3	$-\frac32\rho,\ -\frac32\rho+3\omega_1,\ -\frac32\rho+3\omega_2$	1
	$\mathbb{O}^{(\bar3)}_{[-\frac12\rho]}$	3	$-\frac12\rho,\ -\frac12\rho+3\omega_1,\ -\frac12\rho+3\omega_2$	$-\frac13$
$E^{(\bar3)}_\lambda$	$\mathbb{O}^{(\bar3)}_{[\frac12\rho]}$	1	$\frac12\rho$	$\frac23$
	$\mathbb{O}^{(\bar3)}_{[\frac32\rho]}$	3	$\frac32\rho,\ \frac32\rho-3\omega_1,\ \frac32\rho-3\omega_2$	1
	$\mathbb{O}^{(8)}_{[0]}$	1	0	$-\frac12$
$E^{(8)}_\lambda$	$\mathbb{O}^{(8)}_{[\rho]}$	3	$\rho,\ \rho-3\omega_1,\ \rho-3\omega_2$	$\frac16$
	$\mathbb{O}^{(8)}_{[-\rho]}$	3	$-\rho,\ -\rho+3\omega_1,\ -\rho+3\omega_2$	$\frac16$

Table 1: The dimensions, $\mathfrak{sl}_3$-weights and conformal weights of the top spaces of the (untwisted) irreducible $W_{A_2}(2)$-modules $\mathbb{O}^{(d)}_{[\lambda]}$, where $[\lambda] \in Q/3P$.

(4.7) to compute the character

$$\mathrm{ch}[E^{(1)}_0] = q^{5/12}\big(1 + q^2 + 2q^3 + 5q^4 + \cdots\big) \tag{4.42}$$

and noting that the W_3-symmetry can only account for 3 of the 5 fields of conformal weight 4.

It is now straightforward to induce an irreducible $W^0_{A_2}(2)$-module to obtain a (possibly twisted) irreducible $W_{A_2}(2)$-module. As $Q/3P = \{[0], [\rho], [-\rho]\}$, the $E^{(1)}_\lambda$ yield three induced modules $\mathbb{O}^{(1)}_{[0]}$, $\mathbb{O}^{(1)}_{[\rho]}$ and $\mathbb{O}^{(1)}_{[-\rho]}$, respectively, by fusing with $W_{A_2}(2)$. The first is the vacuum module of $W_{A_2}(2)$ and the others are order-3 simple currents, by (4.23).

To determine if an irreducible induced module is untwisted, it suffices to check if the conformal weights of the component $W^0_{A_2}(2)$-modules differ by integers. This will be the case if and only if the corresponding Fock space conformal weights do. Since

$$\tfrac{1}{3}\|\lambda \pm 3\omega_i\|^2 - \tfrac{1}{3}\|\lambda\|^2 = 2 \pm \langle 2\lambda, \omega_i \rangle, \quad i = 1, 2, 3, \tag{4.43}$$

$E^{(d)}_\lambda$ induces to an untwisted irreducible $W_{A_2}(2)$-module if and only if $\lambda \in \frac{1}{2}Q$. In particular, $\mathbb{O}^{(1)}_{[0]}$, $\mathbb{O}^{(1)}_{[\alpha_3]}$ and $\mathbb{O}^{(1)}_{[-\alpha_3]}$ are untwisted $W_{A_2}(2)$-modules.

Similar analyses result in further untwisted irreducible $W_{A_2}(2)$-modules. Their top-space dimensions, $\mathfrak{sl}_3$-weights and conformal weights are summarised in Table 1. Note that the $E^{(8)}_\lambda$ with $\lambda \in \frac{1}{2}Q$ are reducible unless $\lambda \in Q$. Similarly, the $E^{(4)}_\lambda$ with $\lambda \in \frac{1}{2}Q$ are always reducible. The number of irreducibles is finite in accord with the expectation that the octuplet algebra $W_{A_2}(2)$ is C_2-cofinite.

Predictions for the fusion rules of these irreducible $W_{A_2}(2)$-modules now follow by applying (4.23) to the

conjectural fusion rules of the irreducible $W_{A_2}^0(2)$-modules, themselves deduced from the tensor product computations of Section 3.3 and the Kazhdan–Lusztig correspondence of Conjecture 4.4. Of course, $\mathbb{O}_{[0]}^{(1)}$ is the vacuum module and $\mathbb{O}_{[\pm\rho]}^{(1)}$ is easily seen to be an order-3 simple current:

$$\mathbb{O}_{[\pm\rho]}^{(1)} \,\widetilde{\boxtimes}\, \mathbb{O}_{[\pm\rho]}^{(1)} \cong \mathbb{O}_{[\mp\rho]}^{(1)}, \quad \mathbb{O}_{[\pm\rho]}^{(1)} \,\widetilde{\boxtimes}\, \mathbb{O}_{[\mp\rho]}^{(1)} \cong \mathbb{O}_{[0]}^{(1)}. \tag{4.44}$$

The orbits of the irreducible $W_{A_2}(2)$-modules under fusion with $\mathbb{O}_{[\rho]}^{(1)}$ are as follows:

$$
\begin{aligned}
&\mathbb{O}_{[0]}^{(1)} \to \mathbb{O}_{[\rho]}^{(1)} \to \mathbb{O}_{[-\rho]}^{(1)} \to \mathbb{O}_{[0]}^{(1)}, \qquad
\mathbb{O}_{[-\frac{1}{2}\rho]}^{(3)} \to \mathbb{O}_{[\frac{1}{2}\rho]}^{(3)} \to \mathbb{O}_{[\frac{3}{2}\rho]}^{(3)} \to \mathbb{O}_{[-\frac{1}{2}\rho]}^{(3)}, \\
&\mathbb{O}_{[0]}^{(8)} \to \mathbb{O}_{[\rho]}^{(8)} \to \mathbb{O}_{[-\rho]}^{(8)} \to \mathbb{O}_{[0]}^{(1)}, \qquad
\mathbb{O}_{[-\frac{1}{2}\rho]}^{(\overline{3})} \to \mathbb{O}_{[\frac{1}{2}\rho]}^{(\overline{3})} \to \mathbb{O}_{[\frac{3}{2}\rho]}^{(\overline{3})} \to \mathbb{O}_{[-\frac{1}{2}\rho]}^{(\overline{3})}.
\end{aligned} \tag{4.45}
$$

As fusion is associative, the remaining irreducible fusion rules are determined by the six involving the orbit representatives $\mathbb{O}_{[\frac{3}{2}\rho]}^{(3)}$, $\mathbb{O}_{[\frac{3}{2}\rho]}^{(\overline{3})}$ and $\mathbb{O}_{[0]}^{(8)}$. These rules are

$$
\begin{aligned}
&\mathbb{O}_{[\frac{3}{2}\rho]}^{(3)} \,\widetilde{\boxtimes}\, \mathbb{O}_{[\frac{3}{2}\rho]}^{(3)} \cong \mathbb{N}_{[\frac{3}{2}\rho]}^{(\overline{9})}, \quad
\mathbb{O}_{[\frac{3}{2}\rho]}^{(\overline{3})} \,\widetilde{\boxtimes}\, \mathbb{O}_{[\frac{3}{2}\rho]}^{(\overline{3})} \cong \mathbb{N}_{[\frac{3}{2}\rho]}^{(9)}, \quad
\mathbb{O}_{[\frac{3}{2}\rho]}^{(3)} \,\widetilde{\boxtimes}\, \mathbb{O}_{[\frac{3}{2}\rho]}^{(\overline{3})} \cong \mathbb{O}_{[0]}^{(8)} \oplus \mathbb{O}_{[0]}^{(1)}, \\
&\mathbb{O}_{[\frac{3}{2}\rho]}^{(3)} \,\widetilde{\boxtimes}\, \mathbb{O}_{[0]}^{(8)} \cong \mathbb{P}_{[\frac{3}{2}\rho]}^{(24)}, \quad
\mathbb{O}_{[\frac{3}{2}\rho]}^{(\overline{3})} \,\widetilde{\boxtimes}\, \mathbb{O}_{[0]}^{(8)} \cong \mathbb{P}_{[\frac{3}{2}\rho]}^{(\overline{24})}, \quad
\mathbb{O}_{[0]}^{(8)} \,\widetilde{\boxtimes}\, \mathbb{O}_{[0]}^{(8)} \cong 2\mathbb{O}_{[0]}^{(8)} \oplus \mathbb{P}_{[0]}^{(48)}.
\end{aligned} \tag{4.46}
$$

Here, $\mathbb{P}_{[\frac{3}{2}\rho]}^{(24)}$, $\mathbb{P}_{[\frac{3}{2}\rho]}^{(\overline{24})}$ and $\mathbb{P}_{[0]}^{(48)}$ are the (conjectural) projective covers of $\mathbb{O}_{[\frac{3}{2}\rho]}^{(3)}$, $\mathbb{O}_{[\frac{3}{2}\rho]}^{(\overline{3})}$ and $\mathbb{O}_{[0]}^{(1)}$, respectively, while $\mathbb{N}_{[\frac{3}{2}\rho]}^{(9)}$ and $\mathbb{N}_{[\frac{3}{2}\rho]}^{(\overline{9})}$ are quotients of $\mathbb{O}_{[\frac{3}{2}\rho]}^{(3)}$ and $\mathbb{O}_{[\frac{3}{2}\rho]}^{(\overline{3})}$, respectively.

Loewy diagrams for $\mathbb{N}_{[\frac{3}{2}\rho]}^{(9)}$, $\mathbb{P}_{[\frac{3}{2}\rho]}^{(24)}$ and $\mathbb{P}_{[0]}^{(48)}$ are pictured in Figure 8. Those of $\mathbb{N}_{[\frac{3}{2}\rho]}^{(\overline{9})}$ and $\mathbb{P}_{[\frac{3}{2}\rho]}^{(\overline{24})}$ are obtained from their unbarred cousins by exchanging 3 with $\overline{3}$ everywhere. Those of the projective covers of the remaining irreducibles are then obtained by fusing each composition factor with whichever of the simple currents $\mathbb{O}_{[\rho]}^{(1)}$ or $\mathbb{O}_{[-\rho]}^{(1)}$ is appropriate. Finally, the irreducibles $\mathbb{O}_{[0]}^{(8)}$, $\mathbb{O}_{[\rho]}^{(8)}$ and $\mathbb{O}_{[-\rho]}^{(8)}$ are their own projective covers. The Virasoro zero-mode is expected to act with Jordan blocks of rank 3 on $\mathbb{P}_{[\frac{3}{2}\rho]}^{(24)}$ and $\mathbb{P}_{[0]}^{(48)}$.

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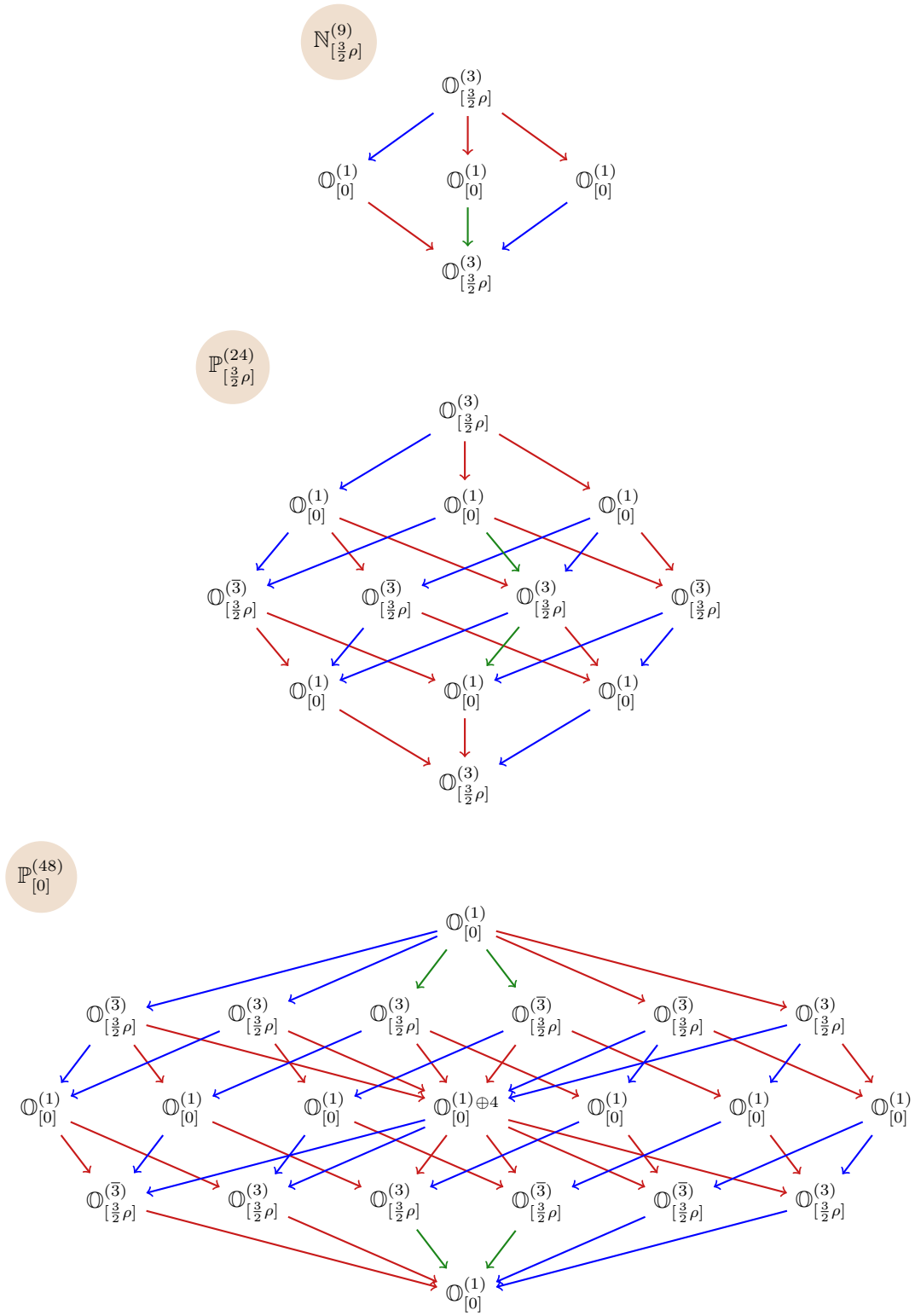
Figure 8: Conjectural Loewy diagrams for the $W_{A_2}(2)$-modules $\mathbb{N}^{(9)}_{[\frac{3}{2}\rho]}$, $\mathbb{P}^{(24)}_{[\frac{3}{2}\rho]}$ and $\mathbb{P}^{(48)}_{[0]}$.

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