WEIGHT MODULE CLASSIFICATIONS FOR BERSHADSKY–POLYAKOV ALGEBRAS

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ABSTRACT. The Bershadsky–Polyakov algebras are the subregular quantum hamiltonian reductions of the affine vertex operator algebras associated with $\mathfrak{sl}_3$. In [5], we realised these algebras in terms of the regular reduction, Zamolodchikov's W$_3$-algebra, and an isotropic lattice vertex operator algebra. We also proved that a natural construction of relaxed highest-weight Bershadsky–Polyakov modules has the property that the result is generically irreducible. Here, we prove that this construction, when combined with spectral flow twists, gives a complete set of irreducible weight modules whose weight spaces are finite-dimensional. This gives a simple independent proof of the main classification theorem of [30] for nondegenerate admissible levels and extends this classification to a category of weight modules. We also deduce the classification for the nonadmissible level $k = -\frac{7}{3}$, which is new.

CONTENTS

1. INTRODUCTION

1.1. **Background.** Among the most important vertex operator algebras are the affine ones. As one might expect, the members of this family that are associated with $\mathfrak{sl}_2$ are the most tractable. In this case, one can distinguish the universal vertex operator algebra $\mathsf{V}^k(\mathfrak{sl}_2)$, where $k \in \mathbb{C} \setminus \{-2\}$ denotes the level, from its simple quotient $\mathsf{L}_k(\mathfrak{sl}_2)$. In fact, these are distinct if and only if k is admissible, a technical condition introduced in [42].

The best understood $\mathsf{L}_k(\mathfrak{sl}_2)$ are those with $k \in \mathbb{Z}_{\geq 0}$. For this subset of admissible levels, $\mathsf{L}_k(\mathfrak{sl}_2)$ is strongly rational [34, 37, 63]. The remaining admissible levels are perhaps even more interesting because then $\mathsf{L}_k(\mathfrak{sl}_2)$ admits finitely many irreducible highest-weight modules but an uncountably infinite number of other irreducible modules [8]. Moreover, the characters of the highest-weight modules span a representation of the modular group (this was the motivation for the introduction of admissibility in [42]). Unfortunately, for admissible levels that are not nonnegative integers, Verlinde's formula [60] for the fusion multiplicities fails [49].

It took twenty years to properly understand the reason behind this failure [55] and another five to fix it [26]. The modern approach to the representation theory of $\mathsf{L}_k(\mathfrak{sl}_2)$ at general admissible levels prioritises the so-called

relaxed highest-weight modules, named in [32] but previously classified in [8], and their images under twisting by spectral flow automorphisms. It is the characters of these modules that carry the true representation of the modular group, consistent with (a mild generalisation of) Verlinde's formula [25, 57].

The characters of the relaxed highest-weight $L_k(\mathfrak{sl}_2)$-modules (and their spectral flows) were proposed in [24, 26] and proven in [46]. Interestingly, they turn out to be proportional to the characters of the irreducible highest-weight modules of a Virasoro minimal model vertex operator algebra. And not just any minimal model, but the quantum hamiltonian reduction of $L_k(\mathfrak{sl}_2)$. This beautiful observation demanded a beautiful explanation and one was subsequently provided in [2] through a functorial construction that we call (following [58]) inverse quantum hamiltonian reduction.

This construction realises a relaxed highest-weight $L_k(\mathfrak{sl}_2)$-module as a tensor product of a highest-weight Virasoro module with a module over a specific lattice vertex operator algebra Π. It has since been generalised to several other affine vertex operator algebras and W-algebras, including $L_k(\mathfrak{osp}(1|2))$ and $N = 1$ super-Virasoro [2], the Bershadsky–Polyakov and Zamolodchikov algebras [5], $L_k(\mathfrak{sl}_3)$ and Bershadsky–Polyakov [3], and the Feigin–Semikhatov and W_n Casimir algebras [29]. The general philosophy here is that the representation theory of a given nonrational affine vertex operator algebra (or W-algebra), which is relatively complicated, should be reconstructed using inverse quantum hamiltonian reduction functors from that of another less complicated (perhaps even rational) W-algebra.

1.2. **The state of the art.** It is natural when faced with an algebraic structure to first try to classify its irreducible modules in an appropriate category. In our case, the algebraic structure is an affine vertex operator algebra or one of its W-algebras and an appropriate category is that consisting of the weight modules with finite-dimensional weight spaces. (This latter condition is reasonable if one wishes to study characters and their modular properties.) The corresponding classifications are known for certain (nonsuper) rational vertex operator algebras including affine ones at nonnegative-integer levels [34], regular W-algebras at nondegenerate admissible levels [12] and (more generally) certain W-algebras said to be exceptional [15, 52].

The situation for nonrational affine vertex operator algebras and W-algebras is not as satisfactory. As noted above, the classification for $L_k(\mathfrak{sl}_2)$, with k admissible, was completed in [8] but only for the full subcategory of relaxed highest-weight modules. More recently, similar classifications have appeared for $L_k(\mathfrak{osp}(1|2))$ [20], $L_k(\mathfrak{sl}_3)$ [14, 48] and the simple minimal W-algebras associated to $\mathfrak{sl}_3$ [30] and $\mathfrak{sl}(2|1)$ [21]. Unfortunately, the methods used in these works appear to be difficult to generalise.

If we further restrict to the full subcategory of highest-weight modules, or more precisely the vertex-algebraic analogue of the BGG category $\mathscr{O}_k$, then the classification was established for all nonsuper affine vertex operator algebras when the level is admissible [13]. The corresponding relaxed classification was subsequently shown to follow algorithmically in [47]. However, it seems that even the highest-weight classification remains out of reach for general W-algebras (and almost all superalgebras).

Our thesis is that the inverse quantum hamiltonian reduction functors of [2] provide a powerful new way to classify irreducible relaxed highest-weight modules of nonrational affine vertex operator algebras and W-algebras. By this, we mean that we expect that applying these functors to irreducible modules will result in generically irreducible relaxed highest-weight modules and that all irreducible relaxed highest-weight modules may be constructed in this manner. (We add the qualifier "generically" here as some of the relaxed highest-weight modules constructed by inverse reduction are necessarily reducible.)

These expectations were shown to be met for $L_k(\mathfrak{sl}_2)$ in [2] by applying inverse quantum hamiltonian reduction functors to irreducible Virasoro modules and comparing with the known character formulae and relaxed classification for $L_k(\mathfrak{sl}_2)$. As the latter results are not available for comparison in general, it becomes desirable to develop proofs that instead rely principally on inverse quantum hamiltonian reduction. In [5], we satisfied a part of this desire by constructing an intrinsic proof that inverse reduction maps irreducible modules to generically irreducible ones. This was presented for the simple Bershadsky–Polyakov vertex operator algebras BP_k of nondegenerate admissible levels k, rather than for $L_k(\mathfrak{sl}_2)$, in order to illustrate the method in a nonaffine example. (The modifications required

for $\mathsf{L_k(sl_2)}$ are very simple and were left to the reader.) The generality of our method was subsequently confirmed in [29], where this generic irreducibility was established for the subregular W-algebras associated to sl_n.

It remains to develop an intrinsic means to prove that inverse quantum hamiltonian reduction constructs all irreducible relaxed highest-weight modules, up to isomorphism. This is the task we set ourselves in this paper. We shall again present the method for $\mathsf{BP_k}$, noting that it may be readily adapted for $\mathsf{L_k(sl_2)}$. The expectation is that it will also readily generalise to higher-rank cases.

1.3. Results.
Recall from [5, Thms. 3.6 and 6.2] that inverse quantum hamiltonian reduction functors are defined for $\mathsf{BP_k}$ if and only if $\mathsf{k} \notin \{-3\} \cup \frac{1}{2}\mathbb{Z}_{\geqslant -3}$. The main results below all assume this restriction on the level. Let $\mathsf{W_{3,k}}$ denote the simple regular W-algebra of level k associated with $\mathsf{sl_3}$. Our first main result is then as follows:

(M1) Every irreducible fully relaxed highest-weight $\mathsf{BP_k}$-module is isomorphic to the result of applying some inverse quantum hamiltonian reduction functor to some irreducible highest-weight $\mathsf{W_{3,k}}$-module.

Here, we use the term "fully relaxed" to exclude the irreducible highest-weight and conjugate highest-weight modules that cannot be so realised (see Definition 2.8). However, these irreducibles are easily brought into the fold because of our second main result:

(M2) Every irreducible highest-weight or conjugate highest-weight $\mathsf{BP_k}$-module is isomorphic to a spectral flow image of a quotient of a reducible fully relaxed highest-weight $\mathsf{BP_k}$-module constructed as in (M1).

In fact, we may equivalently replace "quotient" by "submodule" in this result.

These two results complete the classification of irreducible $\mathsf{BP_k}$-modules in the relaxed category. When k is nondegenerate admissible, this reproduces the main result of [30]. Their proof relies heavily on the special properties of the minimal quantum hamiltonian reduction functor [9, 41, 43] and is therefore difficult to generalise to other nilpotent orbits. Our proof does not have this problem as the quantum hamiltonian reduction functor we use is the regular one, needed only to classify the irreducible highest-weight modules of $\mathsf{W_{3,k}}$, and this classification is known for higher ranks [13]. We expect that our methods will also generalise to degenerate admissible levels using the theory of exceptional W-algebras recently developed in [15].

Our inverse reduction methods also apply to nonadmissible levels of the form $\mathsf{k} = -3 + \frac{2}{\mathsf{v}}$, where $\mathsf{v} \geqslant 3$ is odd. For these levels, the classification given by our two main results is new. When $\mathsf{v} = 3$, hence $\mathsf{k} = -\frac{7}{3}$, we can make this classification explicit because $\mathsf{W_{3,k}}$ then coincides with the singlet algebra [44] of central charge $\mathsf{c} = -2$ whose representation theory is well understood, see [1, 22, 23, 27, 39, 61]. When $\mathsf{v} > 3$, it remains an open problem to make the classification explicit.

Nevertheless, the $\mathsf{k} = -\frac{7}{3}$ results are very interesting. Whereas for nondegenerate levels, one obtains a finite number of highest-weight modules, here we have four one-parameter families of such modules, one of which consists entirely of ordinary modules. Correspondingly, we have a two-parameter family of generically irreducible relaxed highest-weight modules, contrasting with the one-parameter result for nondegenerate levels. In a sense, this combines the nondegenerate result with that obtained for the nonadmissible levels $\mathsf{k} \in \mathbb{Z}_{\geqslant -1}$ in [6, 7]. For these levels, our inverse reduction methods do not apply, but singular vector methods may be used to deduce the existence of one-parameter families of highest-weight modules, all of which are ordinary, and no (fully) relaxed families.

The classification of irreducible $\mathsf{BP_k}$-modules is therefore now very well understood in the relaxed category. However, we are ultimately interested in the larger category of weight $\mathsf{BP_k}$-modules with finite-dimensional weight spaces. Happily, the classification in this category is covered by our third main result:

(M3) Every irreducible weight $\mathsf{BP_k}$-module, with finite-dimensional weight spaces, is isomorphic to a spectral flow of either a fully relaxed highest-weight module or a highest-weight module.

As far as we can tell, this result is also new, as is the corresponding result for $\mathsf{L_k(sl_2)}$ (which is easily obtained using the same methods). Using our method, we can also prove that for $\mathsf{k} \in \mathbb{Z}_{\geqslant -1}$, the irreducible positive-energy $\mathsf{BP_k}$-modules uncovered in [6] likewise give all the irreducible weight $\mathsf{BP_k}$-modules with finite-dimensional weight spaces.

1.4. **Outline.** We commence in Section 2 by reviewing the theory of inverse quantum hamiltonian reduction functors between Zamolodchikov and Bershadsky–Polyakov modules, following [5]. The discussion also serves to fix our notation and conventions. The work begins in Sections 3.1 and 3.2. We first adapt some seminal results of Futorny [36] to the rank-1 Heisenberg vertex algebra. These allow us to prove our main result (M3) above, see Theorem 3.11.

We return to inverse quantum hamiltonian reduction in Section 3.3. It is not difficult to see that these functors produce every relaxed highest-weight module for the universal Bershadsky–Polyakov algebra BP^{k} (Proposition 3.12). The extension, Theorem 3.15, to BP_{k}, $\mathsf{k} \notin \{-3\} \cup \frac{1}{2}\mathbb{Z}_{\geqslant -3}$, is our main result (M1). It requires a technical lemma, which we prove using the string function methods developed in [46, App. A], and a comparison of the maximal ideal of BP^{k} and that of the universal regular W-algebra $\mathsf{W}_3^{\mathsf{k}}$.

Section 3.4 then addresses the irreducible highest-weight modules, noting first (Proposition 3.16) that such a module may always be realised as a quotient of a reducible relaxed highest-weight module if its subspace of minimal conformal weight (equivalently, its image under the Zhu functor) is infinite-dimensional. We then prove that the remaining irreducible highest-weight modules can be obtained from these quotients using spectral flow (Proposition 3.17), thereby establishing our main result (M2). (This proof is the only place in which we need to use the explicit form of the embedding that underlies the inverse reduction functors. It would be nice to dispense with it entirely, assuming that this is possible.)

As a first application of these general results, the classification of irreducible weight modules for nondegenerate levels is quickly described in Section 4.1. The analogous (but new) classification for $\mathsf{k} = -\frac{7}{3}$ is then detailed in Theorem 4.3. We also extract from this theorem the classification of irreducible ordinary $\mathsf{BP}_{-7/3}$-modules (Theorem 4.6). We conclude in Section 5 by proving a few simple consequences of our results for the irreducible ordinary modules of $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_3)$, $\mathsf{k} \notin \{-3\} \cup \frac{1}{2}\mathbb{Z}_{\geqslant -3}$. In particular, we deduce another new result (Corollary 5.5): the classification of irreducible ordinary modules for $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_3)$ at the nonadmissible level $\mathsf{k} = -\frac{7}{3}$.

Finally, let us recall a general principle/conjecture of vertex algebra theory (and conformal field theory) which says that every irreducible module for a vertex subalgebra U of a vertex algebra V may be obtained from V-modules or twisted V-modules. Here, we test this principle when $\mathsf{U} = \mathsf{BP}^{\mathsf{k}}$ and $\mathsf{V} = \mathsf{W}_3^{\mathsf{k}} \otimes \Pi$ in the category of weight modules. We expect that this should be also verified for general affine vertex algebras and W-algebras related by inverse quantum hamiltonian reduction.

2. REALISING BERSHADSKY–POLYAKOV ALGEBRAS AND MODULES

In this section, we review the relationship [5] between the regular and subregular W-algebras associated to $\mathfrak{sl}_3$, also known as the Zamolodchikov [64] and Bershadsky–Polyakov [17, 54] algebras, respectively. We also review an explicit construction [5] of the relaxed highest-weight modules of the latter from those of the former.

Throughout, we shall find it convenient to parametrise our algebras by a level $\mathsf{k} \neq -3$, a complex number that is ultimately identified as the eigenvalue of the central element of $\widehat{\mathfrak{sl}}_3$ on the associated affine vertex algebra. Our primary focus will be rational levels with $\mathsf{k} + 3 > 0$ for which we write

$$(2.1) \qquad \mathsf{k} + 3 = \frac{\mathsf{u}}{\mathsf{v}}, \quad \text{where } \mathsf{u}, \mathsf{v} \in \mathbb{Z}_{>0} \text{ and } \gcd\{\mathsf{u}, \mathsf{v}\} = 1.$$

A level k is said to be admissible if $\mathsf{u} \geqslant 3$ and nondegenerate if, in addition, $\mathsf{v} \geqslant 3$.

2.1. **A lattice vertex operator algebra.** We start with the "half-lattice" vertex operator algebra Π, studied in [16] (see also [35]). Here, and throughout, let $\mathbb{1}$ denote the identity field of a vertex algebra.

Definition 2.1. *Given* $\mathsf{k} \in \mathbb{C}$, *let* Π *denote the universal vertex operator algebra with strong generators* c, d *and* e^{nc}, $n \in \mathbb{Z}$, *subject to the following operator product expansions*

$$(2.2) \quad \begin{aligned} c(z)c(w) &\sim 0, \quad c(z)d(w) \sim \frac{2\,\mathbb{1}}{(z-w)^2}, \quad d(z)d(w) \sim 0, \\ c(z)\mathrm{e}^{nc}(w) &\sim 0, \quad d(z)\mathrm{e}^{nc}(w) \sim \frac{2n\,\mathrm{e}^{nc}(w)}{z-w}, \quad \mathrm{e}^{mc}(z)\mathrm{e}^{nc}(w) \sim 0, \quad m, n \in \mathbb{Z}, \end{aligned}$$

and equipped with the conformal vector

$$(2.3) \quad t = \tfrac{1}{2}{:}cd{:} + \kappa \partial c - \tfrac{1}{2}\partial d, \quad \kappa = \tfrac{1}{3}(2\mathsf{k} + 3).$$

This vertex operator algebra is simple. The conformal weights of the generators c, d and e^{nc} are 1, 1 and n, respectively, while the central charge is

$$(2.4) \quad \mathsf{c}_\mathsf{k}^\Pi = 2 + 24\kappa.$$

We therefore take the corresponding field expansions to be

$$(2.5) \quad c(z) = \sum_{m \in \mathbb{Z}} c_m z^{-m-1}, \quad d(z) = \sum_{m \in \mathbb{Z}} d_m z^{-m-1} \quad \text{and} \quad \mathrm{e}^{nc}(z) = \sum_{m \in \mathbb{Z}} \mathrm{e}_m^{nc} z^{-m-n}.$$

Note that the first three operator product expansions of (2.2) describe a symmetric bilinear form on $\mathrm{span}\{c, d\}$ with $\langle c, c \rangle = \langle d, d \rangle = 0$ and $\langle c, d \rangle = 2$. For later purposes, it will be convenient to introduce an alternative basis to c and d, at least when $\kappa \neq 0$, namely

$$(2.6) \quad a = \tfrac{1}{2}(d - \kappa c) \quad \text{and} \quad b = \tfrac{1}{2}(d + \kappa c).$$

Definition 2.2.

- *The simultaneous eigenspaces of c_0 and d_0, acting on some Π-module, are called weight spaces and their nonzero elements are weight vectors.*
- *A weight Π-module is then a module that is the direct sum of its weight spaces.*
- *A relaxed highest-weight vector for Π is a weight vector that is annihilated by the c_m, d_m and e_m^{nc}, $n \in \mathbb{Z}$, with $m > 0$.*
- *A relaxed highest-weight Π-module is a module that is generated by a relaxed highest-weight vector.*

We remark that a relaxed highest-weight vector is automatically an eigenvector for t_0.

The irreducible relaxed highest-weight Π-modules were classified in [16]. Let $\Pi_{[j]}$, $[j] \in \mathbb{C}/\mathbb{Z}$, denote the relaxed highest-weight Π-module generated by a relaxed highest-weight vector $\mathrm{e}^{-b+(j+\kappa)c}$ on which the zero modes of the generating fields act as follows:

$$(2.7) \quad c_0 \mathrm{e}^{-b+(j+\kappa)c} = -\mathrm{e}^{-b+(j+\kappa)c}, \quad d_0 \mathrm{e}^{-b+(j+\kappa)c} = (2j+\kappa)\mathrm{e}^{-b+(j+\kappa)c}, \quad \mathrm{e}_0^{nc}\mathrm{e}^{-b+(j+\kappa)c} = \mathrm{e}^{-b+(j+n+\kappa)c}.$$

The conformal weight of e^{-b+jc} is then κ. Moreover, we have $\Pi_{[j]} \cong \Pi_{[j+1]}$, explaining the notation. Finally, $\Pi_{[j]}$ is irreducible and every irreducible relaxed highest-weight Π-module is isomorphic to some $\Pi_{[j]}$.

There are also irreducible weight Π-modules that are not relaxed highest-weight. Up to isomorphism, these may all be obtained by twisting the action of Π on some $\Pi_{[j]}$ by spectral flow. Let Y_Π denote the vertex map of Π, so that $A(z) \equiv Y_\Pi(A, z)$ for all $A \in \Pi$. Then, the action of the spectral flow map ς^ℓ, $\ell \in \mathbb{Z}$, on Π is given by [50]

$$(2.8) \quad \varsigma^\ell\big(A(z)\big) = Y_\Pi\big(\Sigma(\ell b, z)A, z\big), \quad \text{where } \Sigma(\ell b, z) = z^{-\ell b_0} \prod_{n=1}^\infty \exp\left(\frac{(-1)^n}{n}\ell b_n z^{-n}\right).$$

There is also a similar spectral flow map given by replacing b in (2.8) by a, but we shall not need it here.

The map ς^ℓ may be naturally lifted to an invertible functor on the category of weight Π-modules that is defined elementwise on objects, $v \in M \mapsto \varsigma^\ell(v) \in \varsigma^\ell(M)$, so that the action on the spectrally flowed module is given by

$$(2.9) \qquad A(z)\varsigma^\ell(v) = \varsigma^\ell\big(\varsigma^{-\ell}(A(z))v\big), \qquad A \in \Pi.$$

Every irreducible weight Π-module is then isomorphic to some $\varsigma^\ell(\Pi_{[j]})$ with $\ell \in \mathbb{Z}$ and $[j] \in \mathbb{C}/\mathbb{Z}$. In fact, it is easy to check that

$$(2.10) \qquad \varsigma^\ell(\mathrm{e}^{-b+(j+\kappa)c}) = \mathrm{e}^{(\ell-1)b+(j+\kappa)c}.$$

In particular, the vacuum state e^0 of Π belongs to the vacuum module $\varsigma(\Pi_{[-\kappa]})$.

2.2. **The Zamolodchikov algebra.** The Zamolodchikov algebra was introduced in [64]. Its universal version $\mathsf{W}_3^{\mathsf{k}}$ coincides with the regular (or principal) level-k W-algebra associated to $\mathfrak{sl}_3$.

Definition 2.3. *The universal Zamolodchikov algebra $\mathsf{W}_3^{\mathsf{k}}$ is the vertex operator algebra strongly generated by two elements T and W, subject to the operator product expansions*

$$(2.11) \quad T(z)T(w) \sim \frac{\mathsf{c}_{\mathsf{k}}^{\mathsf{W}_3}\mathbb{1}}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \qquad T(z)W(w) \sim \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w},$$

$$W(z)W(w) \sim \frac{2\Lambda(w)}{(z-w)^2} + \frac{\partial\Lambda(w)}{z-w} + A_{\mathsf{k}}\left[\frac{\mathsf{c}_{\mathsf{k}}^{\mathsf{W}_3}\mathbb{1}}{3(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} + \frac{\frac{3}{10}\partial^2 T(w)}{(z-w)^2} + \frac{\frac{1}{15}\partial^3 T(w)}{z-w}\right].$$

Here, $\mathsf{k} \in \mathbb{C} \setminus \{-3\}$ is the level, Λ denotes the quasiprimary field $:TT: -\frac{3}{10}\partial^2 T$,

$$(2.12) \qquad \mathsf{c}_{\mathsf{k}}^{\mathsf{W}_3} = -\frac{2(3\mathsf{k}+5)(4\mathsf{k}+9)}{\mathsf{k}+3} \quad and \quad A_{\mathsf{k}} = -\frac{(3\mathsf{k}+4)(5\mathsf{k}+12)}{2(\mathsf{k}+3)} = \frac{22+5\mathsf{c}_{\mathsf{k}}^{\mathsf{W}_3}}{16}.$$

For certain levels, including all nondegenerate ones, the universal Zamolodchikov algebra $\mathsf{W}_3^{\mathsf{k}}$ is not simple [53, 62]. For these levels, its unique simple quotient, which we shall denote by $\mathsf{W}_3(\mathsf{u}, \mathsf{v})$, is called a W_3 minimal model vertex operator algebra. For nondegenerate levels, $\mathsf{W}_3(\mathsf{u}, \mathsf{v})$ is rational and lisse [11, 12]. Moreover, we have $\mathsf{W}_3(\mathsf{u}, \mathsf{v}) = \mathsf{W}_3(\mathsf{v}, \mathsf{u})$ and $\mathsf{W}_3(3, 4) = \mathsf{W}_3(4, 3) \cong \mathbb{C}$.

Define modes T_m and W_m, $m \in \mathbb{Z}$, by expanding the generating fields as

$$(2.13) \qquad T(z) = \sum_{m\in\mathbb{Z}} T_m z^{-m-2} \quad and \quad W(z) = \sum_{m\in\mathbb{Z}} W_m z^{-m-3}.$$

Definition 2.4.

- *The eigenspaces of T_0, acting on a $\mathsf{W}_3^{\mathsf{k}}$-module, are the module's weight spaces and their nonzero elements are its weight vectors.*
- *A weight $\mathsf{W}_3^{\mathsf{k}}$-module is a module that is the direct sum of its weight spaces.*
- *A highest-weight vector for $\mathsf{W}_3^{\mathsf{k}}$ is a simultaneous eigenvector of T_0 and W_0 that is annihilated by the T_m and W_m with $m > 0$.*
- *A highest-weight $\mathsf{W}_3^{\mathsf{k}}$-module is a module that is generated by a highest-weight vector.*

It may seem tempting to refine the definition of a weight vector/space to instead be a simultaneous eigenspace of T_0 and W_0. However, there are natural examples that render this undesirable, see for instance [18, Sec. 2.2.2]. In particular, W_0 need not act semisimply on a highest-weight $\mathsf{W}_3^{\mathsf{k}}$-module, even though it does on the generating highest-weight vector. With the above definitions, a highest-weight $\mathsf{W}_3^{\mathsf{k}}$-module is always a weight module.

An irreducible highest-weight $\mathsf{W}_3^{\mathsf{k}}$-module $\mathcal{W}_{h,w}$ is thus determined, up to isomorphism, by the eigenvalues h of T_0 and w of W_0 on its highest-weight vector $v_{h,w}$. If k is parametrised by coprime integers u and v, as in (2.1), then we let $I_{\mathsf{u},\mathsf{v}}$ denote the set of pairs (h, w) such that $\mathcal{W}_{h,w}$ is a $\mathsf{W}_3(\mathsf{u}, \mathsf{v})$-module. For nondegenerate levels ($\mathsf{u}, \mathsf{v} \geqslant 3$), every irreducible $\mathsf{W}_3(\mathsf{u}, \mathsf{v})$-module is highest-weight; they were first identified in [28]. Here, we use the description of $I_{\mathsf{u},\mathsf{v}}$ given in [31] which is itself an adaptation of the parametrisation used in [19].

For $\ell \in \mathbb{Z}_{\geqslant 0}$, let $\mathsf{P}^\ell_\geqslant$ be the set of triples $t = (t_0, t_1, t_2)$ of nonnegative integers satisfying $t_0 + t_1 + t_2 = \ell$. Given a nondegenerate level, parametrised by $\mathsf{u}, \mathsf{v} \geqslant 3$ as in (2.1), consider the set $(\mathsf{P}^{\mathsf{u}-3}_\geqslant \times \mathsf{P}^{\mathsf{v}-3}_\geqslant)/\mathbb{Z}_3$, where the $\mathbb{Z}_3$-action is simultaneous cyclic permutation:

$$(2.14) \qquad \nabla \colon \big((r_0, r_1, r_2), (s_0, s_1, s_2)\big) \mapsto \big((r_2, r_0, r_1), (s_2, s_0, s_1)\big), \quad r \in \mathsf{P}^{\mathsf{u}-3}_\geqslant, \ s \in \mathsf{P}^{\mathsf{v}-3}_\geqslant.$$

The classifying set $I_{\mathsf{u},\mathsf{v}}$ is, for nondegenerate levels, isomorphic to $(\mathsf{P}^{\mathsf{u}-3}_\geqslant \times \mathsf{P}^{\mathsf{v}-3}_\geqslant)/\mathbb{Z}_3$ and an isomorphism is

$$(2.15a) \qquad h_{[r,s]} = \frac{1}{3\mathsf{uv}}\Big(\big(\mathsf{v}(r_1 + 1) - \mathsf{u}(s_1 + 1)\big)\big(\mathsf{v}(r_2 + 1) - \mathsf{u}(s_2 + 1)\big)$$
$$+ \big(\mathsf{v}(r_1 + 1) - \mathsf{u}(s_1 + 1)\big)^2 + \big(\mathsf{v}(r_2 + 1) - \mathsf{u}(s_2 + 1)\big)^2 - 3(\mathsf{v} - \mathsf{u})^2\Big),$$

$$(2.15b) \qquad w_{[r,s]} = \frac{\big(\mathsf{v}(r_0 - r_1) - \mathsf{u}(s_0 - s_1)\big)\big(\mathsf{v}(r_0 - r_2) - \mathsf{u}(s_0 - s_2)\big)\big(\mathsf{v}(r_1 - r_2) - \mathsf{u}(s_1 - s_2)\big)}{3(3\mathsf{uv})^{3/2}}.$$

We remark that the vacuum module of $\mathsf{W}_3(\mathsf{u}, \mathsf{v})$ is $\mathcal{W}_{0,0}$, corresponding to $[r, s] = [(\mathsf{u} - 3, 0, 0), (\mathsf{v} - 3, 0, 0)]$.

2.3. **The Bershadsky–Polyakov algebra.** The universal Bershadsky–Polyakov algebra BP^k was introduced in [17, 54]. It coincides with the subregular and minimal level-k W-algebra associated with $\mathfrak{sl}_3$ [41].

Definition 2.5. *The universal Bershadsky–Polyakov algebra BP^k is the vertex operator algebra strongly generated by four elements J, L, G^+ and G^-, subject to the operator product expansions*

$$J(z)J(w) \sim \frac{\kappa \mathbb{1}}{(z - w)^2}, \quad L(z)G^+(w) \sim \frac{G^+(w)}{(z - w)^2} + \frac{\partial G^+(w)}{z - w},$$

$$J(z)G^\pm(w) \sim \pm\frac{G^\pm(w)}{z - w}, \quad L(z)G^-(w) \sim \frac{2G^-(w)}{(z - w)^2} + \frac{\partial G^-(w)}{z - w},$$

$$(2.16) \qquad L(z)J(w) \sim -\frac{\kappa \mathbb{1}}{(z - w)^3} + \frac{J(w)}{(z - w)^2} + \frac{\partial J(w)}{z - w},$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad G^\pm(z)G^\pm(w) \sim 0,$$

$$L(z)L(w) \sim \frac{\mathsf{c}^{\mathsf{BP}}_\mathsf{k} \mathbb{1}}{2(z - w)^4} + \frac{2L(w)}{(z - w)^2} + \frac{\partial L(w)}{z - w},$$

$$G^+(z)G^-(w) \sim \frac{(\mathsf{k} + 1)(2\mathsf{k} + 3)\mathbb{1}}{(z - w)^3} + \frac{3(\mathsf{k} + 1)J(w)}{(z - w)^2} + \frac{3{:}J(w)J(w){:} + (2\mathsf{k} + 3)\partial J(w) - (\mathsf{k} + 3)L(w)}{z - w}.$$

Here, $\mathsf{k} \in \mathbb{C} \setminus \{-3\}$ is the level, κ was defined in (2.3) and

$$(2.17) \qquad \mathsf{c}^{\mathsf{BP}}_\mathsf{k} = -\frac{4(\mathsf{k} + 1)(2\mathsf{k} + 3)}{\mathsf{k} + 3}.$$

The universal Bershadsky–Polyakov algebra BP^k is not simple if and only if k has the form (2.1) with $\mathsf{u} \geqslant 2$ and $\mathsf{v} \geqslant 1$ [38]. In particular, this is the case for all admissible levels. When BP^k is not simple, its unique simple quotient, which we shall denote by $\mathsf{BP}(\mathsf{u}, \mathsf{v})$, is called a Bershadsky–Polyakov minimal model vertex operator algebra. Contrary to the case of the $\mathsf{W}_3(\mathsf{u}, \mathsf{v})$, $\mathsf{BP}(\mathsf{u}, \mathsf{v})$ is neither rational nor lisse for nondegenerate levels [5, 30]. The same turns out to be true for admissible levels with $\mathsf{v} = 1$ [6, 7]. However, $\mathsf{BP}(\mathsf{u}, \mathsf{v})$ is rational and lisse for admissible levels with $\mathsf{v} = 2$ [10, 11], these being exceptional levels in the sense of [15]. We remark that unlike the situation for the $\mathsf{W}_3(\mathsf{u}, \mathsf{v})$, there are no isomorphisms between the $\mathsf{BP}(\mathsf{u}, \mathsf{v})$ with different parameters. The trivial case is $\mathsf{BP}(3, 2) \cong \mathbb{C}$.

We have chosen the conformal vector L of the Bershadsky–Polyakov algebra so that the conformal weights of the generating fields are all integral. The corresponding mode expansions take the form

$$(2.18) \qquad J(z) = \sum_{n \in \mathbb{Z}} J_n z^{-n-1}, \quad L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \quad G^+(z) = \sum_{n \in \mathbb{Z}} G^+_n z^{-n-1} \quad \text{and} \quad G^-(z) = \sum_{n \in \mathbb{Z}} G^-_n z^{-n-2}.$$

With this convention, we record the commutation relations of the modes G^+_m and G^-_n for later convenience:

$$(2.19) \qquad \big[G^+_m, G^-_n\big] = 3{:}JJ{:}_{m+n} - (\mathsf{k} + 3)L_{m+n} + \big(\mathsf{k}m - (2\mathsf{k} + 3)(n + 1)\big)J_{m+n}$$
$$+ \tfrac{1}{2}(\mathsf{k} + 1)(2\mathsf{k} + 3)m(m - 1)\delta_{m+n,0}\mathbb{1}.$$

Definition 2.6.

- *The simultaneous eigenspaces of J_0 and L_0, acting on a BP^k-module, are the module's weight spaces and their nonzero elements are its weight vectors. The corresponding weight is the pair (j, Δ) of J_0- and L_0-eigenvalues.*
- *A weight BP^k-module is one that is the direct sum of its weight spaces.*
- *A relaxed highest-weight vector for BP^k is a weight vector that is annihilated by every mode with a positive index.*
- *A highest-weight vector (conjugate highest-weight vector) for BP^k is a relaxed highest-weight vector that is also annihilated by G_0^+ (G_0^-).*
- *A (relaxed/conjugate) highest-weight BP^k-module is then one that is generated by a (relaxed/conjugate) highest-weight vector.*

For future work, it is useful to extend these definitions to include BP^k-modules on which J_0 acts semisimply but L_0 does not.

Definition 2.7.

- *The intersections of the eigenspaces of J_0 and the generalised eigenspaces of L_0, both acting on a BP^k-module, are the module's generalised weight spaces and their nonzero elements are its generalised weight vectors.*
- *A generalised weight BP^k-module is one that is the direct sum of its generalised weight spaces.*

Note that for BP^k, an irreducible generalised weight module is always a weight module.

As usual, an irreducible highest-weight BP^k-module $\mathcal{H}_{j,\Delta}$ is determined, up to isomorphism, by the weight (j, Δ) of its highest-weight vector. One can of course twist the action of BP^k on $\mathcal{H}_{j,\Delta}$ by the conjugation automorphism γ defined by

$$(2.20) \qquad \begin{aligned} \gamma\big(J(z)\big) &= -J(z) + \kappa z^{-1}\mathbb{1}, & \gamma\big(G^+(z)\big) &= zG^-(z), \\ \gamma\big(L(z)\big) &= L(z) - \partial J(z) - z^{-1}J(z), & \gamma\big(G^-(z)\big) &= -z^{-1}G^+(z), \end{aligned}$$

as in (2.9). The corresponding functor, also denoted by γ, on the category of weight BP^k-modules then yields a bijective correspondence between highest-weight and conjugate highest-weight modules. As $\gamma(J_0) = \kappa\mathbb{1} - J_0$ and $\gamma(L_0) = L_0$, the weight of the conjugate highest-weight vector of $\gamma(\mathcal{H}_{j,\Delta})$ is $(\kappa - j, \Delta)$.

The story is a little different for general irreducible relaxed highest-weight BP^k-modules. For this case, it will be convenient to introduce some more terminology.

Definition 2.8.

- *The top space of a relaxed highest-weight BP^k-module is the subspace spanned by its vectors of minimal conformal weight.*
- *We shall say that a relaxed highest-weight BP^k-module is fully relaxed, for brevity, if the eigenvalues of J_0 on its top space fill out an entire coset in $\mathbb{C}/\mathbb{Z}$.*

We remark that highest-weight and conjugate highest-weight modules are relaxed but never fully relaxed.

In the relaxed case, a parametrisation of the irreducibles may be obtained by analysing the Zhu algebra $\mathsf{Zhu}[\mathsf{BP}^\mathsf{k}]$. This is known [7, 10] to be a central extension of a Smith algebra [59]. Here, we shall think of this Zhu algebra as the zero modes of BP^k acting on general relaxed highest-weight vectors (as in [56, App. B]). In this framework, $\mathsf{Zhu}[\mathsf{BP}^\mathsf{k}]$ is generated by J_0, L_0, G_0^+ and G_0^-. As always, L_0 is central in this algebra.

Proposition 2.9 ([30]).

(1) *The centraliser in $\mathsf{Zhu}[\mathsf{BP}^\mathsf{k}]$ of the subalgebra generated by J_0 and L_0 is $\mathbb{C}[J_0, L_0, \Omega]$, where the "cubic Casimir" Ω is central and acts on a relaxed highest-weight vector v as follows:*

$$(2.21) \qquad \Omega v = \Big(G_0^+ G_0^- + G_0^- G_0^+ + 2J_0^3 - (2\mathsf{k}+3)J_0^2 + J_0 - 2(\mathsf{k}+3)J_0 L_0 \Big) v.$$

(2) *The weight spaces of the top space of an irreducible relaxed highest-weight BP^k-module are 1-dimensional.*

(3) *An irreducible relaxed highest-weight BP^k-module is either highest-weight, conjugate highest-weight or fully relaxed.*

(4) *An irreducible fully relaxed BP^k-module is completely characterised, up to isomorphism, by the equivalence class $[j] \in \mathbb{C}/\mathbb{Z}$ of its J_0-eigenvalues, along with the common eigenvalues Δ of L_0 and ω of Ω on its top space.*

Proof. (1) is [30, Lem. 3.20]. It immediately implies (2), which itself implies (3). We therefore prove (4).

It suffices to show [65] that the action of $\mathsf{Zhu}[\mathsf{BP}^k]$ on the top space is determined by the weight (j, Δ) and Ω-eigenvalue ω of an arbitrarily chosen weight vector v in the top space. For this, it is sufficient to show that the actions of J_0, L_0, G_0^+ and G_0^- on a basis of the top space are so determined. If v' is a weight vector in the top space, then its J_0-eigenvalue is $j + n$, for some $n \in \mathbb{Z}$, by irreducibility. Irreducibility also means that v' may be obtained from v by acting with some combination of modes. Since the Poincaré–Birkhoff–Witt theorem holds for the mode algebra of BP^k [43, Thm. 4.1], we can actually obtain v' using only zero modes. If $n \geq 0$, order G_0^+ to the left. As the weight spaces of the top space are 1-dimensional, v' can only be obtained if it is a nonzero multiple of $(G_0^+)^n v$. Similarly, we see that v' is a nonzero multiple of $(G_0^-)^{-n} v$ for $n \leq 0$.

Since our module is fully relaxed, it follows that $\{v\} \cup \{(G_0^+)^n v, (G_0^-)^n v : n > 0\}$ is a basis of its top space. The action of J_0 and L_0 on these basis vectors is thus fixed by (j, Δ). For $n \geq 0$, the action of G_0^+ on the $(G_0^+)^n v$ and G_0^- on the $(G_0^-)^n v$ is also clear. It therefore remains to check if the action of G_0^+ on the $(G_0^-)^n v$ and G_0^- on the $(G_0^+)^n v$, for $n \geq 1$, is likewise fixed. But, this is clearly the case because $G_0^+ G_0^-$ and $G_0^- G_0^+$ act on the top space as a polynomial in J_0, L_0 and Ω, by (2.19) and (2.21). ∎

This almost completes the classification of irreducible relaxed highest-weight BP^k-modules — it only remains to determine which $[j]$, Δ and ω actually correspond to irreducible modules. Rather than delve into the details, we instead make some remarks about the analogous classification for $\mathsf{BP}(\mathsf{u}, \mathsf{v})$.

The classification of irreducible relaxed highest-weight $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-modules was obtained, for nondegenerate levels, in [30, Thm. 4.20] using properties of the minimal quantum hamiltonian reduction functor. The proof given there is quite subtle, but the result involves the same set $I_{\mathsf{u},\mathsf{v}} \cong (\mathsf{P}_{\geq}^{\mathsf{u}-3} \times \mathsf{P}_{\geq}^{\mathsf{v}-3})/\mathbb{Z}_3$ that appears in the classification of irreducible $\mathsf{W}_3(\mathsf{u}, \mathsf{v})$-modules (Section 2.2). One of our aims in what follows is to rederive this classification result for $\mathsf{BP}(\mathsf{u}, \mathsf{v})$ directly from that for $\mathsf{W}_3(\mathsf{u}, \mathsf{v})$, thereby naturally explaining why this set appears.

To achieve this aim, we shall also need the spectral flow functors σ^ℓ, $\ell \in \mathbb{Z}$, on the category of (generalised) weight BP^k-modules. They are defined in the same way as those introduced on the category of weight Π-modules in Section 2.1, except that b is replaced in Equation (2.8) by J. For later convenience, we give the action of spectral flow on the modes of the generating fields:

$$(2.22) \qquad \sigma^\ell(J_n) = J_n - \kappa\ell\delta_{n,0}\mathbb{1}, \quad \sigma^\ell(G_n^-) = G_{n+\ell}^-, \quad \sigma^\ell(G_n^+) = G_{n-\ell}^+, \quad \sigma^\ell(L_n) = L_n - \ell J_n + \tfrac{1}{2}\kappa\ell(\ell+1)\delta_{n,0}\mathbb{1}.$$

It is easy to check that the spectral flow and conjugation automorphisms satisfy the dihedral relation

$$(2.23) \qquad\qquad\qquad \sigma^\ell\gamma = \gamma\sigma^{-\ell}, \quad \ell \in \mathbb{Z}.$$

Let v be a weight vector of weight (j, Δ) in some $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-module $\mathcal{M}$. The spectral flow action (2.9) on Π-module elements generalises immediately to $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-modules (and BP^k-modules) by simply replacing ς by σ. Straightforward computation now verifies that $\sigma^\ell(v) \in \sigma^\ell(\mathcal{M})$ is a weight vector of weight

$$(2.24) \qquad\qquad\qquad \left(j + \kappa\ell, \Delta + \ell j + \tfrac{1}{2}\kappa\ell(\ell-1)\right).$$

This observation will turn out to be extremely useful in what follows.

2.4. **Inverse quantum hamiltonian reduction.** The idea that one could invert quantum hamiltonian reduction, in some sense, goes back to [58]. However, the crucial observation that this extends to functors between module categories is much more recent [2]. This latter observation was generalised to invert the (partial) reduction of BP^k to W_3^k and $\mathsf{BP}(\mathsf{u}, \mathsf{v})$ to $\mathsf{W}_3(\mathsf{u}, \mathsf{v})$ in [5]. Recall the definition (2.6) of $a, b \in \Pi$.

Theorem 2.10 ([5, Thms. 3.6 and 6.2])**.**

(1) *For* $\mathsf{k} \neq -3$, *there is an embedding* $\mathsf{BP}^{\mathsf{k}} \hookrightarrow \Pi \otimes \mathsf{W}_3^{\mathsf{k}}$ *of universal vertex operator algebras given by*

$$G^+ \mapsto \mathrm{e}^c \otimes \mathbb{1}, \quad J \mapsto b \otimes \mathbb{1}, \quad L \mapsto t \otimes \mathbb{1} + \mathbb{1} \otimes T,$$

$$(2.25) \qquad G^- \mapsto \mathrm{e}^{-c} \otimes \left(\tfrac{(\mathsf{k}+3)^{3/2}}{\sqrt{3}} W + \tfrac{1}{2}(\mathsf{k}+2)(\mathsf{k}+3)\partial T \right) + (\mathsf{k}+3)a_{-1}\mathrm{e}^{-c} \otimes T$$
$$- \left(a_{-1}^3 + 3(\mathsf{k}+2)a_{-2}a_{-1} + 2(\mathsf{k}+2)^2 a_{-3} \right) \mathrm{e}^{-c} \otimes \mathbb{1}.$$

(2) *This descends to an embedding* $\mathsf{BP}(\mathsf{u},\mathsf{v}) \hookrightarrow \Pi \otimes \mathsf{W}_3(\mathsf{u},\mathsf{v})$ *of minimal model vertex operator algebras unless* $\mathsf{u} \geqslant 2$ *and* $\mathsf{v} = 1$ *or* 2. *For these* u *and* v, *no such embedding of minimal model vertex operator algebras exists.*

Because J is identified with b in (2.25), Theorem 2.10 also identifies the spectral flow maps/functors ς and σ. We remark that the embedding of L implies the easily checked identity $\mathsf{c}_{\mathsf{k}}^{\Pi} + \mathsf{c}_{\mathsf{k}}^{\mathsf{W}_3} = \mathsf{c}_{\mathsf{k}}^{\mathsf{BP}}$. This identity dictated the choice of conformal structure made in (2.3) for Π.

Corollary 2.11.

(1) *For* $\mathsf{k} \neq -3$, *every* $(\Pi \otimes \mathsf{W}_3^{\mathsf{k}})$-*module is a* BP^{k}-*module by restriction. In particular,*

$$(2.26) \qquad \qquad \mathcal{R}_{[j],h,w} = \Pi_{[j]} \otimes \mathcal{W}_{h,w}$$

is a BP^{k}-*module, for any* $[j] \in \mathbb{C}/\mathbb{Z}$ *and* $h,w \in \mathbb{C}$.

(2) *For* $\mathsf{u} \geqslant 2$ *and* $\mathsf{v} \geqslant 3$, *every* $(\Pi \otimes \mathsf{W}_3(\mathsf{u},\mathsf{v}))$-*module is a* $\mathsf{BP}(\mathsf{u},\mathsf{v})$-*module by restriction. In particular,* $\mathcal{R}_{[j],h,w}$ *is a* $\mathsf{BP}(\mathsf{u},\mathsf{v})$-*module for all* $(h,w) \in I_{\mathsf{u},\mathsf{v}}$.

Recalling that $v_{h,w}$ denotes the highest-weight vector of $\mathcal{W}_{h,w}$, we see that the eigenvalue of $J_0 = b_0 \otimes \mathbb{1}$ on the relaxed highest-weight vector $\mathrm{e}^{-b+(j+\kappa)c} \otimes v_{h,w} \in \Pi_{[j]} \otimes \mathcal{W}_{h,w}$ is j, explaining the conventions that we chose for the $\Pi_{[j]}$ in Section 2.1.

Tensoring with a fixed $\Pi_{[j]}$ thus defines a functor from the weight module category of $\mathsf{W}_3^{\mathsf{k}}$ to that of BP^{k}, respectively $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ and $\mathsf{BP}(\mathsf{u},\mathsf{v})$. We call these the inverse quantum hamiltonian reduction functors (or just inverse reduction functors for short). Happily, the modules constructed by these functors turn out to be relevant for classifications.

We recall a useful definition from [5].

Definition 2.12. *A relaxed highest-weight* BP^{k}-*module is said to be almost irreducible if it is generated by its top space and all of its nonzero submodules have nonzero intersections with its top space.*

Of course, an irreducible relaxed highest-weight BP^{k}-module is almost irreducible. However, the existence of other almost irreducible BP^{k}-modules will be crucial for what follows.

Theorem 2.13 ([5, Cor. 5.11 and Thms. 5.12 and 6.3])**.** *For* $\mathsf{k} \neq -3$, *the* BP^{k}-*module* $\mathcal{R}_{[j],h,w}$:

(1) *is indecomposable, almost irreducible and fully relaxed*;

(2) *has a bijective action of* G_0^+;

(3) *is, for fixed* h *and* w, *irreducible for all but at least one, and at most three,* $[j] \in \mathbb{C}/\mathbb{Z}$.

Inverse reduction therefore allows us to construct a huge range of irreducible fully relaxed BP^{k}- and $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules (as well as a few reducible ones) from the irreducible highest-weight modules of $\mathsf{W}_3^{\mathsf{k}}$ and $\mathsf{W}_3(\mathsf{u},\mathsf{v})$, respectively. A natural question is whether every irreducible fully relaxed module is isomorphic to one that may be so constructed. When k is nondegenerate, the answer is of course yes, by the classification results of [30]. However, we seek an answer to this question that is intrinsic to inverse reduction, meaning that it does not rely on comparing with an independent classification theorem. As further motivation, we want to develop tools to extend the results of [30] to nonadmissible levels for which the classification is not presently known.

3. CLASSIFYING IRREDUCIBLE WEIGHT MODULES

We begin by specifying the module categories of interest.

Definition 3.1. *Let $\mathscr{W}^k$ and $\mathscr{W}_{u,v}$ denote the categories of generalised weight* BP^k*- and* $\mathsf{BP}(u,v)$*-modules, respectively, with finite-dimensional generalised weight spaces* (*see Definition* 2.7).

$\mathscr{W}_{u,v}$ is then a full subcategory of $\mathscr{W}^k$, where we assume that k, u and v are related by (2.1). Much is already known about these categories:

- For $k \in \mathbb{C} \setminus \{-3\}$, $\mathscr{W}^k$ is nonsemisimple with uncountably many irreducible modules (up to isomorphism).
- For $u, v \geqslant 3$ (nondegenerate levels), $\mathscr{W}_{u,v}$ is also nonsemisimple with uncountably many irreducibles [5, 30].
- For $u \geqslant 3$, $\mathscr{W}_{u,2}$ is semisimple with finitely many irreducibles [10] (in fact, it is a modular tensor category [40]).
- For $u \geqslant 2$, $\mathscr{W}_{u,1}$ has uncountably many irreducibles [6, 7]. $\mathscr{W}_{2,1}$ is semisimple, while the $\mathscr{W}_{n,1}$ with $n \geqslant 3$ are not.

Our aim here is to use inverse reduction to classify the irreducibles in $\mathscr{W}_{u,v}$. This requires the embedding of Theorem 2.10 to exist, so we are limited to studying $\mathscr{W}_{u,v}$ for nondegenerate levels and nonadmissible levels with $u = 2$ and $v \geqslant 3$. The classification for these latter levels is currently unknown.

Remark 3.2. *The methods introduced in this section may be straightforwardly adapted to prove the analogous classification of irreducible generalised weight modules, with finite-dimensional generalised weight spaces, for the simple affine vertex operator algebra* $\mathsf{L}_k(\mathfrak{sl}_2)$ *with* k *nondegenerate* (*meaning now that* $k + 2 = \frac{u}{v}$ *with* $u, v \geqslant 2$ *coprime*). *We leave the easy details to the reader.*

3.1. **Weight modules for the Heisenberg vertex algebra.** We start with a few useful results concerning the Heisenberg vertex subalgebra H of BP^k generated by J. Abstractly, this vertex algebra admits many choices of conformal vector, each of which yields a nonnegative-integer grading of H through the eigenvalues of the associated Virasoro zero mode L_0^{H}. Given a choice of grading operator L_0^{H}, a graded H-module is then just a module that decomposes as a direct sum of its generalised L_0^{H}-eigenspaces.

In this section, any operator L_0^{H} satisfying $[L_0^{\mathsf{H}}, J_n] = -n J_n$ will suffice. For our subsequent applications to BP^k-modules, we will therefore always take the grading operator to be L_0 (even though $L \notin \mathsf{H}$).

The results of this section are minor modifications of results of Futorny [36]; we provide proofs for completeness. For these, recall that the mode algebra of H is (an appropriate completion of) the universal enveloping algebra of the affine Kac–Moody algebra $\widehat{\mathfrak{gl}}_1$ (modulo the ideal in which the central element $\mathbb{1}$ is identified with the universal enveloping algebra's unit). The latter Lie algebra is spanned by the J_n and $\mathbb{1}$, with Lie bracket

$$(3.1) \qquad [J_m, J_n] = m\delta_{m+n,0}\kappa\mathbb{1}, \quad [J_m, \mathbb{1}] = 0, \qquad m, n \in \mathbb{Z}.$$

The parameter κ will be assumed in this section to be nonzero. Note that if v is a nonzero vector in an H-module satisfying $J_n v = 0$ for some $n \neq 0$, then $\kappa \neq 0$ forces $J_{-n} v \neq 0$.

We will also make much use of the operator $A = J_{-1}J_1 \in \mathsf{U}(\widehat{\mathfrak{gl}}_1)$. Its action on a Fock space (highest-weight Verma module) $\mathcal{F}_j$, with highest-weight vector v_j of J_0-eigenvalue $j \in \mathbb{C}$, picks out the number of J_{-1}-modes in each Poincaré–Birkhoff–Witt monomial: $A(\cdots J_{-2}^m J_{-1}^n v_j) = n\kappa(\cdots J_{-2}^m J_{-1}^n v_j)$. Up to the omnipresent factor of κ, the eigenvalues of A are thus nonnegative integers.

Lemma 3.3 ([36, Lem. 4.2]). *Assuming $\kappa \neq 0$, let $\mathcal{V}$ be a graded H-module with a nonzero finite-dimensional graded subspace $\mathcal{V}_\Delta$. Then, the eigenvalues of A on $\mathcal{V}_\Delta$ lie in $\kappa\mathbb{Z}_{\geqslant 0}$.*

Proof. Since $\mathcal{V}_\Delta$ is finite-dimensional and preserved by the A-action, A possesses an eigenvector $v \in \mathcal{V}_\Delta$. Let λ denote the associated eigenvalue and assume that $\lambda \notin \kappa\mathbb{Z}_{\geqslant 0}$. Since $\mathcal{V}$ is a module for a vertex operator algebra, we must have $J_n v = 0$ for $n \gg 0$. It follows that $J_{-n} v \neq 0$ for $n \gg 0$. Now consider

$$(3.2) \qquad J_{-1}J_1^{m+1}J_{-n}v = [J_{-1}, J_1^m]J_1 J_{-n}v + J_1^m A J_{-n}v = (\lambda - m\kappa)J_1^m J_{-n}v,$$

which holds for all $m \geqslant 0$ and $n > 1$. Since $\lambda \neq 0$, substituting $m = 0$ shows that $J_{-1}J_1J_{-n}v = \lambda J_{-n}v \neq 0$ for $n \gg 0$, hence that $J_1 J_{-n}v \neq 0$ for $n \gg 0$. Substituting successively larger values of m, we conclude inductively from $\lambda - m\kappa \neq 0$ that $J_1^m J_{-n}v \neq 0$ for all $m \geqslant 0$ and $n \gg 0$. In particular, $J_1^n J_{-n}v \in \mathcal{V}_\Delta$ is nonzero for all $n \gg 0$. But,

$$(3.3) \qquad A J_1^n J_{-n}v = J_{-1}J_1^{n+1}J_{-n}v = (\lambda - n\kappa)J_1^n J_{-n}v,$$

so A has infinitely many distinct eigenvalues on $\mathcal{V}_\Delta$. This contradicts $\dim \mathcal{V}_\Delta < \infty$. ∎

Lemma 3.4 ([36, Prop. 4.3])**.** *Assuming $\kappa \neq 0$, let $\mathcal{V}$ be a graded H-module with a nonzero finite-dimensional graded subspace $\mathcal{V}_\Delta$. Then, $\mathcal{V}$ has a submodule isomorphic to a Fock space whose highest-weight vector has grade $\Delta' \leqslant \Delta$.*

Proof. Again, A has eigenvectors in $\mathcal{V}_\Delta$ and the eigenvalues all have the form $r\kappa$, with $r \in \mathbb{Z}_{\geqslant 0}$, by Lemma 3.3. Choose an eigenvector v whose eigenvalue $r\kappa$ is such that r is maximal. We also assume, without loss of generality, that v is a J_0-eigenvector.

We claim that $J_n v = 0$ for all $n > 1$. To prove this, suppose that there exists $n > 1$ such that $J_n v \neq 0$. Then, $J_1 J_{-1}^{m+1} J_n v = (r+m+1)\kappa J_{-1}^m J_n v$ shows inductively that $J_{-1}^m J_n v \neq 0$ for all $m \geqslant 0$, because $r+m+1 > 0$. In particular, $J_{-1}^n J_n v \in \mathcal{V}_\Delta$ is nonzero, but calculation shows that it is an eigenvector of A with eigenvalue $(r + n)\kappa$. Since $n > 1$, this contradicts the maximality of r and the claim is proved.

Consider now the $J_1^m v$ with $m \geqslant 0$. If none of these vanish, then $J_n J_1^m v = J_1^m J_n v = 0$ for all $m \geqslant 0$ and $n > 1$ implies that $J_{-n}J_1^m v \neq 0$ for all $m \geqslant 0$ and $n > 1$. But then, $J_{-n}J_1^n v \neq 0$ is an A-eigenvector of eigenvalue $(r - n)\kappa$ for all $n > 1$, hence this again contradicts $\dim \mathcal{V}_\Delta < \infty$. We conclude that there exists a minimal $m > 0$ such that $J_1^m v = 0$. It follows that $w = J_1^{m-1}v \neq 0$ is a highest-weight vector of grade $\Delta' = \Delta - m + 1 \leqslant \Delta$. Clearly, it generates the desired Fock space as a submodule of $\mathcal{V}$. ∎

Proposition 3.5. *Assuming $\kappa \neq 0$, let $\mathcal{V}$ be a nonzero graded H-module whose grades all lie in $\Delta + \mathbb{Z}$, for some $\Delta \in \mathbb{C}$. Suppose further that all graded subspaces are finite-dimensional. Then, the grades of $\mathcal{V}$ are bounded below.*

Proof. Choose Δ so that $\mathcal{V}_\Delta \neq 0$. By Lemma 3.4, $\mathcal{V}^0 = \mathcal{V}$ has a Fock submodule, $\mathcal{F}_{j_0}$ say, whose highest-weight vector has grade $\Delta_0 \leqslant \Delta$. Since $\mathcal{F}_{j_0}$ is graded with $(\mathcal{F}_{j_0})_{\Delta_0+m} \neq 0$ for all $m \in \mathbb{Z}_{\geqslant 0}$, it follows that the quotient module $\mathcal{V}^1 = \mathcal{V}^0/\mathcal{F}_{j_0}$ has $\dim \mathcal{V}^1_\Delta < \dim \mathcal{V}^0_\Delta$. If $\mathcal{V}^1 \neq 0$, then Lemma 3.4 applies and we conclude that $\mathcal{V}^1$ has a Fock submodule, $\mathcal{F}_{j_1}$ say, whose highest-weight vector has grade $\Delta_1 \leqslant \Delta$. Moreover, the quotient $\mathcal{V}^2 = \mathcal{V}^1/\mathcal{F}_{j_1}$ has $\dim \mathcal{V}^2_\Delta < \dim \mathcal{V}^1_\Delta$. Continuing, we obtain a sequence of quotient H-modules $\mathcal{V}^m = \mathcal{V}^{m-1}/\mathcal{F}_{j_{m-1}}$ and Fock submodules $\mathcal{F}_{j_m} \subseteq \mathcal{V}^m$ whose highest-weight vectors have grades $\Delta_m \leqslant \Delta$. Because the dimension of $\mathcal{V}^m_\Delta$ is strictly decreasing, there exists n such that $\mathcal{V}^n_\Delta = 0$.

We claim that in fact $\mathcal{V}^n_{\Delta'} = 0$ for all $\Delta' \leqslant \Delta$. Suppose not, so that there exists $\Delta' < \Delta$ with $\mathcal{V}^n_{\Delta'} \neq 0$. Then, $\mathcal{V}^n_{\Delta'}$ is finite-dimensional, because $\mathcal{V}_{\Delta'}$ is, hence Lemma 3.4 applies and $\mathcal{V}^n$ has a Fock submodule $\mathcal{F}_{j_n}$ whose highest-weight vector has grade $\Delta_n \leqslant \Delta' < \Delta$. But, this is impossible because $(\mathcal{F}_{j_n})_\Delta \neq 0$ while $\mathcal{V}^n_\Delta = 0$. This proves that $\mathcal{V}^n_{\Delta'} = 0$ for all $\Delta' \leqslant \Delta$, hence that $\mathcal{V}$ has a minimal grade (the minimum of the Δ_m, $m = 0, 1, \ldots, n-1$). ∎

It is perhaps useful to finish with an example that illustrates the need for a finite-dimensionality hypothesis in Proposition 3.5. Consider the triangular decomposition of $\widehat{\mathfrak{gl}}_1$ into the following three Lie subalgebras:

$$(3.4) \qquad \mathfrak{n}_- = \mathrm{span}\{J_{-n}, J_1 : n \geqslant 2\}, \quad \mathfrak{h} = \mathrm{span}\{J_0, \mathbb{1}\}, \quad \mathfrak{n}_+ = \mathrm{span}\{J_n, J_{-1} : n \geqslant 2\}.$$

Setting $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$, we consider the $\mathfrak{b}$-module $\mathbb{C}v$ defined by $J_0 v = \mathfrak{n}_+ v = 0$ and $\mathbb{1}v = v$. The associated Verma module $\mathsf{U}(\widehat{\mathfrak{gl}}_1) \otimes_{\mathsf{U}(\mathfrak{b})} \mathbb{C}v$ has a Poincaré–Birkhoff–Witt basis consisting of monomials of the form $\cdots J_{-3}^\ell J_{-2}^m J_1^n v$. This $\widehat{\mathfrak{gl}}_1$-module is clearly graded with grades that differ by integers. However, the grades are neither bounded above nor below. More interestingly, it is a smooth $\widehat{\mathfrak{gl}}_1$-module (in the sense of [33]), hence it is an H-module. This is nevertheless consistent with Proposition 3.5 because its graded subspaces are all infinite-dimensional.

3.2. **Extremal weights.** We now return to our study of the categories $\mathscr{W}^{\mathsf{k}}$ and $\mathscr{W}_{\mathsf{u},\mathsf{v}}$ of generalised weight BP^{k}- and $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules, respectively, with finite-dimensional generalised weight spaces.

Definition 3.6.
- *An extremal weight of a BP^{k}-module $\mathcal{M}$ is a weight (j, Δ) whose L_0-eigenvalue Δ is minimal among those of all weights sharing the same J_0-eigenvalue j.*
- *$\mathcal{M}$ is said to admit extremal weights if there is an extremal weight for each eigenvalue of J_0 on $\mathcal{M}$.*

Consider any BP^{k}-module in $\mathscr{W}^{\mathsf{k}}$ whose L_0-eigenvalues all lie in $\Delta + \mathbb{Z}$, for some $\Delta \in \mathbb{C}$. For example, any indecomposable module in $\mathscr{W}^{\mathsf{k}}$ has this property. Then, its J_0-eigenspaces are H-modules to which Proposition 3.5 applies, as long as $\kappa \neq 0$. Assuming this, it follows that each J_0-eigenspace has an extremal weight, hence the BP^{k}-module admits extremal weights.

A slightly more general consequence of Proposition 3.5 is then as follows.

Proposition 3.7.
(1) *For $\mathsf{k} \neq -3, -\frac{3}{2}$, every finitely generated module in $\mathscr{W}^{\mathsf{k}}$ admits extremal weights.*
(2) *For coprime integers $\mathsf{u} \geqslant 2$ and $\mathsf{v} \geqslant 1$, every finitely generated module in $\mathscr{W}_{\mathsf{u},\mathsf{v}}$ admits extremal weights.*

Proof. These follow immediately as above, except when $\mathsf{u} = 3$ and $\mathsf{v} = 2$, hence $\mathsf{k} = -\frac{3}{2}$ and $\kappa = 0$. In this case, the Bershadsky–Polyakov minimal model vertex operator algebra is trivial: $\mathsf{BP}(3,2) \cong \mathbb{C}$. The finitely generated $\mathsf{BP}(3,2)$-modules are thus finite direct sums of the 1-dimensional module and they clearly admit extremal weights. In fact, they have a unique extremal weight: $(0, 0)$. ∎

Remark 3.8. *A second exceptional case occurs when $\mathsf{k} = -1$, equivalently $\mathsf{u} = 2$ and $\mathsf{v} = 1$, because the Bershadsky–Polyakov minimal model then reduces to the Heisenberg vertex algebra H [4]. In this case, the Fock modules are the irreducible modules in $\mathscr{W}_{2,1}$ and they also have a unique extremal weight: $(j, \frac{1}{2}j(3j-1))$.*

Lemma 3.9. *For $\mathsf{k} \neq -3, -1, -\frac{3}{2}$, the extremal weights of any irreducible module in $\mathscr{W}^{\mathsf{k}}$ have the form (j, Δ_j), where j runs over a complete equivalence class in $\mathbb{C}/\mathbb{Z}$.*

Proof. Obviously, the set of J_0-eigenvalues on any irreducible weight BP^{k}-module must be contained in a single equivalence class in $\mathbb{C}/\mathbb{Z}$. Suppose that the set of J_0-eigenvalues of the extremal weights of an irreducible module $\mathcal{M}$ in $\mathscr{W}^{\mathsf{k}}$ has a "gap" for which j belongs to this set but $j - 1$ does not. (The other possibility, that $j + 1$ does not belong, follows from this one by applying conjugation.)

Then, there exists a weight vector $v \in \mathcal{M}$ of J_0-eigenvalue j and we must have $G_m^- v = 0$ for all $m \in \mathbb{Z}$. As $\mathcal{M}$ is a module over a vertex operator algebra, we also have $G_n^+ v = 0$ for all $n \gg 0$. This implies that $[G_n^+, G_{-n}^-]v = 0$ for all $n \gg 0$. In particular, (2.19) gives

$$(3.5) \qquad 0 = \big([G_{n+1}^+, G_{-n-1}^-] - [G_n^+, G_{-n}^-]\big)v = 3(\mathsf{k}+1)J_0 v + (\mathsf{k}+1)(2\mathsf{k}+3)nv = (\mathsf{k}+1)\big(3j + (2\mathsf{k}+3)n\big)v,$$

for all $n \gg 0$. This is only possible if either $\mathsf{k} = -1$ or both $\mathsf{k} = -\frac{3}{2}$ and $j = 0$ hold. Otherwise, the set of J_0-eigenvalues cannot have such a gap. ∎

Lemma 3.10. *If $(j-1, \Delta - m)$, (j, Δ) and $(j+1, \Delta + n)$ are extremal weights of an irreducible module $\mathcal{M} \in \mathscr{W}^{\mathsf{k}}$, then $m \leqslant n$.*

Proof. Since the Poincaré–Birkhoff–Witt theorem holds for the mode algebra of BP^{k} [43, Thm. 4.1], we may choose an ordering so that monomials have the G_r^-, with $r > n$, as the rightmost modes and the G_r^-, with $r \leqslant n$, as the leftmost. With this ordering, every monomial that maps the extremal weight $(j+1, \Delta + n)$ into the extremal weight (j, Δ) has G_n^- as its leftmost mode. Similarly, every monomial mapping the extremal weight $(j+1, \Delta + n)$

into the extremal weight $(j-1, \Delta-m)$ has $G^-_{n_1} G^-_{n_2}$ as its two leftmost modes, where $n_1, n_2 \leqslant n$ and $n_1 + n_2 = m+n$. If $m > n$, then there are no such monomials. However, this contradicts the assumption that $\mathcal{M}$ is irreducible. $\blacksquare$

Note that Lemma 3.9 establishes that the hypothesised extremal weights in Lemma 3.10 always exist as long as $\mathsf{k} \neq -3, -1, -\frac{3}{2}$.

Theorem 3.11.

(1) *For* $\mathsf{k} \neq -3, -1, -\frac{3}{2}$, *every irreducible module in* $\mathscr{W}^{\mathsf{k}}$ *is the spectral flow of a relaxed highest-weight module.*

(2) *For coprime integers* $\mathsf{u} \geqslant 2$ *and* $\mathsf{v} \geqslant 1$, *every irreducible module in* $\mathscr{W}_{\mathsf{u},\mathsf{v}}$ *is the spectral flow of a relaxed highest-weight module.*

Proof. We prove the statement for $\mathscr{W}^{\mathsf{k}}$, noting that the statement for $\mathscr{W}_{\mathsf{u},\mathsf{v}}$ follows because we have already noted that each irreducible module in $\mathscr{W}_{2,1}$ and $\mathscr{W}_{3,2}$ is highest-weight (see Proposition 3.7 and Remark 3.8).

So take $\mathsf{k} \neq -3, -1, -\frac{3}{2}$ and fix an irreducible module $\mathcal{M} \in \mathscr{W}^{\mathsf{k}}$. Let (j, Δ_j) denote its extremal weights, where j runs over an equivalence class in $\mathbb{C}/\mathbb{Z}$ (Lemma 3.9). Defining $\delta_j(\mathcal{M}) = \Delta_{j+1} - \Delta_j$, it follows from Lemma 3.10 that $\delta_j(\mathcal{M})$ is weakly increasing with j. The limiting values $\delta_\infty(\mathcal{M})$ and $\delta_{-\infty}(\mathcal{M})$ are then defined, though they may be ∞ and $-\infty$, respectively.

Suppose first that $\delta_\infty(\mathcal{M}) \geqslant 0$ and $\delta_{-\infty}(\mathcal{M}) \leqslant 0$. Then, it follows that Δ_j must take a minimal value. Choose any j such that Δ_j achieves this global minimum. Then, the corresponding weight vectors are relaxed highest-weight vectors. As $\mathcal{M}$ is irreducible, it is thus a relaxed highest-weight module.

Suppose next that $\delta_{-\infty}(\mathcal{M}) > 0$, hence that $\delta_\infty(\mathcal{M}) > 0$ too. Then, Δ_j has no minima and $\mathcal{M}$ is not relaxed highest-weight. However, (2.24) shows that spectral flow maps extremal weights to extremal weights. It also shows that applying the functor σ^ℓ increases δ_j by ℓ:

$$(3.6) \qquad\qquad \delta_{j+(2\mathsf{k}+3)\ell/3}\big(\sigma^\ell(\mathcal{M})\big) = \delta_j(\mathcal{M}) + \ell.$$

Taking $\ell = -\delta_{-\infty}(\mathcal{M})$ and $j \to \pm\infty$ then gives $\delta_\infty\big(\sigma^\ell(\mathcal{M})\big) = \delta_\infty(\mathcal{M}) - \delta_{-\infty}(\mathcal{M}) \geqslant 0$ and $\delta_{-\infty}\big(\sigma^\ell(\mathcal{M})\big) = 0$. We therefore conclude that $\sigma^{-\delta_{-\infty}(\mathcal{M})}(\mathcal{M})$ is a relaxed highest-weight module, by the previous part.

The only remaining possibility is that $\delta_\infty(\mathcal{M}) < 0$, which requires that $\delta_{-\infty}(\mathcal{M}) < 0$ as well. In this case, a similar argument shows that $\sigma^{-\delta_\infty(\mathcal{M})}(\mathcal{M})$ is relaxed highest-weight. $\blacksquare$

3.3. **Completeness for irreducible fully relaxed modules.** The previous section reduced the classification of irreducible modules in $\mathscr{W}^{\mathsf{k}}$ and $\mathscr{W}_{\mathsf{u},\mathsf{v}}$ to the classification of relaxed highest-weight modules. In this section, we shall establish that the inverse reduction functors, when defined by (2.26), construct a complete set of irreducible fully relaxed $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules.

Recall that $\mathcal{R}_{[j],h,w} = \Pi_{[j]} \otimes \mathcal{W}_{h,w}$ is a BP^{k}-module, for all $\mathsf{k} \neq -3$, by Corollary 2.11(1).

Proposition 3.12. *For* $\mathsf{k} \neq -3$, *every irreducible fully relaxed* BP^{k}-*module* $\mathcal{M}$ *is isomorphic to* $\mathcal{R}_{[j],h,w}$, *for some* $[j] \in \mathbb{C}/\mathbb{Z}$ *and* $h, w \in \mathbb{C}$.

Proof. As $\mathcal{M}$ is irreducible and relaxed highest-weight, it is determined up to isomorphism by the eigenvalues of J_0, L_0 and Ω on some weight vector v in its top space, by Proposition 2.9(4). Let j', Δ and ω denote these eigenvalues, respectively. Then, we need only match them with those of some relaxed highest-weight vector $\mathrm{e}^{-b+(j+n+\kappa)c} \otimes v_{h,w}$, $n \in \mathbb{Z}$, in $\mathcal{R}_{[j],h,w}$. Here, $v_{h,w}$ is the highest-weight vector of $\mathcal{W}_{h,w}$.

As noted after Corollary 2.11, the J_0-eigenvalue is $j + n$. This means that we must choose $j' = j + n$, for some $n \in \mathbb{Z}$, hence $[j'] = [j]$ in $\mathbb{C}/\mathbb{Z}$. A similar computation with L_0 instead of J_0 leads to $\Delta = h + \kappa$. The computation for Ω is complicated by the form of G^-_0 in (2.25). However, it is enough to note that

$$(3.7\mathrm{a}) \qquad G^-_0\big(\mathrm{e}^{-b+(j+n+\kappa)c} \otimes v_{h,w}\big) = \big(\alpha_{\mathsf{k}} w + P_{\mathsf{k}}(j+n, h)\big)\mathrm{e}^{-b+(j+n-1+\kappa)c} \otimes v_{h,w},$$

where $\alpha_k = \frac{(k+3)^{3/2}}{\sqrt{3}}$ and P_k is the polynomial

$$(3.7b) \qquad P_k(j, h) = -(k+2)(k+3)h + \big((k+3)h - 2(k+2)^2\big)(j+\kappa) + 3(k+2)(j+\kappa)^2 - (j+\kappa)^3.$$

In fact, the precise form of this polynomial is unimportant here. All we need is that the consequent identification for the Ω-eigenvalue has the form $\omega = 2\alpha_k w + Q_k(j, h)$, where Q_k is a (different) polynomial in j and h, by (2.21). (Because Ω is central in the Zhu algebra of BP^k and $\mathcal{M}$ is irreducible, this polynomial is in fact independent of j.)

We conclude that any choice of $[j']$, Δ and ω corresponds to some (unique) choice of $[j]$, h and w. ∎

We mention that while the precise form of the polynomial P_k was not important for the proof of Proposition 3.12, it will be important in some of the finer classification analyses in Section 4.

Remark 3.13. *There is an alternative means to prove Proposition* 3.12 *that may be more useful when generalising to higher rank W-algebras. First, prove the corresponding statement for highest-weight (or conjugate highest-weight) modules. This is somewhat easier because the eigenvalues that one is required to match will not include those of any "higher Casimir" operators. Then, extend the proof to fully relaxed modules using the analogue of Mathieu's twisted localisation functors* [51] *for the W-algebra's Zhu algebra, as in* [45, 47].

This establishes the desired completeness result for the universal Bershadsky–Polyakov vertex operator algebras. We next turn to its analogue for irreducible fully relaxed $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-modules. This means restricting to $\mathsf{u} \geqslant 2$ and $\mathsf{v} \geqslant 3$, by Theorem 2.10(2). We mention again that the irreducible relaxed highest-weight $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-modules with $\mathsf{u} \geqslant 2$ and $\mathsf{v} = 1$ or 2 were already shown to be highest-weight in [7].

We start with a technical lemma about the embedding (2.25) of universal vertex operator algebras given in Theorem 2.10(1). Our proof involves characters and string functions, although it is also easy to give an equivalent combinatorial proof using Poincaré–Birkhoff–Witt bases.

Lemma 3.14. *For every $v \in \mathsf{W}_3^k$, we have $\mathrm{e}^{nc} \otimes v \in \mathsf{BP}^k$ for all $n \gg 0$.*

Proof. We may assume, without loss of generality, that v is a weight vector of weight h (say). Then, the statement of the lemma will follow if we can show that the dimensions of the weight spaces of $\Pi \otimes \mathsf{W}_3^k$ and BP^k, with weight $(n, h + n)$, match for $n \gg 0$. For this, it suffices to show that the string functions $s_n^{\mathsf{BP}}(q)$ of BP^k converge to the string functions $s_n(q)$ of $\Pi \otimes \mathsf{W}_3^k$ as $n \to \infty$. (We define these string functions below and refer to [46, App A] for further details.)

Define the characters of Π and W_3^k as follows:

$$
\begin{aligned}
\mathrm{ch}\big[\Pi\big](z; q) &= \mathrm{tr}_\Pi \, z^{b_0} q^{t_0 - c_k^\Pi/24} = \frac{\sum_{n \in \mathbb{Z}} z^n q^{n - c_k^\Pi/24}}{\prod_{i=1}^{\infty}(1 - q^i)^2}, \\
\mathrm{ch}\big[\mathsf{W}_3^k\big](q) &= \mathrm{tr}_{\mathsf{W}_3^k} q^{T_0 - c_k^{\mathsf{W}_3}/24} = \frac{q^{-c_k^{\mathsf{W}_3}/24}}{\prod_{i=1}^{\infty}(1 - q^{i+1})(1 - q^{i+2})}.
\end{aligned}
\tag{3.8}
$$

The string function $s_n(q)$ of $\Pi \otimes \mathsf{W}_3^k$ is then the coefficient of z^n in its character:

$$(3.9) \qquad s_n(q) = \frac{q^{n - c_k^{\mathsf{BP}}/24}}{\prod_{i=1}^{\infty}(1 - q^i)^2 (1 - q^{i+1})(1 - q^{i+2})}.$$

We note that $q^{-n} s_n(q)$ is independent of n. For this reason, we shall actually prove that the string function of BP^k, normalised by a factor of q^{-n}, converges as $n \to \infty$ to $q^{-n} s_n(q)$.

To do this, we employ the method of [46, App. A]. First, note that $J \mapsto b \otimes \mathbb{1}$ in (2.25) implies that the appropriate definition of character for BP^k is

$$(3.10) \qquad \mathrm{ch}\big[\mathsf{BP}^k\big](z; q) = \mathrm{tr}_{\mathsf{BP}^k} z^{J_0} q^{L_0 - c_k^{\mathsf{BP}}/24} = \frac{q^{-c_k^{\mathsf{BP}}/24}}{\prod_{i=1}^{\infty}(1 - q^i)(1 - zq^i)(1 - q^{i+1})(1 - z^{-1}q^{i+1})}.$$

(This is in fact the standard definition, explaining why we defined $\mathrm{ch}[\Pi]$ as we did above.) Next, note that as BP^{k} has finite-dimensional L_0-eigenspaces, its character (as a power series) must converge when $z = 1$. Looking at the poles in (3.10), we conclude that the right-hand side will give the correct power series when expanded in the region $|q| < 1$ and $|q|^2 < |z| < |q|^{-1}$. In particular, we may assume that $|zq| < 1$.

This motivates writing (3.10) in the form

$$
(3.11) \qquad \mathrm{ch}\big[\mathsf{BP}^{\mathsf{k}}\big](z; q) = \frac{q^{-c_{\mathsf{k}}^{\mathsf{BP}}/24}}{1 - zq} \frac{1}{\prod_{i=1}^{\infty}(1 - q^i)(1 - q^{i+1})(1 - zq^{i+1})(1 - z^{-1}q^{i+1})}
$$

$$
= q^{-c_{\mathsf{k}}^{\mathsf{BP}}/24} \sum_{\ell=0}^{\infty} z^{\ell} q^{\ell} \sum_{m=0}^{\infty} p_m(z) q^m,
$$

where the p_m are Laurent polynomials. The string function is then obtained as a residue about 0:

$$
(3.12) \qquad s_n^{\mathsf{BP}}(q) = \oint_0 \mathrm{ch}\big[\mathsf{BP}^{\mathsf{k}}\big](z; q)\, z^{-n-1}\, \frac{\mathrm{d}z}{2\pi\mathrm{i}} = \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} \oint_0 p_m(z) z^{\ell-n-1}\, \frac{\mathrm{d}z}{2\pi\mathrm{i}}\, q^{\ell+m-c_{\mathsf{k}}^{\mathsf{BP}}/24}.
$$

For each m, we see that the ℓ-sum may be extended to include the negative integers, provided that n is larger than the maximal power of z appearing in $p_m(z)$. In particular, this extension is justified in the $n \to \infty$ limit, giving

$$
(3.13) \qquad \lim_{n\to\infty} q^{-n} s_n^{\mathsf{BP}}(q) = \lim_{n\to\infty} q^{-n-c_{\mathsf{k}}^{\mathsf{BP}}/24} \oint_0 \sum_{\ell=-\infty}^{\infty} z^{\ell} q^{\ell} \sum_{m=0}^{\infty} p_m(z) q^m\, z^{-n-1}\, \frac{\mathrm{d}z}{2\pi\mathrm{i}}
$$

$$
= \lim_{n\to\infty} q^{-c_{\mathsf{k}}^{\mathsf{BP}}/24} \oint_0 \frac{\delta(zq)\,(zq)^{-n} z^{-1}}{\prod_{i=1}^{\infty}(1 - q^i)(1 - q^{i+1})(1 - zq^{i+1})(1 - z^{-1}q^{i+1})}\, \frac{\mathrm{d}z}{2\pi\mathrm{i}}
$$

$$
= \frac{q^{-c_{\mathsf{k}}^{\mathsf{BP}}/24}}{\prod_{i=1}^{\infty}(1 - q^i)^2 (1 - q^{i+1})(1 - q^{i+2})} \oint_0 \delta(zq)\, z^{-1}\, \frac{\mathrm{d}z}{2\pi\mathrm{i}} = q^{-n} s_n(q),
$$

as desired. Here, $\delta(x) = \sum_{\ell \in \mathbb{Z}} x^{\ell}$ denotes the delta function of formal power series. ∎

Recall that for coprime integers $\mathsf{u} \geqslant 2$ and $\mathsf{v} \geqslant 3$, $I_{\mathsf{u},\mathsf{v}}$ is the set of pairs $(h, w) \in \mathbb{C}^2$ such that the irreducible highest-weight $\mathsf{W}_3^{\mathsf{k}}$-module $\mathcal{W}_{h,w}$ is a $\mathsf{W}_3(\mathsf{u}, \mathsf{v})$-module. Corollary 2.11(2) then guarantees that the $\mathcal{R}_{[j],h,w}$, with $(h, w) \in I_{\mathsf{u},\mathsf{v}}$, are $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-modules. We now prove a converse.

Theorem 3.15. *For coprime integers $\mathsf{u} \geqslant 2$ and $\mathsf{v} \geqslant 3$, every irreducible fully relaxed $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-module is isomorphic to $\mathcal{R}_{[j],h,w}$, for some $[j] \in \mathbb{C}/\mathbb{Z}$ and $(h, w) \in I_{\mathsf{u},\mathsf{v}}$.*

Proof. By Proposition 3.12, every irreducible fully relaxed $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-module is isomorphic to $\mathcal{R}_{[j],h,w}$, for some $[j] \in \mathbb{C}/\mathbb{Z}$ and $(h, w) \in \mathbb{C}^2$. (Obviously, this means that this $\mathcal{R}_{[j],h,w}$ is also irreducible and fully relaxed.) Our task is thus to prove that in fact $(h, w) \in I_{\mathsf{u},\mathsf{v}}$.

Suppose that this is not the case, so that $\mathcal{W}_{h,w}$ is not a $\mathsf{W}_3(\mathsf{u}, \mathsf{v})$-module. We will show that this implies that $\mathcal{R}_{[j],h,w} = \Pi_{[j]} \otimes \mathcal{W}_{h,w}$ is not a $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-module. To see this, let J^{k} be the (unique) maximal ideal of $\mathsf{W}_3^{\mathsf{k}}$. Then, $\mathsf{J}^{\mathsf{k}} \cdot \mathcal{W}_{h,w} \neq 0$. In fact, as $\mathcal{W}_{h,w}$ is generated by its highest-weight vector $v_{h,w}$, we must have $\mathsf{J}^{\mathsf{k}} v_{h,w} \neq 0$. In other words, there exists $\chi \in \mathsf{J}^{\mathsf{k}}$ and $m \in \mathbb{Z}$ such that $\chi_m v_{h,w} \neq 0$. We shall choose m to be maximal with this property.

By Lemma 3.14, there exists $n \in \mathbb{Z}$ such that $\mathrm{e}^{nc} \otimes \chi \in \mathsf{BP}^{\mathsf{k}}$. We claim that for this n, $\mathrm{e}^{nc} \otimes \chi$ is necessarily in the (unique) maximal ideal K^{k} of BP^{k}. For if this were not the case, then we could act with BP^{k}-modes to bring $\mathrm{e}^{nc} \otimes \chi$ to the vacuum vector $\mathbb{1}_{\mathsf{BP}} = \mathbb{1}_{\Pi} \otimes \mathbb{1}_{\mathsf{W}_3}$. However, this is impossible because $\chi \in \mathsf{J}^{\mathsf{k}}$. On the other hand, the maximality of m gives

$$
(3.14) \qquad (\mathrm{e}^{nc} \otimes \chi)_m (\mathrm{e}^{-b+(j+\kappa)c} \otimes v_{h,w}) = \mathrm{e}_0^{nc}\, \mathrm{e}^{-b+(j+\kappa)c} \otimes \chi_m v_{h,w} = \mathrm{e}^{-b+(j+n+\kappa)c} \otimes \chi_m v_{h,w} \neq 0.
$$

This shows that there is an element of K^{k} acting nontrivially on an element of $\mathcal{R}_{[j],h,w}$, proving that $\mathcal{R}_{[j],h,w}$ is not a $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-module, as required. ∎

We recall from Section 2.2 that for nondegenerate levels, $I_{\mathsf{u},\mathsf{v}}$ is known to be isomorphic to $(\mathsf{P}_{\geqslant}^{\mathsf{u}-3} \times \mathsf{P}_{\geqslant}^{\mathsf{v}-3})/\mathbb{Z}_3$, see Equations (2.14) and (2.15). Consequently, Theorem 3.15 recovers the fully relaxed part of the irreducible

classification in $\mathscr{W}_{u,v}$ for these levels, as was first established in [30] using different methods. Crucially, the inverse reduction arguments given here explain why the set $(\mathsf{P}^{u-3}_{\geqslant} \times \mathsf{P}^{v-3}_{\geqslant})/\mathbb{Z}_3$ appears in this result.

3.4. **Completeness for irreducible highest-weight modules.** Having classified the irreducible fully relaxed $\mathsf{BP}(u, v)$-modules, for $u \geqslant 2$ and $v \geqslant 3$, we turn to the remaining irreducible relaxed highest-weight modules. As noted in Proposition 2.9(3), these are either highest-weight or conjugate highest-weight. We shall start by classifying the highest-weight $\mathsf{BP}(u, v)$-modules with an infinite-dimensional top space.

Proposition 3.16. *Let* $\mathsf{k} \neq -3$ *(coprime integers* $u \geqslant 2$ *and* $v \geqslant 3$*). Then*:

(1) *Every irreducible conjugate highest-weight* BP^{k}-*module* $(\mathsf{BP}(u, v)$-*module*$)$ $\mathcal{C}_{j,\Delta}$ *with an infinite-dimensional top space is isomorphic to a submodule of* $\mathcal{R}_{[j],h,w}$, *for some unique* $(h, w) \in \mathbb{C}^2$ $((h, w) \in I_{u,v})$.

(2) *Every irreducible highest-weight* BP^{k}-*module* $(\mathsf{BP}(u, v)$-*module*$)$ $\mathcal{H}_{j,\Delta}$ *with an infinite-dimensional top space is isomorphic to a quotient of* $\mathcal{R}_{[j],h,w}$, *for some unique* $(h, w) \in \mathbb{C}^2$ $((h, w) \in I_{u,v})$.

Proof. As in the proof of Proposition 3.12, we search for a conjugate highest-weight vector of weight $(j + 1, \Delta)$ among the relaxed highest-weight vectors $\mathrm{e}^{-b+(j'+n+\kappa)c} \otimes v_{h,w}$, $n \in \mathbb{Z}$, of $\mathcal{R}_{[j'],h,w}$. Calculation with (2.25) shows that the weights match if we take $j' + n = j + 1$, hence $[j'] = [j]$, and $h = \Delta - \kappa$, while being a conjugate highest-weight vector fixes w uniquely in terms of j and Δ, by (3.7). This vector then generates a conjugate highest-weight submodule $\mathcal{C}$ of $\mathcal{R}_{[j],h,w}$. Evidently, $\mathcal{C}_{j+1,\Delta}$ is a quotient of $\mathcal{C}$. However, every nonzero submodule of $\mathcal{R}_{[j],h,w}$ has a nonzero intersection with its top space (Theorem 2.13(1)), hence the same is true for its submodule $\mathcal{C}$. If $\mathcal{C}_{j+1,\Delta} \not\cong \mathcal{C}$, then $\mathcal{C}$ has a submodule whose intersection with its top space is nonzero. However, this is impossible as the top spaces of both $\mathcal{C}_{j+1,\Delta}$ and $\mathcal{C}$ have the same weights, $(j + n + 1, \Delta)$ for all $n \geqslant 0$, appearing with the same multiplicities, while $\mathcal{C}_{j+1,\Delta}$ is irreducible. We conclude that $\mathcal{C}_{j+1,\Delta} \not\cong \mathcal{C}$, proving (1) for BP^{k}.

For (2), note that the top space of the quotient $\mathcal{Q} = \mathcal{R}_{[j],h,w}/\mathcal{C}$ has weights $(j - n, \Delta)$, for all $n \geqslant 0$. Consequently, $\mathcal{Q}$ has a highest-weight vector of weight (j, Δ). Let $\mathcal{H}$ be the highest-weight submodule of $\mathcal{Q}$ generated by this highest-weight vector. As $\mathcal{H}_{j,\Delta}$ is irreducible, it is a quotient of $\mathcal{H}$. Because its top space is infinite-dimensional, its top space's weights precisely match those of $\mathcal{Q}$, hence so do those of $\mathcal{H}$. By Proposition 2.9(2), the top spaces of $\mathcal{H}$ and $\mathcal{Q}$ therefore coincide. But, $\mathcal{R}_{[j],h,w}$ is generated by its top space, by Theorem 2.13(1), hence the same is true for $\mathcal{Q}$. It follows that $\mathcal{H} = \mathcal{Q}$, hence that $\mathcal{H}_{j,\Delta}$ is a quotient of $\mathcal{Q}$ and, thus, also of $\mathcal{R}_{[j],h,w}$. This completes the proof for BP^{k}-modules.

To finish, we only need to show that $\mathcal{C}_{j+1,\Delta}$ or $\mathcal{H}_{j,\Delta}$ being a $\mathsf{BP}(u, v)$-module implies that $\mathcal{R}_{[j'],h,w}$ is too. This is essentially [30, Prop. 4.22] (see also [47, Thm. 5.3]). We sketch the proof for $\mathcal{H}_{j,\Delta}$ for completeness, leaving that for $\mathcal{C}_{j+1,\Delta}$ as an exercise.

Recall that $\mathcal{R}_{[j],h,w}$ is generated by its top space. In fact, it is generated by any of its top space weight vectors as long as the J_0-eigenvalue is at most j. This follows as G_0^+ acts bijectively on the weight spaces of the top space while G_0^- acts bijectively on those with J_0-eigenvalue at most j (because the quotient $\mathcal{H}_{j,\Delta}$ is irreducible).

Since $\mathsf{Zhu}[\mathsf{BP}^{\mathsf{k}}]$ is noetherian [59], its maximal ideal is generated by a finite number of J_0-eigenvectors $A_0^{(i)}$, say. Choose a positive n greater than the J_0-eigenvalues of the $A_0^{(i)}$ and pick a weight vector v of weight $(j - n, \Delta)$ in the top space of $\mathcal{R}_{[j],h,w}$. Then, v generates $\mathcal{R}_{[j],h,w}$. Its image $\bar{v}$ in $\mathcal{H}_{j,\Delta}$ is annihilated by the $A_0^{(i)}$ because $\mathcal{H}_{j,\Delta}$ is a $\mathsf{BP}(u, v)$-module. It follows that $A_0^{(i)}v$ must lie in a weight space of the maximal submodule of $\mathcal{R}_{[j],h,w}$, the quotient by which is $\mathcal{H}_{j,\Delta}$. However, the J_0-eigenvalue of $A_0^{(i)}v$ is not greater than j, for all i, by construction. The weight space of the maximal submodule is therefore 0, so $A_0^{(i)}v = 0$ for all i. We conclude that the maximal ideal of $\mathsf{Zhu}[\mathsf{BP}^{\mathsf{k}}]$ annihilates a vector v in the top space of $\mathcal{R}_{[j],h,w}$ that generates the entire module. This proves that $\mathcal{R}_{[j],h,w}$ is a $\mathsf{BP}(u, v)$-module, as desired. ∎

This implies that we can obtain a complete set of irreducible highest-weight $\mathsf{BP}(u, v)$-modules, with infinite-dimensional top spaces, by identifying the irreducible quotient of each reducible $\mathcal{R}_{[j],h,w}$. A complete set of irreducible conjugate highest-weight modules, again with infinite-dimensional top spaces, is then obtained by

applying the conjugation functor. It only remains to study the irreducible highest-weight $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-modules, with finite-dimensional top spaces.

Proposition 3.17. *For* $\mathsf{k} \notin \{-3\} \cup \frac{1}{2}\mathbb{Z}_{\geqslant -3}$, *the spectral flow orbit* $\mathbb{O}_{\mathcal{H}} = \{\sigma^\ell(\mathcal{H}) : \ell \in \mathbb{Z}\}$ *of any irreducible highest-weight* BP^k-*module* $\mathcal{H}$ *contains*:

(1) *a unique highest-weight module whose top space is infinite-dimensional*;

(2) *a unique conjugate highest-weight module whose top space is infinite-dimensional*;

(3) *at most two highest-weight modules with finite-dimensional top spaces*.

Proof. We start with some choice of highest weight $(j, \Delta) \in \mathbb{C}^2$ and aim to show that the spectral flow orbit of $\mathcal{H}_{j,\Delta}$ has a highest-weight module with an infinite-dimensional top space. If $\mathcal{H}_{j,\Delta}$ already satisfies this requirement, then there is nothing to prove. So suppose that its top space is finite-dimensional and let v denote its highest-weight vector. Then, $(G_0^-)^n v = 0$ for some minimal $n \geqslant 1$. We set

$$f(j, \Delta) = 3j^2 - (\mathsf{k} + 3)\Delta - (2\mathsf{k} + 3)j$$

(3.15)
$$\text{and} \quad g_n(j, \Delta) = \frac{1}{n} \sum_{m=0}^{n-1} f(j - m, \Delta) = 3j^2 - (\mathsf{k} + 3)\Delta - (2\mathsf{k} + 3n)j + (n - 1)(\mathsf{k} + n + 1),$$

so that (2.19) gives

(3.16)
$$0 = \left[G_0^+, (G_0^-)^n\right]v = \sum_{m=0}^{n-1} (G_0^-)^{n-1-m}\left[G_0^+, G_0^-\right](G_0^-)^m v = \sum_{m=0}^{n-1} (G_0^-)^{n-1-m} f(J_0, L_0)(G_0^-)^m v$$

$$= \sum_{m=0}^{n-1} f(j - m, \Delta)(G_0^-)^{n-1}v = n g_n(j, \Delta)(G_0^-)^{n-1}v,$$

hence $g_n(j, \Delta) = 0$.

As $\mathcal{H}_{j,\Delta}$ is irreducible with finite-dimensional top space, its image under the spectral flow functor σ is also irreducible and highest-weight, with highest-weight vector $\sigma\left((G_0^-)^{n-1}v\right)$. Equation (2.24) then gives

(3.17)
$$\sigma(\mathcal{H}_{j,\Delta}) \cong \mathcal{H}_{j-n+1+\kappa, \Delta+j-n+1}.$$

If $\sigma(\mathcal{H}_{j,\Delta})$ has an infinite-dimensional top space, then we are done. If not, then $g_m(j - n + 1 + \kappa, \Delta + j - n + 1) = 0$ for some minimal $m \geqslant 1$. However, this implies that

(3.18)
$$0 = g_m(j - n + 1 + \kappa, \Delta + j - n + 1) - g_n(j, \Delta) = (3j + 3 - m - 2n)(\mathsf{k} + 3 - m - n).$$

Noting that the last factor on the right-hand side can only vanish if k lies in $\mathbb{Z}_{\geqslant -1} \subset \frac{1}{2}\mathbb{Z}_{\geqslant -3}$, we conclude that

(3.19)
$$h_{m,n}(j) = 3j + 3 - m - 2n = 0.$$

Continuing, $\sigma(\mathcal{H}_{j,\Delta})$ having a finite-dimensional top space means that $\sigma^2(\mathcal{H}_{j,\Delta}) \cong \sigma(\mathcal{H}_{j-n+1+\kappa, \Delta+j-n+1})$ is another irreducible highest-weight module. If its top space were also finite-dimensional, then we would conclude as above that $h_{\ell,m}(j - n + 1 + \kappa) = 0$ for some minimal $\ell \geqslant 1$. However, this contradicts $\mathsf{k} \notin \frac{1}{2}\mathbb{Z}_{\geqslant -3}$:

(3.20)
$$0 = h_{\ell,m}(j - n + 1 + \kappa) - h_{m,n}(j) = 2(\mathsf{k} + 3) - \ell - m - n.$$

This establishes the existence of a highest-weight module with infinite-dimensional top space in $\mathbb{O}_{\mathcal{H}_{j,\Delta}}$.

We next claim that $\mathbb{O}_{\mathcal{H}_{j,\Delta}}$ also contains a conjugate highest-weight module with infinite-dimensional top space. This follows from the easily checked fact that applying σ to an irreducible conjugate highest-weight module results in a highest-weight module:

$\mathbb{O}_{\mathcal{H}_{j,\Delta}}$ contains an irreducible highest-weight module

$\Rightarrow \quad \gamma\left(\mathbb{O}_{\mathcal{H}_{j,\Delta}}\right)$ contains an irreducible conjugate highest-weight module $\mathcal{C}$

$\Rightarrow \quad \mathcal{H} = \sigma(\mathcal{C})$ is an irreducible highest-weight module in $\gamma\left(\mathbb{O}_{\mathcal{H}_{j,\Delta}}\right)$

$\Rightarrow$ $\mathbb{O}_{\mathcal{H}} = \gamma(\mathbb{O}_{\mathcal{H}_{j,\Delta}})$ contains an irreducible highest-weight module $\mathcal{H}'$ with infinite-dimensional top space

$\Rightarrow$ $\gamma(\mathcal{H}')$ is an irreducible conjugate highest-weight module with infinite-dimensional top space in $\mathbb{O}_{\mathcal{H}_{j,\Delta}}$.

Finally, the uniqueness of this highest-weight and conjugate highest-weight module in $\mathbb{O}_{\mathcal{H}_{j,\Delta}}$ follows from the fact that applying σ^n, $n > 0$ ($n < 0$), to a highest-weight BP^{k}-module (conjugate highest-weight BP^{k}-module) with infinite-dimensional top space results in a BP^{k}-module that is not relaxed highest-weight. This proves (1) and (2), while (3) now follows from the contradiction of Equation (3.20). ∎

Remark 3.18. *Note that* $\mathsf{k} \in \frac{1}{2}\mathbb{Z}_{\geqslant -3}$ *is equivalent to* $\mathsf{u} \geqslant 2$ *and* $\mathsf{v} = 1$ *or* 2. *Moreover, for these* u *and* v, *every irreducible highest-weight* $\mathsf{BP}(\mathsf{u},\mathsf{v})$-*module has a finite-dimensional top space* [7, 10]. *In particular, the spectral flow orbits never include modules with infinite-dimensional top spaces.*

It follows from Proposition 3.17 that we will obtain a complete set of irreducible highest-weight BP^{k}- or $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules with finite-dimensional top spaces, the latter assuming $\mathsf{u} \geqslant 2$ and $\mathsf{v} \geqslant 3$, by looking at the spectral flow orbits of the irreducible highest-weight modules with infinite-dimensional top spaces. Indeed, it follows from the above analysis that if $\mathcal{H}_{j,\Delta}$ has an infinite-dimensional top space, then the only possible candidates for finite-dimensional top spaces are $\sigma^{-1}(\mathcal{H}_{j,\Delta})$ and $\sigma^{-2}(\mathcal{H}_{j,\Delta})$.

We assemble the main results thus far, namely Proposition 2.9(4) as well as Theorems 3.11 and 3.15 and Propositions 3.12, 3.16 and 3.17(3), as a theorem.

Theorem 3.19. *For* $\mathsf{k} \neq -3, -1, -\frac{3}{2}$ *(coprime integers* $\mathsf{u} \geqslant 2$ *and* $\mathsf{v} \geqslant 3$*), every simple object of the category* $\mathscr{W}^{\mathsf{k}}$ *($\mathscr{W}_{\mathsf{u},\mathsf{v}}$) of generalised weight* BP^{k}-*modules ($\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules), with finite-dimensional generalised weight spaces, is isomorphic to either*:

- *A spectral flow of an irreducible fully relaxed module* $\mathcal{R}_{[j],h,w}$ *with* $[j] \in \mathbb{C}/\mathbb{Z}$ *and* $h, w \in \mathbb{C}$ *(($h, w) \in I_{\mathsf{u},\mathsf{v}}$).*
- *A spectral flow of an irreducible (highest-weight) quotient* $\mathcal{H}_{j,\Delta}$ *of a reducible fully relaxed module* $\mathcal{R}_{[j'],h,w}$ *with* $[j'] \in \mathbb{C}/\mathbb{Z}$ *and* $h, w \in \mathbb{C}$ *(($h, w) \in I_{\mathsf{u},\mathsf{v}}$).*

Remark 3.20. *Considering Proposition 3.17(2) instead of (3) (or applying conjugation), it is clear that we can alternatively characterise the simple objects of* $\mathscr{W}^{\mathsf{k}}$ *and* $\mathscr{W}_{\mathsf{u},\mathsf{v}}$ *as spectral flows of irreducible fully relaxed modules and irreducible (conjugate highest-weight) submodules of reducible fully relaxed modules.*

Algorithmically, this theorem allows us to classify (subject to the stated restrictions on k, u and v) the irreducible BP^{k}- and $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules in $\mathscr{W}^{\mathsf{k}}$ and $\mathscr{W}_{\mathsf{u},\mathsf{v}}$, respectively, using inverse quantum hamiltonian reduction:

- For each (h, w), determine for which $[j] \in \mathbb{C}/\mathbb{Z}$, $\mathcal{R}_{[j],h,w} = \Pi_{[j]} \otimes \mathcal{W}_{h,w}$ is irreducible.
- For each of the (up to 3) $[j] \in \mathbb{C}/\mathbb{Z}$ with $\mathcal{R}_{[j],h,w}$ reducible, identify its (unique) irreducible quotient $\mathcal{H}_{j,\Delta}$.
- Apply σ^{ℓ}, for all $\ell \in \mathbb{Z}$, to all the irreducible $\mathcal{R}_{[j],h,w}$ and $\mathcal{H}_{j,\Delta}$.

We shall see how to implement this algorithm with examples in the next section.

Remark 3.21. *A natural question is whether inverse quantum hamiltonian reduction also allows one to analyse the possibility of nonsplit extensions between irreducible modules. For example, for nondegenerate levels, can one use the semisimplicity of the category of* $\mathsf{W}_3(\mathsf{u},\mathsf{v})$-*modules to prove the semisimplicity of the analogue of the BGG category* $\mathcal{O}_{\mathsf{k}}$ *for* $\mathsf{BP}(\mathsf{u},\mathsf{v})$? *The latter fact was in fact established in* [30], *but by using minimal quantum hamiltonian reduction to relate it back to the semisimplicity* [13] *of* $\mathcal{O}_{\mathsf{k}}$ *for the simple affine vertex operator algebra* $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_3)$. *However, we expect that this method will be difficult to generalise.*

4. EXAMPLES

We apply the general results of the previous Section 3 to $\mathsf{BP}(\mathsf{u},\mathsf{v})$ for two classes of (u,v). The first, $\mathsf{u}, \mathsf{v} \geqslant 3$, corresponds to k being nondegenerate. The second, $(\mathsf{u},\mathsf{v}) = (2,3)$, corresponds to the nonadmissible level $\mathsf{k} = -\frac{7}{3}$.

4.1. **Nondegenerate levels.** In this section, we classify irreducible relaxed highest-weight $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-modules when $\mathsf{u}, \mathsf{v} \geqslant 3$ (k is nondegenerate). This result was originally obtained in [30] using properties of the minimal quantum hamiltonian reduction functor. Here, we obtain it straightforwardly using inverse quantum hamiltonian reduction and lift it to a classification of the simple objects of $\mathscr{W}_{\mathsf{u}, \mathsf{v}}$, again when $\mathsf{u}, \mathsf{v} \geqslant 3$.

Recall that for nondegenerate levels, $I_{\mathsf{u}, \mathsf{v}}$ is isomorphic, via the parametrisations $h_{[r,s]}$ and $w_{[r,s]}$ of (2.15), to $(\mathsf{P}^{\mathsf{u}-3}_{\geqslant} \times \mathsf{P}^{\mathsf{v}-3}_{\geqslant})/\mathbb{Z}_3$, where the $\mathbb{Z}_3$-action is effected by the permutation ∇ of (2.14). We define

$$(4.1) \qquad j_{(r,s)} = \tfrac{1}{3}\big(r_2 - r_1 - \tfrac{\mathsf{u}}{\mathsf{v}}(s_2 - s_1 - 1)\big), \quad (r, s) \in \mathsf{P}^{\mathsf{u}-3}_{\geqslant} \times \mathsf{P}^{\mathsf{v}-3}_{\geqslant},$$

and recall that $[r, s] = \big\{(r, s), \nabla(r, s), \nabla^2(r, s)\big\}$.

Theorem 4.1. *Let k be nondegenerate, so that $\mathsf{u}, \mathsf{v} \geqslant 3$. Then, every irreducible $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-module in $\mathscr{W}_{\mathsf{u}, \mathsf{v}}$ is isomorphic to one, and only one, of the following:*

- *The $\sigma^\ell\big(\mathcal{R}_{[j], h_{[r,s]}, w_{[r,s]}}\big)$ with $\ell \in \mathbb{Z}$, $[r, s] \in (\mathsf{P}^{\mathsf{u}-3}_{\geqslant} \times \mathsf{P}^{\mathsf{v}-3}_{\geqslant})/\mathbb{Z}_3$ and $[j] \notin \big\{[j_{(r',s')}] : (r', s') \in [r, s]\big\}$.*
- *The $\sigma^\ell\big(\mathcal{H}_{j_{(r,s)}-1, h_{[r,s]}+\kappa}\big)$ with $\ell \in \mathbb{Z}$ and $(r, s) \in \mathsf{P}^{\mathsf{u}-3}_{\geqslant} \times \mathsf{P}^{\mathsf{v}-3}_{\geqslant}$.*

Proof. As $\mathcal{R}_{[j], h, w}$ is almost irreducible with a top space possessing 1-dimensional weight spaces (Proposition 2.9(2)) and a bijective action of G_0^+ (Theorem 2.13(2)), it is reducible if and only if it has a conjugate highest-weight vector in its top space. We test for such vectors by letting G_0^- act, as per (2.25), on the top space weight vector $\mathrm{e}^{-b+(j+\kappa)c} \otimes v_{h,w}$. The result is

$$(4.2) \qquad G_0^-\big(\mathrm{e}^{-b+(j+\kappa)c} \otimes v_{h,w}\big) = \big(\alpha_{\mathsf{k}} w + P_{\mathsf{k}}(j, h)\big)\mathrm{e}^{-b+(j-1+\kappa)c} \otimes v_{h,w},$$

where α_{k} and P_{k} were defined in (3.7). Substituting the parametrisations (2.15) and simplifying, we obtain

$$(4.3) \qquad \alpha_{\mathsf{k}} w_{[r,s]} + P_{\mathsf{k}}(j, h_{[r,s]}) = -\prod_{(r',s') \in [r,s]} (j - j_{(r',s')}),$$

whence $\mathcal{R}_{[j], h_{[r,s]}, w_{[r,s]}}$ is reducible if and only if $[j] = [j_{(r',s')}]$ for some $(r', s') \in [r, s]$.

Fixing $[r, s] \in (\mathsf{P}^{\mathsf{u}-3}_{\geqslant} \times \mathsf{P}^{\mathsf{v}-3}_{\geqslant})/\mathbb{Z}_3$, hence $(h_{[r,s]}, w_{[r,s]}) \in I_{\mathsf{u}, \mathsf{v}}$, it is easy to check that the three zeroes $j_{(r',s')}$, $(r', s') \in [r, s]$ of (4.3) belong to distinct cosets in $\mathbb{C}/\mathbb{Z}$. For example,

$$(4.4) \qquad j_{\nabla(r,s)} - j_{(r,s)} = r_1 + 1 - \tfrac{\mathsf{u}}{\mathsf{v}}(s_1 + 1)$$

is not an integer because u and v are coprime and $0 \leqslant s_1 \leqslant \mathsf{v} - 3$. We therefore have three distinct reducible fully relaxed modules $\mathcal{R}_{[j_{(r',s')}], h_{[r,s]}, w_{[r,s]}}$, $(r', s') \in [r, s]$, for each choice of $[r, s]$. Since $j_{(r',s')}$ is the weight of the conjugate highest-weight vector in the top space, the irreducible quotient of $\mathcal{R}_{[j_{(r',s')}], h_{[r,s]}, w_{[r,s]}}$ is isomorphic to the highest-weight $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-module $\mathcal{H}_{j_{(r',s')}-1, h_{[r,s]}+\kappa}$, by Proposition 3.16(2). Moreover, the top space of the latter is clearly infinite-dimensional. The result now follows from Theorem 3.19. $\blacksquare$

Remark 4.2. *For $\mathsf{u}, \mathsf{v} \geqslant 3$ and $[r, s] \in (\mathsf{P}^{\mathsf{u}-3}_{\geqslant} \times \mathsf{P}^{\mathsf{v}-3}_{\geqslant})/\mathbb{Z}_3$, it is easy to see that the conjugate highest-weight submodule of $\mathcal{R}_{[j_{(r',s')}], h_{[r,s]}, w_{[r,s]}}$ constructed in the proof of Theorem 4.1 is irreducible, hence isomorphic to $\mathcal{C}_{j_{(r',s')}, h_{[r,s]}+\kappa}$. It is also true, but less easy to see, that*

$$(4.5) \qquad 0 \longrightarrow \mathcal{C}_{j_{(r',s')}, h_{[r,s]}+\kappa} \longrightarrow \mathcal{R}_{[j_{(r',s')}], h_{[r,s]}, w_{[r,s]}} \longrightarrow \mathcal{H}_{j_{(r',s')}-1, h_{[r,s]}+\kappa} \longrightarrow 0$$

is exact. This can be shown using an analogous argument to that of [46, Sec. 4], see [30, Thm. 4.24].

This theorem then classifies the irreducible $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-modules in $\mathscr{W}_{\mathsf{u}, \mathsf{v}}$ when k is nondegenerate. One may of course continue the analysis, calculating how many highest-weight modules with finite-dimensional top spaces are in each spectral flow orbit and identifying their highest weights explicitly. This is straightforward and we refer the interested reader to [31, Sec. 2.3].

4.2. **Irreducible** $\mathsf{BP}(2,3)$**-modules.** We turn to the classification of irreducible modules in $\mathscr{W}_{2,3}$. The level $\mathsf{k} = -\frac{7}{3}$, corresponding to $\mathsf{u} = 2$, $\mathsf{v} = 3$, $\kappa = -\frac{5}{9}$ and $\mathsf{c}_\mathsf{k}^{\mathsf{BP}} = -\frac{40}{3}$, is nonadmissible but may still be tackled using inverse quantum hamiltonian reduction, see Theorem 2.10(2). What makes this an ideal case to study is that $\mathsf{c}_\mathsf{k}^{\mathsf{W}_3} = -2$ for this level and so the W_3 minimal model $\mathsf{W}_3(2,3)$ coincides with Kausch's singlet algebra [44].

The irreducible highest-weight $\mathsf{W}_3(2,3)$-modules were classified by Wang in [61], see also [1, 27, 39]. Here, we review this classification following [25, Sec. 3.3]. First, recall that $\mathsf{W}_3(2,3)$ is a vertex subalgebra of a rank-1 Heisenberg vertex algebra. The latter's Fock spaces $\mathcal{F}_\lambda$, $\lambda \in \mathbb{C}$, are thus $\mathsf{W}_3(2,3)$-modules by restriction. A little calculation shows that the highest-weight vector $v_\lambda \in \mathcal{F}_\lambda$ satisfies

(4.6) $\qquad T_0 v_\lambda = h_\lambda v_\lambda, \quad h_\lambda = \tfrac{1}{2}\lambda(\lambda+1), \qquad \text{and} \qquad W_0 v_\lambda = w_\lambda v_\lambda, \quad w_\lambda = -\tfrac{1}{6\sqrt{2}}\lambda(\lambda+1)(2\lambda+1).$

The $\mathcal{F}_\lambda$ turn out to be irreducible, as $\mathsf{W}_3(2,3)$-modules, if and only if $\lambda \notin \mathbb{Z}$. We therefore have the identification

(4.7) $$\mathcal{F}_\lambda \cong \mathcal{W}_{h_\lambda, w_\lambda}, \quad \lambda \notin \mathbb{Z}.$$

These Fock spaces are sometimes referred to as the *typical* irreducible $\mathsf{W}_3(2,3)$-modules. The $\mathcal{F}_\lambda$ with $\lambda \in \mathbb{R}$ are then the *standard* $\mathsf{W}_3(2,3)$-modules, according to the standard module formalism of [25, 57].

For $\lambda \in \mathbb{Z}$, $\mathcal{F}_\lambda$ has a unique irreducible submodule that we shall denote by $\mathcal{S}_\lambda$. Moreover, the following short sequence is nonsplit and exact:

(4.8) $$0 \longrightarrow \mathcal{S}_\lambda \longrightarrow \mathcal{F}_\lambda \longrightarrow \mathcal{S}_{\lambda+1} \longrightarrow 0.$$

The $\mathcal{S}_\lambda$ are also highest-weight and we have

(4.9) $$\mathcal{S}_\lambda \cong \begin{cases} \mathcal{W}_{h_\lambda, w_\lambda}, & \lambda \in \mathbb{Z}_{\geqslant 0}, \\ \mathcal{W}_{h_{\lambda-1}, w_{\lambda-1}} & \lambda \in \mathbb{Z}_{<0}. \end{cases}$$

These are then the *atypical* irreducible $\mathsf{W}_3(2,3)$-modules.

It is easy to check from (4.6) that the only nontrivial coincidence $(h_\lambda, w_\lambda) = (h_\mu, w_\mu)$, $\lambda \neq \mu$, of highest weights occurs with $(h_0, w_0) = (0,0) = (h_{-1}, w_{-1})$. A complete set of mutually nonisomorphic irreducible highest-weight $\mathsf{W}_3(2,3)$-modules is thus given by the $\mathcal{W}_{h_\lambda, w_\lambda}$ with $\lambda \in \mathbb{C} \setminus \{-1\}$.

Theorem 4.3.

(1) *Every irreducible fully relaxed* $\mathsf{BP}(2,3)$*-module is isomorphic to one, and only one, of the* $\mathcal{R}_{[j], h_\lambda, w_\lambda}$ *with* $\lambda \in \mathbb{C} \setminus \{-1\}$ *and* $[j] \notin \left\{ \left[\frac{3\lambda+5}{9}\right], \left[\frac{3\lambda+2}{9}\right], \left[-\frac{6\lambda+1}{9}\right] \right\}$.

(2) *Every irreducible highest-weight* $\mathsf{BP}(2,3)$*-module with an infinite-dimensional top space is isomorphic to one, and only one, of the following modules*:
   (i) *The* $\mathcal{H}_{(3\lambda-4)/9, h_\lambda-5/9}$ *with* $\lambda \in \mathbb{C} \setminus \left(\{-1\} \cup (\mathbb{Z}_{\geqslant 0} + \frac{1}{3})\right)$.
   (ii) *The* $\mathcal{H}_{(3\lambda-7)/9, h_\lambda-5/9}$ *with* $\lambda \in \mathbb{C} \setminus \left(\{-1\} \cup (\mathbb{Z}_{\geqslant 0} + \frac{2}{3})\right)$.
   (iii) *The* $\mathcal{H}_{-(6\lambda+10)/9, h_\lambda-5/9}$ *with* $\lambda \in \mathbb{C} \setminus \left(\{-1\} \cup (-\mathbb{Z}_{\geqslant 0} - \frac{1}{3}) \cup (-\mathbb{Z}_{\geqslant 0} - \frac{2}{3})\right)$.

(3) *Every irreducible* $\mathsf{BP}(2,3)$*-module in* $\mathscr{W}_{2,3}$ *is isomorphic to a spectral flow of one, and only one, of these modules.*

Proof. We again look for conjugate highest-weight vectors in the top space of $\mathcal{R}_{[j], h_\lambda, w_\lambda}$, as in the proof of Theorem 4.1. This time, the existence of such a vector is equivalent to the vanishing of

(4.10) $\qquad \alpha_{-7/3} w_\lambda + P_{-7/3}(j, h_\lambda) = -\left(j - \tfrac{3\lambda+5}{9}\right)\left(j - \tfrac{3\lambda+2}{9}\right)\left(j + \tfrac{6\lambda+1}{9}\right).$

This determines when the fully relaxed $\mathsf{BP}(2,3)$-module $\mathcal{R}_{[j], h_\lambda, w_\lambda}$ is irreducible, proving (1). Note that the roots of (4.10) are the same for $\lambda = 0$ and -1.

Unlike the nondegenerate case studied in Theorem 4.1, the three zeroes of (4.10) need not belong to different cosets in $\mathbb{C}/\mathbb{Z}$. Indeed, we have $\left[\frac{3\lambda+5}{9}\right] = \left[-\frac{6\lambda+1}{9}\right]$ for $\lambda \in \mathbb{Z} + \frac{1}{3}$ and $\left[\frac{3\lambda+2}{9}\right] = \left[-\frac{6\lambda+1}{9}\right]$ for $\lambda \in \mathbb{Z} - \frac{1}{3}$. For

$\lambda \notin \mathbb{Z} \pm \frac{1}{3}$, it therefore follows that there are three irreducible highest-weight quotients, namely $\mathcal{H}_{(3\lambda-4)/9, h_\lambda - 5/9}$, $\mathcal{H}_{(3\lambda-7)/9, h_\lambda - 5/9}$ and $\mathcal{H}_{-(6\lambda+10)/9, h_\lambda - 5/9}$, and that each has an infinite-dimensional top space.

Suppose now that $\lambda \in \mathbb{Z} + \frac{1}{3}$. Then, there is a single zero of (4.10) in $[\frac{3\lambda+2}{9}]$ and so $\mathcal{H}_{(3\lambda-7)/9, h_\lambda - 5/9}$ is the irreducible highest-weight quotient (with infinite-dimensional top space). However, there are two zeroes in $[\frac{3\lambda+5}{9}] = [-\frac{6\lambda+1}{9}]$, hence two conjugate highest-weight vectors in the top space of $\mathcal{R}_{[j], h_\lambda, w_\lambda}$. In other words, $\mathcal{R}_{[j], h_\lambda, w_\lambda}$ has two conjugate highest-weight submodules, one of which contains the other. We want the quotient by the larger of the two, which is the one whose conjugate highest-weight vector has the smallest J_0-eigenvalue. If $\lambda < 0$, then this eigenvalue is $\frac{3\lambda+5}{9}$, hence $\mathcal{H}_{(3\lambda-4)/9, h_\lambda - 5/9}$ is the irreducible highest-weight quotient (with infinite-dimensional top space). Otherwise, it is $-\frac{6\lambda+1}{9}$ and the desired quotient is $\mathcal{H}_{-(6\lambda+10)/9, h_\lambda - 5/9}$.

The analysis for $\lambda \in \mathbb{Z} - \frac{1}{3}$ is very similar. To complete the proof of (2), we only have to check that the members of the three highest-weight families are all distinct. This is easily verified. For example, $\left(\frac{3\lambda-4}{9}, h_\lambda - \frac{5}{9}\right) = \left(-\frac{6\mu+10}{9}, h_\mu - \frac{5}{9}\right)$ gives two solutions: $\lambda = 0$, $\mu = -1$; and $\lambda = \mu = -\frac{2}{3}$. In both cases, λ corresponds to a family member but μ does not.

Finally, (3) now follows from (1), (2) and Theorem 3.19.   ∎

Remark 4.4. *The exclusions for the parameter λ in the families of Theorem 4.3(2) avoid the following coincidences*:

- $\mathcal{H}_{-7/9, -5/9}$ *belongs to family* (i) *with* $\lambda = -1$ *and family* (ii) *with* $\lambda = 0$.
- $\mathcal{H}_{-10/9, -5/9}$ *belongs to family* (ii) *with* $\lambda = -1$ *and family* (iii) *with* $\lambda = 0$.
- $\mathcal{H}_{-4/9, -5/9}$ *belongs to family* (iii) *with* $\lambda = -1$ *and family* (i) *with* $\lambda = 0$.
- $\mathcal{H}_{-8/9, -2/3}$ *belongs to family* (iii) *with* $\lambda = -\frac{1}{3}$ *and family* (ii) *with* $\lambda = -\frac{1}{3}$.
- $\mathcal{H}_{-2/3, -2/3}$ *belongs to family* (iii) *with* $\lambda = -\frac{2}{3}$ *and family* (i) *with* $\lambda = -\frac{2}{3}$.

Remark 4.5. *The proof of Theorem 4.3 shows that there exist reducible conjugate highest-weight $\mathsf{BP}(2, 3)$-modules. Conjugating therefore gives the same conclusion in the highest-weight case. The analogue of the BGG category $\mathcal{O}_\mathsf{k}$ for $\mathsf{BP}(2, 3)$ is consequently nonsemisimple.*

Conjecture. *The analogue of the BGG category $\mathcal{O}_\mathsf{k}$ for $\mathsf{BP}(\mathsf{u}, \mathsf{v})$ is semisimple if and only if $\mathsf{u} = 2$ and $\mathsf{v} = 1$, $\mathsf{u} \geqslant 3$ and $\mathsf{v} = 2$, or $\mathsf{u}, \mathsf{v} \geqslant 3$.*

While Theorem 4.3 classifies the irreducibles in $\mathscr{W}_{2,3}$, it may be made more explicit by determining those (j, Δ) for which $\mathcal{H}_{j, \Delta}$ is an irreducible highest-weight $\mathsf{BP}(2, 3)$-module with a finite-dimensional top space. These are precisely the weight modules whose L_0-eigenvalues are bounded below and whose L_0-eigenspaces are finite-dimensional, that is they are ordinary modules.

Theorem 4.6. *Every irreducible ordinary $\mathsf{BP}(2, 3)$-module is isomorphic to one, and only one, of the following*:

(1) *The $\mathcal{H}_{\lambda/3, h_\lambda + \lambda/3}$ with $\lambda \in \mathbb{C} \setminus \{-\frac{5}{3}\}$ and top space dimension 1.*
(2) *The $\mathcal{H}_{(3\lambda-4)/9, h_\lambda - 5/9}$ with $\lambda \in \mathbb{Z}_{\geqslant 0} + \frac{4}{3}$ and top space dimension $\lambda + \frac{2}{3}$.*
(3) *The $\mathcal{H}_{(3\lambda-7)/9, h_\lambda - 5/9}$ with $\lambda \in \mathbb{Z}_{\geqslant 0} + \frac{2}{3}$ and top space dimension $\lambda + \frac{1}{3}$.*
(4) *The $\mathcal{H}_{-(6\lambda+10)/9, h_\lambda - 5/9}$ with $\lambda \in -\mathbb{Z}_{\geqslant 0} - \frac{8}{3}$ and top space dimension $-\lambda - \frac{2}{3}$.*
(5) *The $\mathcal{H}_{-(6\lambda+10)/9, h_\lambda - 5/9}$ with $\lambda \in -\mathbb{Z}_{\geqslant 0} - \frac{7}{3}$ and top space dimension $-\lambda - \frac{1}{3}$.*

Proof. Suppose that the irreducible highest-weight $\mathsf{BP}(2, 3)$-module $\mathcal{H}_{j, \Delta}$ has a top space of dimension n. By Equation (3.15), this is equivalent to n being the smallest positive integer satisfying $g_n(j, \Delta) = 0$. Moreover, either $\sigma(\mathcal{H}_{j, \Delta})$ or $\sigma^2(\mathcal{H}_{j, \Delta})$ is highest-weight with an infinite-dimensional top space, by Proposition 3.17.

Suppose that it is $\sigma(\mathcal{H}_{j, \Delta})$. Then, we recall that $\sigma(\mathcal{H}_{j, \Delta}) \cong \mathcal{H}_{j-n+4/9, \Delta+j-n+1}$, by (3.17), and compare with the classification in Theorem 4.3(2). There are thus three possibilities:

- $j - n + \frac{4}{9} = \frac{3\lambda-4}{9}$ and $\Delta + j - n + 1 = h_\lambda - \frac{5}{9}$ for some $\lambda \notin \{-1\} \cup (\mathbb{Z}_{\geqslant 0} + \frac{1}{3})$. In this case, solving for j and Δ results in $g_n(j, \Delta) = (n - \frac{1}{3})(n + \lambda)$. As n must be a positive integer, this only vanishes when

$\lambda = -n \in \mathbb{Z}_{\leqslant -1}$. However, $\lambda = -1$ is explicitly excluded, so we only take $\lambda \in \mathbb{Z}_{\leqslant -2}$. Substituting back, our solution becomes $\mathcal{H}_{j,\Delta} = \mathcal{H}_{-2(3\lambda+4)/9,(\lambda-1)(3\lambda+4)/6}$. If we set $\lambda = \mu + \frac{1}{3}$, we recognise the family (5) module $\mathcal{H}_{j,\Delta} = \mathcal{H}_{-(6\mu+10)/9,h_\mu-5/9}$. Its top space has dimension $n = -\lambda = -\mu - \frac{1}{3}$, where $\mu \in -\mathbb{Z}_{\geqslant 0} - \frac{7}{3}$.

- $j - n + \frac{4}{9} = \frac{3\lambda-7}{9}$ and $\Delta + j - n + 1 = h_\lambda - \frac{5}{9}$ for some $\lambda \notin \{-1\} \cup (\mathbb{Z}_{\geqslant 0} + \frac{2}{3})$. Following the same steps as in the previous case now gives $g_n(j,\Delta) = (n-1)(n+\lambda-\frac{1}{3})$. The smallest positive-integer solution for n is therefore always 1, so $\mathcal{H}_{j,\Delta} = \mathcal{H}_{(3\lambda-2)/9,(\lambda+1)(3\lambda-2)/6}$ has a 1-dimensional top space. Setting $\mu = \lambda - \frac{2}{3}$, we recognise these modules as belonging to family (1) with $\mu \notin \{-\frac{5}{3}\} \cup \mathbb{Z}_{\geqslant 0}$.

- $j - n + \frac{4}{9} = -\frac{6\lambda+10}{9}$ and $\Delta + j - n + 1 = h_\lambda - \frac{5}{9}$ for some $\lambda \notin \{-1\} \cup (-\mathbb{Z}_{\geqslant 0} - \frac{1}{3}) \cup (-\mathbb{Z}_{\geqslant 0} - \frac{2}{3})$. This time, we get $g_n(j,\Delta) = (n-\lambda-1)(n-\lambda-\frac{4}{3})$, hence two distinct families of solutions: $\lambda = n-1 \in \mathbb{Z}_{\geqslant 0}$ and $\lambda = n-\frac{4}{3} \in \mathbb{Z}_{\geqslant 0} + \frac{2}{3}$ (we have to exclude $\lambda = -\frac{1}{3}$). For $\lambda \in \mathbb{Z}_{\geqslant 0}$, we set $\mu = \lambda + \frac{2}{3}$ to recognise the $\mathcal{H}_{j,\Delta}$ as belonging to family (3) with $\mu \in \mathbb{Z}_{\geqslant 0} + \frac{2}{3}$. For $\lambda \in \mathbb{Z}_{\geqslant 0} + \frac{2}{3}$, $\mu = \lambda + \frac{2}{3}$ instead results in the $\mathcal{H}_{j,\Delta}$ belonging to family (2) with $\mu \in \mathbb{Z}_{\geqslant 0} + \frac{4}{3}$.

The only alternative is that $\sigma^2(\mathcal{H}_{j,\Delta})$ is highest-weight with an infinite-dimensional top space. Then, $\sigma(\mathcal{H}_{j,\Delta})$ must belong to one of the four families of highest-weight $\mathsf{BP}(2,3)$-modules with finite-dimensional top spaces that we have already discovered. The analysis for families (2), (3) and (5) is then the same as above, except that λ is now required to lie in $\mathbb{Z}_{\geqslant 0} + \frac{4}{3}$, $\mathbb{Z}_{\geqslant 0} + \frac{2}{3}$ and $-\mathbb{Z}_{\geqslant 0} - \frac{7}{3}$, respectively. The results are that this is impossible when $\sigma(\mathcal{H}_{j,\Delta})$ belongs to families (2) and (5), but for family (3) the $\mathcal{H}_{j,\Delta}$ are found to belong to family (1) with $\lambda \in \mathbb{Z}_{\geqslant 0}$.

It only remains to consider if $\sigma(\mathcal{H}_{j,\Delta})$ can belong to family (1) (with $\lambda \notin \{-\frac{5}{3}\} \cup \mathbb{Z}_{\geqslant 0}$). Setting $j - n + \frac{4}{9} = \frac{\lambda}{3}$ and $\Delta + j - n + 1 = h_\lambda + \frac{\lambda}{3}$, we deduce that $g_n(j,\Delta) = (n+\frac{1}{3})(n+\lambda+\frac{2}{3})$. Noting that $\lambda = -n - \frac{2}{3} \in -\mathbb{Z}_{\geqslant 0} - \frac{8}{3}$, because $-\frac{5}{3}$ must be excluded, we conclude that $\mathcal{H}_{j,\Delta}$ belongs to family (4). ∎

Corollary 4.7. *Every irreducible highest-weight* $\mathsf{BP}(2,3)$*-module is isomorphic to one in the set*

$$(4.11) \qquad \left\{ \mathcal{H}_{(3\lambda-4)/9,h_\lambda-5/9}, \mathcal{H}_{(3\lambda-7)/9,h_\lambda-5/9}, \mathcal{H}_{-(6\lambda+10)/9,h_\lambda-5/9}, \mathcal{H}_{\lambda/3,h_\lambda+\lambda/3} : \lambda \in \mathbb{C} \right\}.$$

We may equivalently reparametrise the four families in (4.11) using the J_0-eigenvalue of the highest-weight vector:

$$(4.12) \qquad \left\{ \mathcal{H}_{j,(j+1)(9j+2)/2}, \mathcal{H}_{j,(3j+4)(9j+5)/6}, \mathcal{H}_{j,j(9j+14)/8}, \mathcal{H}_{j,j(9j+5)/2} : j \in \mathbb{C} \right\}.$$

For convenience, (a part of) this set is plotted (for real j) in Figure 1.

Remark 4.8. *The exclusions for the parameter λ in the families of Theorem 4.6 avoid the following coincidences*:

- $\mathcal{H}_{-1/3,-1/3}$ *belongs to family (1) with $\lambda = -1$ and family (2) with $\lambda = \frac{1}{3}$.*
- $\mathcal{H}_{-5/9,0}$ *belongs to family (1) with $\lambda = -\frac{5}{3}$ and family (3) with $\lambda = \frac{2}{3}$.*
- $\mathcal{H}_{0,0}$ *belongs to family (1) with $\lambda = 0$ and family (4) with $\lambda = -\frac{5}{3}$.*
- $\mathcal{H}_{-2/9,-1/3}$ *belongs to family (1) with $\lambda = -\frac{2}{3}$ and family (5) with $\lambda = -\frac{4}{3}$.*

Remark 4.9. *We record for completeness the result of applying spectral flow to a highest-weight* $\mathsf{BP}(2,3)$*-module with finite-dimensional top space*:

- *For $\lambda \in \mathbb{C}$, $\sigma(\mathcal{H}_{\lambda/3,h_\lambda+\lambda/3}) \cong \mathcal{H}_{(3\mu-7)/9,h_\mu-5/9}$, where $\mu = \lambda + \frac{2}{3}$.*
- *For $\lambda \in \mathbb{Z}_{\geqslant 0} + \frac{1}{3}$, $\sigma(\mathcal{H}_{(3\lambda-4)/9,h_\lambda-5/9}) \cong \mathcal{H}_{-(6\mu+10)/9,h_\mu-5/9}$, where $\mu = \lambda - \frac{2}{3}$.*
- *For $\lambda \in \mathbb{Z}_{\geqslant 0} + \frac{2}{3}$, $\sigma(\mathcal{H}_{(3\lambda-7)/9,h_\lambda-5/9}) \cong \mathcal{H}_{-(6\mu+10)/9,h_\mu-5/9}$, where $\mu = \lambda - \frac{2}{3}$.*
- *For $\lambda \in -\mathbb{Z}_{\geqslant 0} - \frac{5}{3}$, $\sigma(\mathcal{H}_{-(6\lambda+10)/9,h_\lambda-5/9}) \cong \mathcal{H}_{\mu/3,h_\mu+\mu/3}$, where $\mu = \lambda$.*
- *For $\lambda \in -\mathbb{Z}_{\geqslant 0} - \frac{4}{3}$, $\sigma(\mathcal{H}_{-(6\lambda+10)/9,h_\lambda-5/9}) \cong \mathcal{H}_{(3\mu-4)/9,h_\mu-5/9}$, where $\mu = \lambda + \frac{1}{3}$.*

We can thus roughly summarise the corresponding spectral flow orbits in terms of the families of Theorems 4.3 and 4.6 as follows (*ignoring modules that are not highest-weight*):

$$(4.13) \qquad (1) \xrightarrow{\sigma} (3) \xrightarrow{\sigma} (\text{iii}), \qquad (2) \xrightarrow{\sigma} (\text{iii}), \qquad (4) \xrightarrow{\sigma} (1) \xrightarrow{\sigma} (\text{ii}), \qquad (5) \xrightarrow{\sigma} (\text{i}).$$

Of course, there are also an uncountably infinite number of orbits with a single highest-weight module.
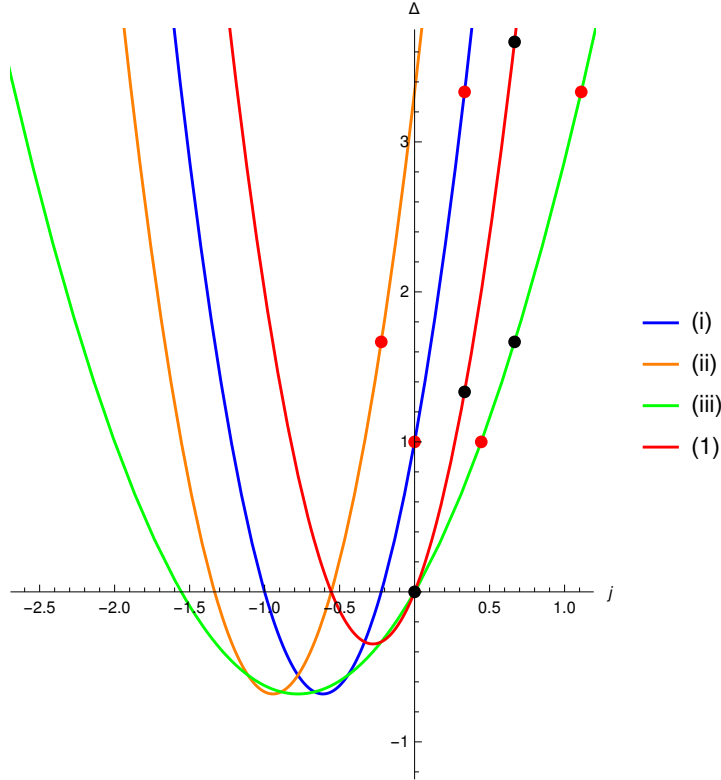
FIGURE 1. A picture of the highest weights (j, Δ) appearing in the set (4.12), with j real. The blue, orange and green curves indicate the families (i), (ii) and (iii) of highest-weight $\mathsf{BP}(2,3)$-modules described in Theorem 4.3. These modules generically have infinite-dimensional top spaces. The red curve indicates family (1) in Theorem 4.6. Its modules have finite-dimensional top spaces and their images under σ have infinite-dimensional top spaces. A red dot indicates another module with these properties (families (2), (3) and (5) in Theorem 4.6). On the other hand, a black dot indicates a module with a finite-dimensional top space whose image under σ also has a finite-dimensional top space (family (1) with $j \in \frac{1}{3}\mathbb{Z}_{\geqslant 0}$ and family (4)).

5. AN APPLICATION TO $\mathfrak{sl}_3$ MINIMAL MODELS

We finish by studying some of the implications of our results, when combined with other known relationships, to $\mathfrak{sl}_3$ minimal models. We denote the universal level-k affine vertex operator algebra associated with $\mathfrak{sl}_3$ by $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$ and its simple quotient by $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_3)$. When k is expressed in terms of u and v, as in (2.1), we shall also write $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_3) = \mathsf{A}_2(\mathsf{u}, \mathsf{v})$ and refer to the latter as an $\mathfrak{sl}_3$ minimal model vertex operator algebra.

Recall that BP^{k} is the quantum hamiltonian reduction of $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$ corresponding to the minimal (and subregular) nilpotent orbit [17, 41, 54]. We restrict the corresponding reduction functor $\Phi^{\mathrm{min.}}$ to the Kazhdan–Lusztig category $\mathscr{KL}^{\mathsf{k}}$ of ordinary $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$-modules, these being the weight modules with bounded-below L_0-eigenvalues and finite-dimensional L_0-eigenspaces. The simple objects of $\mathscr{KL}^{\mathsf{k}}$ are thus the irreducible highest-weight modules whose highest weights have the form $(\mathsf{k} - r - s + 2)\omega_0 + (r - 1)\omega_1 + (s - 1)\omega_2$, for some $r, s \in \mathbb{Z}_{\geqslant 1}$. Here, ω_i, $i = 0, 1, 2$, denotes the i-th fundamental weight of $\widehat{\mathfrak{sl}}_3$. We denote the irreducible highest-weight $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$-module of this highest weight by $\mathcal{L}_{r,s}$.

Proposition 5.1. *For $\mathsf{k} \notin \{-3\} \cup \mathbb{Z}_{\geqslant -1}$ and $r, s \in \mathbb{Z}_{\geqslant 1}$, the minimal quantum hamiltonian reduction of $\mathcal{L}_{r,s}$ is the irreducible highest-weight BP^{k}-module $\Phi^{\mathrm{min.}}(\mathcal{L}_{r,s}) = \mathcal{H}_{j_{r,s}, \Delta_{r,s}}$, where*

$$(5.1) \qquad j_{r,s} = \frac{r + 2s - 3}{3} \quad and \quad \Delta_{r,s} = \frac{r^2 + rs + s^2 - 3}{3(\mathsf{k} + 3)} - \frac{2r + s - 3}{3}.$$

Moreover, the top space of $\mathcal{H}_{j_{r,s}, \Delta_{r,s}}$ has dimension s.

Proof. Since $\mathsf{k} \notin \mathbb{Z}_{\geqslant 0}$, the zeroth Dynkin label of the highest weight of $\mathcal{L}_{r,s}$ is not in $\mathbb{Z}_{\geqslant 0}$. The minimal reduction of $\mathcal{L}_{r,s}$ is thus an irreducible highest-weight module, by [9, Thm. 6.7.4]. Moreover, its highest weight corresponds to the quoted formulae for $j_{r,s}$ and $\Delta_{r,s}$, by [43, Thm. 6.3]. It remains to check that its top space has dimension s. This follows from (3.15) and $\mathsf{k} \notin \mathbb{Z}_{\geqslant -1}$, because

$$(5.2) \qquad g_n(j_{r,s}, \Delta_{r,s}) = (n - r - s + \mathsf{k} + 3)(n - s). \qquad \blacksquare$$

Proposition 5.2. *For $\mathsf{k} \notin \{-3\} \cup \mathbb{Z}_{\geqslant -1}$ and $r, s \in \mathbb{Z}_{\geqslant 1}$, the (irreducible) BP^{k}-module $\mathcal{H}_{j_{r,s}, \Delta_{r,s}}$ may be realised as a submodule of $\sigma(\mathcal{R}_{[j'_{r,s}], h_{r,s}, w_{r,s}})$, where*

$$(5.3) \qquad \begin{aligned} & j'_{r,s} = \frac{r + 2s - 2(\mathsf{k} + 3)}{3}, \quad h_{r,s} = \frac{r^2 + rs + s^2 - 3}{3(\mathsf{k} + 3)} - r - s + 2 \\ & \text{and} \quad w_{r,s} = -\frac{\sqrt{3}}{(\mathsf{k} + 3)^{3/2}} \frac{r - s}{3} \left(\frac{2r + s}{3} - \mathsf{k} - 3 \right) \left(\frac{r + 2s}{3} - \mathsf{k} - 3 \right). \end{aligned}$$

Proof. As $\mathcal{H}_{j_{r,s}, \Delta_{r,s}}$ is highest-weight, with a finite-dimensional top space, it is isomorphic to either $\sigma(\mathcal{C})$ or $\sigma^2(\mathcal{C})$, where $\mathcal{C}$ is a conjugate highest-weight module with an infinite-dimensional top space (Proposition 3.17). Suppose that it is $\sigma(\mathcal{C})$. Then, $\mathcal{H}_{j_{r,s}, \Delta_{r,s}}$ is isomorphic to a submodule of $\sigma(\mathcal{R}_{[j'_{r,s}], h_{r,s}, w_{r,s}})$, for some $[j'_{r,s}] \in \mathbb{C}/\mathbb{Z}$ and $h_{r,s}, w_{r,s} \in \mathbb{C}$, by Proposition 3.16(1). The highest-weight vector of $\mathcal{H}_{j_{r,s}, \Delta_{r,s}}$ is mapped to the conjugate highest-weight vector of $\mathcal{C}$ by σ^{-1} and the weight of the latter is $(j_{r,s} - \kappa, \Delta_{r,s} - j_{r,s} + \kappa)$, by (2.24). As in the proof of Proposition 3.16, this identifies $[j'_{r,s}] = [j_{r,s} - \kappa]$ and $h_{r,s} = \Delta_{r,s} - j_{r,s}$. To obtain $w_{r,s}$, substitute $j'_{r,s}$ and $h_{r,s}$ into (3.7). As we have found a solution, there is no need to consider the possibility that $\mathcal{H}_{j_{r,s}, \Delta_{r,s}} \cong \sigma^2(\mathcal{C})$. $\qquad \blacksquare$

Remark 5.3. *Proposition 5.2 constructs an embedding $\mathcal{H}_{j_{r,s}, \Delta_{r,s}} \hookrightarrow \sigma(\mathcal{R}_{[j'_{r,s}], h_{r,s}, w_{r,s}}) = \varsigma(\Pi_{[j'_{r,s}]}) \otimes \mathcal{W}_{h_{r,s}, w_{r,s}}$. However, $\Pi_{[j'_{r,s}]} \cong \Pi\, e^{-b + (j'_{r,s} + \kappa)c} = \Pi\, e^{-b + (r + 2s - 3)c/3}$ and thus*

$$(5.4) \qquad \varsigma(\Pi_{[j'_{r,s}]}) = \Pi\, e^{(r + 2s - 3)c/3} \in \Pi^{1/3}, \quad \Pi^{1/3} = \Pi \oplus \Pi\, e^{c/3} \oplus \Pi\, e^{2c/3},$$

by (2.10). It follows that this Proposition 5.2 constructs the ordinary BP^{k}-modules $\mathcal{H}_{j_{r,s}, \Delta_{r,s}}$ as submodules of $\Pi^{1/3} \otimes \mathcal{W}_{h_{r,s}, w_{r,s}}$. This is thus the analogue of the realisation of ordinary $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)$-modules presented in [2, Sec. 6].

We have the following important consequence.

Theorem 5.4. *Assume that $\mathsf{u} \geqslant 2$ and $\mathsf{v} \geqslant 3$ are coprime and that $r, s \in \mathbb{Z}_{\geqslant 1}$. Let $j_{r,s}, \Delta_{r,s}, h_{r,s}$ and $w_{r,s}$ be defined by (5.1) and (5.3). Then, the following conditions are equivalent:*

(1) *$\mathcal{L}_{r,s}$ is an $\mathsf{A}_2(\mathsf{u}, \mathsf{v})$-module.*
(2) *$\mathcal{H}_{j_{r,s}, \Delta_{r,s}}$ is a $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-module.*
(3) *$\mathcal{W}_{h_{r,s}, w_{r,s}}$ is a $\mathsf{W}_3(\mathsf{u}, \mathsf{v})$-module.*

Proof. (1) $\Rightarrow$ (2) is a standard result about quantum hamiltonian reduction, see for example [30, Prop. 4.7].

For (2) $\Rightarrow$ (1), there also exists an inverse reduction embedding [3, Thm. 5.2]

$$(5.5) \qquad \mathsf{A}_2(\mathsf{u}, \mathsf{v}) \hookrightarrow \mathsf{BP}(\mathsf{u}, \mathsf{v}) \otimes \mathsf{SB} \otimes \Pi,$$

where SB denotes the symplectic bosons vertex operator algebra (also known as bosonic ghosts). Moreover, calculation shows that $\mathcal{L}_{r,s}$ may be explicitly realised [3, Thm. 6.3(2)] a submodule of the tensor product of $\mathcal{H}_{j_{r,s}, \Delta_{r,s}}$, SB and a direct summand of $\Pi^{1/3}$.

So far, the proven implications hold for $\mathsf{u}, \mathsf{v} \geqslant 2$. For (2) $\Leftrightarrow$ (3), note that Proposition 5.2 shows that $\sigma^{-1}(\mathcal{H}_{j_{r,s}, \Delta_{r,s}})$ is an irreducible conjugate highest-weight submodule of a fully relaxed module. By Theorem 3.19 and Remark 3.20, which require $\mathsf{v} \geqslant 3$, $\sigma^{-1}(\mathcal{H}_{j_{r,s}, \Delta_{r,s}})$ is a $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-module if and only if this fully relaxed module is. But, the latter condition is equivalent to $(h_{r,s}, w_{r,s}) \in I_{\mathsf{u}, \mathsf{v}}$. $\qquad \blacksquare$

When k is nondegenerate ($u, v \geqslant 3$), (1) $\Leftrightarrow$ (2) is exactly [30, Thm. 4.8]. For $u = 2$, we believe that this equivalence is new. Here is an interesting corollary for the Kazhdan–Lusztig category $\mathscr{KL}_{2,3}$ of ordinary $A_2(2,3)$-modules.

Corollary 5.5. *Every simple object in $\mathscr{KL}_{2,3}$ is isomorphic to a module from the set*

$$(5.6) \qquad \left\{ \mathcal{L}_{n,1}, \mathcal{L}_{1,n} : n \in \mathbb{Z}_{\geqslant 1} \right\}.$$

Proof. This follows from Theorem 5.4 by comparing the formulae in (5.1) with the classification of irreducible ordinary $BP(2,3)$-modules in Theorem 4.6. The result is that the only solutions with $r, s \in \mathbb{Z}_{\geqslant 1}$ correspond to families (1) and (4) of the latter theorem, the former with $\lambda \in \mathbb{Z}_{\geqslant 0}$, $r = \lambda + 1$, $s = 1$ and the latter with $\lambda \in -\mathbb{Z}_{\geqslant 0} - \frac{5}{3}$, $r = 1$, $s = -\lambda - \frac{2}{3}$. ∎

Remark 5.6. *Note that the two families of irreducible ordinary $BP(2,3)$-modules that arise as minimal quantum hamiltonian reductions of the irreducible ordinary $A_2(2,3)$-modules are precisely those whose images under σ are again ordinary. Indeed, for $n \in \mathbb{Z}_{\geqslant 1}$, Remark 4.9 and Proposition 5.1 give*

$$(5.7) \quad \begin{aligned} \mathcal{L}_{n,1} &\xrightarrow{\Phi^{min.}} \mathcal{H}_{(n-1)/3, h_{n-1} + (n-1)/3} \xrightarrow{\sigma} \mathcal{H}_{(3\lambda-7)/9, h_\lambda - 5/9}, \quad \text{where } \lambda = n - \tfrac{1}{3}, \\ \mathcal{L}_{1,n} &\xrightarrow{\Phi^{min.}} \mathcal{H}_{-(6\lambda+10)/9, h_\lambda - 5/9} \xrightarrow{\sigma} \mathcal{H}_{\lambda/3, h_\lambda + \lambda/3}, \quad \text{where } \lambda = -n - \tfrac{2}{3}. \end{aligned}$$

Remark 5.7. *We believe that $\mathscr{KL}_{2,3}$ is semisimple. We will study this category in forthcoming publications.*

REFERENCES

[1] D Adamović. Classification of irreducible modules of certain subalgebras of free boson vertex algebra. *J. Algebra*, 270:115–132, 2003. arXiv:math.QA/0207155.

[2] D Adamović. Realizations of simple affine vertex algebras and their modules: the cases $\widehat{sl(2)}$ and $\widehat{osp(1,2)}$. *Comm. Math. Phys.*, 366:1025–1067, 2019. arXiv:1711.11342 [math.QA].

[3] D Adamović, T Creutzig and N Genra. Relaxed and logarithmic modules of $\widehat{sl}_3$. *Math. Ann.* (to appear). https://doi.org/10.1007/s00208-023-02634-6, arXiv:2110.15203 [math.RT].

[4] D Adamović, V Kac, P Möseneder Frajria, P Papi and O Perše. Conformal embeddings of affine vertex algebras in minimal W-algebras I: structural results. *J. Algebra*, 500:117–152, 2018. arXiv:1602.04687 [math.RT].

[5] D Adamović, K Kawasetsu and D Ridout. A realisation of the Bershadsky–Polyakov algebras and their relaxed modules. *Lett. Math. Phys.*, 111:38, 2021. arXiv:2007.00396 [math.QA].

[6] D Adamović and A Kontrec. Bershadsky–Polyakov vertex algebras at positive integer levels and duality. *Transform. Groups* (to appear). https://doi.org/10.1007/s00031-022-09721-z, arXiv:2011.10021 [math.QA].

[7] D Adamović and A Kontrec. Classification of irreducible modules for Bershadsky–Polyakov algebra at certain levels. *J. Algebra Appl.*, 20:2150102, 2021. arXiv:1910.13781 [math.QA].

[8] D Adamović and A Milas. Vertex operator algebras associated to modular invariant representations of $A_1^{(1)}$. *Math. Res. Lett.*, 2:563–575, 1995. arXiv:q-alg/9509025.

[9] T Arakawa. Representation theory of superconformal algebras and the Kac–Roan–Wakimoto conjecture. *Duke Math. J.*, 130:435–478, 2005. arXiv:math-ph/0405015.

[10] T Arakawa. Rationality of Bershadsky–Polyakov vertex algebras. *Comm. Math. Phys.*, 323:627–633, 2013. arXiv:1005.0185 [math.QA].

[11] T Arakawa. Associated varieties of modules over Kac–Moody algebras and C_2-cofiniteness of W-algebras. *Int. Math. Res. Not.*, 2015:11605–11666, 2015. arXiv:1004.1554 [math.QA].

[12] T Arakawa. Rationality of W-algebras: Principal nilpotent cases. *Ann. Math.*, 182:565–604, 2015. arXiv:1211.7124 [math.QA].

[13] T Arakawa. Rationality of admissible affine vertex algebras in the category $\mathcal{O}$. *Duke Math. J.*, 165:67–93, 2016. arXiv:1207.4857 [math.QA].

[14] T Arakawa, V Futorny and L-E Ramirez. Weight representations of admissible affine vertex algebras. *Comm. Math. Phys.*, 353:1151–1178, 2017. arXiv:1605.07580 [math.RT].

[15] T Arakawa and J van Ekeren. Rationality and fusion rules of exceptional W-algebras. *J. Eur. Math. Soc.*, 25:2763–2813, 2023. arXiv:1905.11473 [math.RT].

[16] S Berman, C Dong and S Tan. Representations of a class of lattice type vertex algebras. *J. Pure Appl. Algebra*, 176:27–47, 2002. arXiv:math.QA/0109215.

[17] M Bershadsky. Conformal field theories via Hamiltonian reduction. *Comm. Math. Phys.*, 139:71–82, 1991.

[18] P Bouwknegt, J McCarthy and K Pilch. *The $\mathcal{W}_3$ Algebra: Modules, Semi-Infinite Cohomology and BV Algebras*, volume m42 of *Lecture Notes in Physics*. Springer, Berlin, 1996. arXiv:hep-th/9509119.

[19] P Bouwknegt and K Schoutens. $\mathcal{W}$ symmetry in conformal field theory. *Phys. Rep.*, 223:183–276, 1993. arXiv:hep-th/9210010.

[20] T Creutzig, S Kanade, T Liu and D Ridout. Cosets, characters and fusion for admissible-level $\mathfrak{osp}(1|2)$ minimal models. *Nucl. Phys.*, B938:22–55, 2018. arXiv:1806.09146 [hep-th].

[21] T Creutzig, T Liu, D Ridout and S Wood. Unitary and non-unitary $N = 2$ minimal models. *J. High Energy Phys.*, 1906:024, 2019. arXiv:1902.08370 [math-ph].

[22] T Creutzig, R McRae and J Yang. On ribbon categories for singlet vertex algebras. *Comm. Math. Phys.*, 387:865–925, 2021. arXiv:2007.12735 [math.QA].

[23] T Creutzig, R McRae and J Yang. Ribbon tensor structure on the full representation categories of the singlet vertex algebras. *Adv. Math.*, 413:108828, 2023. arXiv:2202.05496 [math.QA].

[24] T Creutzig and D Ridout. Modular data and Verlinde formulae for fractional level WZW models I. *Nucl. Phys.*, B865:83–114, 2012. arXiv:1205.6513 [hep-th].

[25] T Creutzig and D Ridout. Logarithmic conformal field theory: beyond an introduction. *J. Phys.*, A46:494006, 2013. arXiv:1303.0847 [hep-th].

[26] T Creutzig and D Ridout. Modular data and Verlinde formulae for fractional level WZW models II. *Nucl. Phys.*, B875:423–458, 2013. arXiv:1306.4388 [hep-th].

[27] W Eholzer, M Flohr, A Honecker, R Hübel, W Nahm and R Varnhagen. Representations of $\mathcal{W}$-algebras with two generators and new rational models. *Nucl. Phys.*, B383:249–288, 1992.

[28] V Fateev and A Zamolodchikov. Conformal quantum field theory models in two dimensions having Z_3 symmetry. *Nucl. Phys.*, B280:644–660, 1987.

[29] Z Fehily. Subregular W-algebras of type A. *Comm. Contemp. Math.*, 25:2250049, 2023. arXiv:2111.05536 [math.QA].

[30] Z Fehily, K Kawasetsu and D Ridout. Classifying relaxed highest-weight modules for admissible-level Bershadsky–Polyakov algebras. *Comm. Math. Phys.*, 385:859–904, 2021. arXiv:2007.03917 [math.RT].

[31] Z Fehily and D Ridout. Modularity of Bershadsky–Polyakov minimal models. *Lett. Math. Phys.*, 122:46, 2022. arXiv:2110.10336 [math.QA].

[32] B Feigin, A Semikhatov and I Yu Tipunin. Equivalence between chain categories of representations of affine $sl(2)$ and $N = 2$ superconformal algebras. *J. Math. Phys.*, 39:3865–3905, 1998. arXiv:hep-th/9701043.

[33] E Frenkel and D Ben-Zvi. *Vertex Algebras and Algebraic Curves*, volume 88 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, 2001.

[34] I Frenkel and Y Zhu. Vertex operator algebras associated to representations of affine and Virasoro algebras. *Duke Math. J.*, 66:123–168, 1992.

[35] D Friedan, E Martinec and S Shenker. Conformal invariance, supersymmetry and string theory. *Nucl. Phys.*, B271:93–165, 1986.

[36] V Futorny. Irreducible non-dense $A_1^{(1)}$-modules. *Pacific J. Math.*, 172:83–99, 1996.

[37] D Gepner and E Witten. String theory on group manifolds. *Nucl. Phys.*, B278:493–549, 1986.

[38] M Gorelik and V Kac. On simplicity of vacuum modules. *Adv. Math.*, 211:621–677, 2007. arXiv:math-ph/0606002.

[39] A Honecker. Automorphisms of W-algebras and extended rational conformal field theories. *Nucl. Phys.*, B400:574–596, 1993. arXiv:hep-th/9211130.

[40] Y-Z Huang. Vertex operator algebras, the Verlinde conjecture, and modular tensor categories. *Proc. Natl. Acad. Sci. USA*, 102:5352–5356, 2005. arXiv:math.QA/0412261.

[41] V Kac, S Roan and M Wakimoto. Quantum reduction for affine superalgebras. *Comm. Math. Phys.*, 241:307–342, 2003. arXiv:math-ph/0302015.

[42] V Kac and M Wakimoto. Modular invariant representations of infinite-dimensional Lie algebras and superalgebras. *Proc. Nat. Acad. Sci. USA*, 85:4956–4960, 1988.

[43] V Kac and M Wakimoto. Quantum reduction and representation theory of superconformal algebras. *Adv. Math.*, 185:400–458, 2004. arXiv:math-ph/0304011.

[44] H Kausch. Extended conformal algebras generated by a multiplet of primary fields. *Phys. Lett.*, B259:448–455, 1991.

[45] K Kawasetsu. Relaxed highest-weight modules III: character formulae. *Adv. Math.*, 393:108079, 2021. arXiv:2003.10148 [math.RT].

[46] K Kawasetsu and D Ridout. Relaxed highest-weight modules I: rank 1 cases. *Comm. Math. Phys.*, 368:627–663, 2019. arXiv:1803.01989 [math.RT].

[47] K Kawasetsu and D Ridout. Relaxed highest-weight modules II: classifications for affine vertex algebras. *Comm. Contemp. Math.*, 24:2150037, 2022. arXiv:1906.02935 [math.RT].

[48] K Kawasetsu, D Ridout and S Wood. An admissible-level $\mathfrak{sl}_3$ model. *Lett. Math. Phys.*, 112:96, 2022. arXiv:2107.13204 [math.QA].

[49] I Koh and P Sorba. Fusion rules and (sub)modular invariant partition functions in nonunitary theories. *Phys. Lett.*, B215:723–729, 1988.

[50] H Li. The physics superselection principle in vertex operator algebra theory. *J. Algebra*, 196:436–457, 1997.

[51] O Mathieu. Classification of irreducible weight modules. *Ann. Inst. Fourier (Grenoble)*, 50:537–592, 2000.

[52] R McRae. On rationality for C_2-cofinite vertex operator algebras. arXiv:2108.01898 [math.QA].

[53] S Mizoguchi. Determinant formula and unitarity for the W_3 algebra. *Phys. Lett.*, B222:226–230, 1989.

[54] A Polyakov. Gauge transformations and diffeomorphisms. *Int. J. Mod. Phys.*, A5:833–842, 1990.

[55] D Ridout. $\widehat{\mathfrak{sl}}(2)_{-1/2}$: A case study. *Nucl. Phys.*, B814:485–521, 2009. arXiv:0810.3532 [hep-th].

[56] D Ridout and S Wood. Relaxed singular vectors, Jack symmetric functions and fractional level $\widehat{\mathfrak{sl}}(2)$ models. *Nucl. Phys.*, B894:621–664, 2015. arXiv:1501.07318 [hep-th].

[57] D Ridout and S Wood. The Verlinde formula in logarithmic CFT. *J. Phys. Conf. Ser.*, 597:012065, 2015. arXiv:1409.0670 [hep-th].

[58] A Semikhatov. Inverting the Hamiltonian reduction in string theory. In *28th International Symposium on Particle Theory, Wendisch-Rietz, Germany*, pages 156–167, 1994. arXiv:hep-th/9410109.

[59] S Smith. A class of algebras similar to the enveloping algebra of $sl(2)$. *Trans. Amer. Math. Soc.*, 322:285–314, 1990.

[60] E Verlinde. Fusion rules and modular transformations in 2D conformal field theory. *Nucl. Phys.*, B300:360–376, 1988.

[61] W Wang. Classification of irreducible modules of W_3 algebra with $c = -2$. *Commun. Math. Phys.*, 195:113–128, 1998. arXiv:9708016 [math.QA].

[62] G Watts. Determinant formulae for extended algebras in two-dimensional conformal field theory. *Nucl. Phys.*, B326:648–672, 1989.

[63] E Witten. Non-abelian bosonization in two dimensions. *Comm. Math. Phys.*, 92:455–472, 1984.

[64] A Zamolodchikov. Infinite additional symmetries in two-dimensional conformal quantum field theory. *Theoret. and Math. Phys.*, 65:1205–1213, 1985.

[65] Y Zhu. Modular invariance of characters of vertex operator algebras. *J. Amer. Math. Soc.*, 9:237–302, 1996.

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