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EXCLUSION STATISTICS IN CONFORMAL FIELD THEORY AND THE UCPF FOR WZW MODELS

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Dedicated to the memory of Prof. H.S. Green

ABSTRACT

In this paper we further elaborate on the notion of fractional exclusion statistics, as introduced by Haldane, in two-dimensional conformal field theory, and its connection to the Universal Chiral Partition Function as defined by McCoy and collaborators. We will argue that in general, besides the pseudo-particles introduced recently by Guruswamy and Schoutens, one needs additional 'null quasi-particles' to account for the null-states in the quasi-particle Fock space. We illustrate this in several examples of WZW-models.

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1. Introduction

Elementary excitations in low dimensional quantum many-body systems may exhibit 'fractional statistics' generalizing the usual Bose and Fermi statistics. In such cases the single particle states available to an excitation may depend on the entire particle content of the multi-particle state. To handle such systems Haldane [1] introduced a particular form of, so-called, 'fractional exclusion statistics' where the statistical interactions are encoded into a matrix G_{ab}. The thermodynamics of an ideal gas of particles satisfying Haldane's fractional exclusion statistics was subsequently worked out in a series of papers [2,3].

Fractional exclusion statistics arises naturally in quasi-particle descriptions of two-dimensional Conformal Field Theories (CFTs). Here quasi-particles correspond to intertwiners (Chiral Vertex Operators or CVOs) between the various representations of the chiral algebra and the (chiral) spectrum is constructed by repeated application of the modes of a preferred set of CVOs on the vacuum. Inspired by [4,5], such a basis was first constructed for the $(\widehat{\mathfrak{sl}_2})_{k\geq 1}$ WZW models [6,7] in terms of a $j = 1/2$ spinon field. This basis in particular illuminates the appearance of fundamental spinons in spin-S integrable spin chains whose effective CFT is an $(\widehat{\mathfrak{sl}_2})_{k=2S}$ WZW model [8].

The virtue of the quasi-particle approach to CFT is that there is a simple method, developed in [9], to compute the thermodynamical properties of the quasi-particles and expose their 'fractional exclusion statistics'. This method involves truncating the quasi-particle basis in momentum space and finding recursion relations for the associated finitized chiral characters. From the recursion relations one immediately derives the (total) single-level grand partition function $\lambda_{\rm tot}$ for the quasi-particles and hence their statistical properties. A large number of examples were subsequently worked out [9–14]. Particularly interesting applications of this approach include the study of the fractional exclusion statistics of the edge quasi-particle excitations over abelian quantum Hall states [13].

In many cases, including $(\widehat{\mathfrak{sl}_2})_{k=1}$ WZW models [1,9] and $\mathbb{Z}_k$ parafermions [12,14], it was discovered that the exclusion statistics of these CFT quasi-particles is indeed of the type introduced by Haldane. All these examples involved quasi-particles with abelian braid statistics, corresponding in the CFT to intertwiners (CVOs) with a unique fusion path. The corresponding statistics is referred to as abelian exclusion statistics. At the same time it was obvious that quasi-particles with non-abelian exclusion statistics, corresponding to non-abelian braid group representations, do not satisfy exclusion statistics of the type originally envisaged in [1,2]. Recently, however, it was realized [15] that these cases could be incorporated into Haldane's scheme as well by allowing for pseudo-particles, i.e., particles that do not carry any bare mass or energy. In particular, the non-abelian exclusion statistics of $(\widehat{\mathfrak{sl}_2})_{k>1}$ spinons (for $k = 2, 3, 4$) and generalized fermions in minimal models $\mathcal{M}_{k+2}$ of CFT (for $k = 1, 2, 3$) was shown to agree with the pseudo-particle generalization of Haldane statistics [15].

In a parallel development much progress has been made in the last few years in the analytic calculation of the character formulas directly from a statistical mechanics approach. These works generally involve the classification of all the eigenvalues of the transfer matrix and the computation of their finite-size corrections. This was first carried out by the Stony Brook group by solving the Bethe-Ansatz type equations [16], and was followed by the work of [17,18] which deals directly with the functional relations for the eigenvalues. Typically these calculations lead to the so-called 'fermionic' (or quasi-particle) type expressions for the characters of the representations of the chiral algebra. One can again identify the quasi-particles in these fermionic characters, and they seem to be related to the particle spectrum appearing in certain integrable perturbations of the underlying CFT [19,20].

Based on the many known examples (see, e.g., [20–28,7] and references therein), McCoy et al. (see, in particular, [29]) conjectured that all CFT characters can be written in the so-called 'Universal Chiral Partition Function' (UCPF) form, which can be interpreted as the grand partition function for a system of chiral particles with fugacities, and whose single particle momenta satisfy certain fermionic counting rules. Actually, it was noted a few years earlier by the Stony Brook group [30] that such counting rules are very similiar to Haldane's exclusion statistics. The relation of exclusion statistics to models solvable by the Thermodynamic Bethe Ansatz (TBA) was also noticed by Bernard and Wu [31]. Thus it became natural to conjecture that the quasi-particles underlying the UCPF satisfy Haldane exclusion statistics with a statistical interaction matrix G_{ab} given by the bilinear form matrix entering the UCPF. This was successfully demonstrated in a number of cases corresponding to abelian exclusion statistics [32,13,12,29,14], but it was realized [14], and confirmed for $(\widehat{\mathfrak{sl}_2})_{k>1}$ spinons and generalized fermions in [15], that the most general form of the UCPF involves quasi-particles with non-abelian exclusion statistics.

In this paper we will further elaborate on exclusion statistics in CFT, and the connection with the UCPF. In particular, in Sections 2 and 3, we will show that both lead to the same effective central charge. Furthermore, in Section 4, we will argue that, in general, to write characters of WZW models in UCPF form one needs to introduce, besides the pseudo-particles, yet another kind of quasi-particles. These particles are composites of the basic quasi-particles and account for the null-states in the quasi-particle Fock spaces [33]. We will refer to these as null-particles. They can contribute to the UCPF either with strictly positive or with alternating signs. We will incorporate these null-particles in Haldane's scheme from the outset. In Section 5 we discuss various examples, corresponding to both abelian and non-abelian exclusion statistics. We compare the exclusion statistics defined by the interaction matrix G_{ab} in the UCPF with the results obtained from the recursion relation approach and find complete agreement in all cases. We end with some conclusions.

2. Exclusion statistics with pseudo- and null-particles

Exclusion statistics, as introduced by Haldane [1] (and generalized to the multi-component case in [3][1]), is based on the idea that the number of accessible states $d_{(a,k)}$ for a particle of species a and momentum k depends on the occupation number $N_{(a,k)}$ of all other particles though a statistical interaction matrix $g_{(a,k)(b,k')}$ by

$$\Delta d_{(a,k)} \;=\; -\sum_{(b,k')} \; g_{(a,k)(b,k')}\, \Delta N_{(b,k')}\,. \tag{2.1}$$

It follows that the total number of states $W(\{N_{(a,k)}\})$, at fixed occupation numbers $\{N_{(a,k)}\}$ is given by

$$W(\{N_{(a,k)}\}) \;=\; \prod_{(a,k)} \begin{pmatrix} D^0_{(a,k)} + N_{(a,k)} - 1 - \sum_{(b,k')} g_{(a,k)(b,k')} N_{(b,k')} \\ N_{(a,k)} \end{pmatrix}, \tag{2.2}$$

where $D^0_{(a,k)}$ is the total number of states available to particles of species a with momentum k when there are no particles in the system. Thus a gas of particles satisfying the above

[1] For a different approach to exclusion statistics with internal degrees of freedom, see [34].

exclusion statistics would have a grand canonical partition function given by

$$Z = \sum_{\{N_{(a,k)}\}} \left(\prod_{(a,k)} (\tau_a)^{N_{(a,k)}} \right) W(\{N_{(a,k)}\}) \exp \left(\sum_{(a,k)} N_{(a,k)}(\mu_a - \epsilon_a^0(k))/k_B T \right) , \quad (2.3)$$

where $\epsilon_a^0(k)$ and μ_a are, respectively, the bare energy and chemical potential of the particle of species a. In the sequel we shall specialize to the case of a one-dimensional ideal gas where the particle interaction is localized in momentum space and encoded in a finite matrix G_{ab}, i.e.,

$$g_{(a,k)(b,k')} = \delta_{k,k'} G_{ab} . \quad (2.4)$$

We have allowed for particles that contribute to the partition function (2.3) with alternating signs, i.e. $\tau_a = -1$, as opposed to the 'real particles' with $\tau_a = 1$. We will see that they occur naturally in the quasi-particle description of certain conformal field theories.[2] We also partition the full set of particle species $\mathcal{S}$ into a set of 'physical particles' $\mathcal{S}^{\mathrm{ph}}$, and a set of 'pseudo-particles' $\mathcal{S}^{\mathrm{ps}}$. The pseudo-particles do not carry any bare mass or energy (i.e. $\epsilon_a^0(k) = 0$), but have the unique role of exchanging internal degrees of freedom (color) between the physical particles. In TBA literature they arise in models with non-diagonal scattering (see, for example, [35–38]). Pseudo-particles were recently introduced in Haldane's framework in [15]. It also seems that they have been anticipated in [3] where the case of one physical particle and several pseudo-particles was referred to as a hierarchical basis.

In the thermodynamic limit where the system size $M \to \infty$ with finite fixed temperature $T > 0$, a saddle point approach to the partition function (2.3) yields the following most probable 1-particle distribution function [2]

$$n_a(k) = z_a \frac{\partial}{\partial z_a} \log \lambda_a(z) \Big|_{z_b = \tau_b e^{\beta(\mu_b - \epsilon_b^0(k))}} , \quad (2.5)$$

where λ_a is determined by

$$\left(\frac{\lambda_a - 1}{\lambda_a} \right) \prod_b \lambda_b^{G_{ab}} = z_a , \quad (2.6)$$

and $z_a = \tau_a \exp(\beta(\mu_a - \epsilon_a^0(k)))$. From a TBA point of view, $\lambda_a = 1 + \exp(-\beta \epsilon_a)$ where ϵ_a is the dressed energy for species a. One could proceed further, generalizing the computation

[2] Of course, the alternating sign can be absorbed in the exponent by adding an imaginary part to the chemical potential.

in [29], by using n_a to calculate the internal energy per unit length. Alternatively, we could work with the expression for the free energy obtained in [2,3]

$$F \;=\; -k_B T \sum_{(a,k)} D^0_{(a,k)} \log \lambda_a \,. \tag{2.7}$$

For a gas with linear dispersion relation, i.e.,

$$(\epsilon^0_a(k),\, D^0_{(a,k)}) \;=\; \begin{cases} (v|k|,\, M\Delta k/2\pi) & \text{for } a \in \mathcal{S}^{\rm ph}\,, \\ (0,0) & \text{for } a \in \mathcal{S}^{\rm ps}\,, \end{cases} \tag{2.8}$$

where we assumed the speed of sound v is independent of the species, we obtain

$$F \;=\; -\frac{k_B^2 T^2 M}{v} \sum_{a\in\mathcal{S}^{\rm ph}} \int_0^{y_a} \frac{dz_a}{z_a}\, \log \lambda_a(z)\,, \tag{2.9}$$

where $y_a = \tau_a e^{\beta\mu_a}$ is the fugacity of species a. Thus the specific heat per unit length (at constant fugacity) for a one-dimensional ideal gas with exclusion statistics (2.1) is

$$C \;=\; \frac{\pi^2 k_B^2 T}{3v}\, \tilde{c}\,, \tag{2.10}$$

with

$$\frac{\pi^2}{6}\tilde{c} \;=\; \sum_{a\in\mathcal{S}^{\rm ph}} \int_0^{y_a} \frac{dz_a}{z_a}\, \log \lambda_a(z)\,. \tag{2.11}$$

We have written the specific heat in a form where $\tilde{c}$ admits an interpretation as the effective central charge for systems with conformal symmetry.

The integral (2.11) may be evaluated along the lines of [14] (similar computations are of course well-known from related TBA equations, cf. [38,39,36,40,34]) and leads to[3]

$$\left(\frac{\pi^2}{6}\right) \tilde{c}(y) \;=\; \sum_a \left(L(\xi_a) - L(\eta_a) - \tfrac{1}{2}\log y_a \log\left(\frac{1-\xi_a}{1-\eta_a}\right)\right)\,, \tag{2.12}$$

where (ξ_a, η_a) are solutions to the equations

$$\xi_a \;=\; y_a \prod_b (1-\xi_b)^{G_{ab}}\,, \qquad \eta_a \;=\; y_a \sigma_a \prod_b (1-\eta_b)^{G_{ab}}\,, \tag{2.13}$$

where $\sigma_a = 0$ ($\sigma_a = 1$) for $a \in \mathcal{S}^{\rm ph}$ ($a \in \mathcal{S}^{\rm ps}$), and $L(x)$ is Rogers' dilogarithm defined by

$$L(x) \;=\; -\tfrac{1}{2} \int_{\mathcal{C}_{0,x}} dz \left(\frac{\log z}{1-z} + \frac{\log(1-z)}{z} \right)\,, \tag{2.14}$$

where $\log z$ (for $z \neq 0$) signifies the logarithm in the branch $-\pi < \arg z \leq \pi$ and $\mathcal{C}_{0,x}$ is a contour in $\mathbb{C}$ from 0 to x which does not go across the branch cuts of $\log z$ and $\log(1-z)$. Thus, in contrast to the case with no pseudo-particles [14], the presence of the pseudo-particles induces subtraction terms in the effective central charge (2.12).

[3] Here we have assumed that G_{ab} is symmetric.

3. Exclusion statistics in conformal field theory

Suppose we have a two-dimensional conformal field theory with chiral algebra $\mathcal{A}$, a set of $\mathcal{A}$-modules V_i, labeled by some index set $i \in \mathcal{I}$, and intertwiners (CVOs)

$$\phi^a \begin{pmatrix} & j & \\ i' & & i \end{pmatrix}(z) \;=\; \sum_{n\in\mathbb{Z}} \phi^a \begin{pmatrix} & j & \\ i' & & i \end{pmatrix}_{-n-(\Delta_{i'}-\Delta_i)} z^{n+(\Delta_{i'}-\Delta_i-\Delta_j)}\,, \tag{3.1}$$

where Δ_i denotes the conformal dimension of V_i and $a = 1,\ldots,\dim V_j$. The number of intertwiners $i \times j \to i'$ is given by the fusion rules $N_{ij}{}^{i'}$. In a quasi-particle approach to conformal field theory the (chiral) spectrum is constructed by repeated application of the modes of a preferred set of CVOs on the vacuum $|0\rangle$, i.e., a set of quasi-particle states of type

$$\phi^{a_N} \begin{pmatrix} & j_N & \\ i_N & & i_{N-1} \end{pmatrix}_{-n_N-\Delta(N)} \ldots \phi^{a_2} \begin{pmatrix} & j_2 & \\ i_2 & & i_1 \end{pmatrix}_{-n_2-\Delta(2)} \phi^{a_1} \begin{pmatrix} & j_1 & \\ i_1 & & 0 \end{pmatrix}_{-n_1-\Delta(1)} |0\rangle\,, \tag{3.2}$$

where $\Delta(k) = \Delta_{i_k} - \Delta_{i_{k-1}}$, constitute a basis of the $\mathcal{A}$-module V_i. This basis is overcomplete unless we put restrictions on the mode sequences $(n_1,\ldots,n_N)$. These restrictions are obtained both from the braiding and fusion relations satisfied by the intertwiners (3.1) as well as by possible null-states in the Fock space of intertwiners and may depend on the fusion path $(0, i_1, i_2, \ldots, i_N)$.

It has been observed in a variety of approaches – TBA approaches, integrable spin chains and also in the context of conformal field theory – that the degrees of freedom contained in (3.2) can be separated into physical excitations and pseudo-particle excitations. Loosely speaking, the physical excitations correspond to excitations over some reference fusion path, while the pseudo-particles create 'excited fusion paths'. While this separation might, strictly speaking, not hold in the conformal field theory, it may hold in some crystal limit (cf. [41]) which is sufficient as far as the discussion of the partition function is concerned. Thus, if the quasi-particles in a conformal field theory are described by Haldane exclusion statistics, we need to distinguish two kinds of particles: 'pseudo-particles' with vanishing bare energy and 'physical particles' with an infinite range of energy levels separated by integers, i.e., we have the dispersion relation (2.8).

Let us consider the partition function $\mathrm{ch}_i(y;q)$, i.e., the character of the $\mathcal{A}$-module V_i. If we assume that the quasi-particle interaction is purely statistical according to (2.1), and that λ_a of (2.6) can be interpreted as the single quasi-particle grand partition function, the character will have the following approximate form in the thermodynamic limit[4]

$$\mathrm{ch}_i(y;q) \;\sim\; \prod_{a\in\mathcal{S}^{\mathrm{ps}}} \lambda_a(y_a q^{\Delta_a}) \prod_{a\in\mathcal{S}^{\mathrm{ph}}} \prod_{l\geq 0} \lambda_a(y_a q^{\Delta_a+l})\,. \tag{3.3}$$

Of course the expression (3.3) is not meant to be exact but only valid, in general, in the thermodynamic limit. We have chosen to write it in a discrete form, rather than in an integral form like (2.9), to emphasize the discrete energy spectrum of the CFT. Also, the product over a and l will be subject to restrictions depending on the sector i. Cases where the CFT characters (or sums thereof) do admit exact factorizations of the form (3.3) have recently been studied in [42].

The character $\mathrm{ch}_i(y;q)$ may be expanded as a power series in q

$$\mathrm{ch}_i(y;q) \;=\; \sum_{N\geq 0} a_N(y)\, q^N\,, \tag{3.4}$$

It is well-known, of course, that modular transformations relate the asymptotic behaviour of $a_N(y)$, for $N \gg 0$, to the specific heat (2.10). For definiteness, let us see how this works out using (3.3). Asymptotically, we may approximate

$$\begin{aligned} a_N(y) &= \frac{1}{2\pi i}\oint \frac{dq}{q^{N+1}}\,\mathrm{ch}_i(y;q) \;=\; \frac{1}{2\pi i}\oint dq\;\exp\left(-(N+1)\log q + \log \mathrm{ch}_i(y;q)\right) \\ &\sim \oint dq \exp\left(-(N+1)\log q - (\log q)^{-1}\sum_{a\in\mathcal{S}^{\mathrm{ph}}}\int_0^{y_a} \frac{dz_a}{z_a}\log\lambda_a(z)\right)\,. \end{aligned} \tag{3.5}$$

In the last step, we have omitted all terms that do not contribute to the leading N behaviour of $a_N(y)$. The integral can be evaluated using a saddle point approximation, and we find

$$\log a_N(y) \;\sim\; 2\pi\sqrt{\frac{c_{\mathrm{eff}}(y)N}{6}}\,, \tag{3.6}$$

with

$$\frac{\pi^2}{6}\,c_{\mathrm{eff}}(y) \;=\; \sum_{a\in\mathcal{S}^{\mathrm{ph}}}\int_0^{y_a}\frac{dz_a}{z_a}\log\lambda_a(z)\,. \tag{3.7}$$

[4] The modular parameter q is related to the quantum spin chain quantities by $q = \exp(-2\pi v/Mk_BT)$, thus $M\to\infty$ (at fixed $T>0$) corresponds to $q\to 1^-$.

From (3.6) we can identify $c_{\rm eff}(y)$ with the effective central charge of the partition function (3.3) as shown in, e.g., [43,44]. Note that (2.11) and (3.7) indeed imply that we can also identify $c_{\rm eff}(y)$ with the $\tilde{c}$ computed in the previous section.

For future use, note that if all the chemical potentials μ_a, for $a \in \mathcal{S}^{\rm ph}$, are given in terms of a single chemical potential μ as $\mu_a = \ell_a \mu$, then we may write (3.7) as

$$\frac{\pi^2}{6}\, c_{\rm eff}(y) \;=\; \int_0^y \frac{dz}{z} \log \lambda_{\rm tot}(z)\,, \tag{3.8}$$

where

$$\lambda_{\rm tot}(z) \;=\; \prod_{a\in\mathcal{S}^{\rm ph}} \lambda_a(z)^{\ell_a}\,. \tag{3.9}$$

The central charge $c = c_{\rm eff} + \Delta_{\rm min}$ of the conformal field theory follows from (2.12) and (2.13) in the limit of vanishing chemical potentials, i.e. $y_a = \tau_a$, where we need to take those solutions to (2.13) that satisfy $0 \le \xi_a, \eta_a \le 1$ for $\tau_a = 1$ and $-1 \le \xi_a, \eta_a \le 0$ for $\tau_a = -1$. For the latter case, the imaginary part of the corresponding dilogarithm is precisely canceled by the logarithm term in (2.12). Indeed, for $x < 0$,

$$L(x) - \frac{\pi i}{2}\log(1-x) \;=\; L\left(\frac{1}{1-x}\right) - L(1) \;=\; -L\left(\frac{-x}{1-x}\right)\,. \tag{3.10}$$

An interesting development over the last few years has been the derivation of quasi-particle type character formulas for the modules of chiral algebras (see, e.g., [20–28,7]). This work has led to the conjecture that all conformal field theory characters can be written in the so-called ‘Universal Chiral Partition Function’ (UCPF) form (see, in particular, [29])

$$\mathrm{ch}_i(y;q) \;=\; \sum_{\substack{m_1,\ldots,m_n\ge 0\\ \text{restrictions}}} \left(\prod_a y_a^{m_a}\right) q^{\frac{1}{2}\mathbf{m}\cdot\mathbf{G}\cdot\mathbf{m} - \frac{1}{2}\mathbf{A}\cdot\mathbf{m}} \prod_a \begin{bmatrix} ((\mathbf{1}-\mathbf{G})\cdot\mathbf{m} + \frac{\mathbf{u}}{2})_a \\ m_a \end{bmatrix}\,, \tag{3.11}$$

where $\mathbf{G}$ is an $n\times n$ matrix and $\mathbf{A}$ and $\mathbf{u}$ are certain n-vectors. Both $\mathbf{A}$ and $\mathbf{u}$ as well as the restrictions on the summations over the quasi-particle numbers m_a will in general depend on the sector i, while $\mathbf{G}$ will be independent of i. Furthermore, we have defined

$$\begin{bmatrix} m \\ n \end{bmatrix} \;=\; \frac{(q)_m}{(q)_n (q)_{m-n}}\,, \qquad (q)_n \;=\; \prod_{k=1}^{n} (1-q^k)\,. \tag{3.12}$$

It has been conjectured by various groups (see, in particular, [29]) that the quasi-particles underlying (3.11) satisfy Haldane exclusion statistics with statistical interaction matrix given by the *same* matrix $\mathbf{G}$ as the one entering (3.11). We will refer to this conjecture as the 'UCPF-exclusion statistics' conjecture.

A convincing piece of evidence in support of this conjecture is that the asymptotics of the character (3.11) (in the thermodynamic limit $q \to 1^-$) is given by exactly the same formulas (2.12) and (2.13) where $\sigma_a = 0$ for $u_a = \infty$ (physical particles), while $\sigma_a = 1$ for $u_a < \infty$ (pseudo-particles). The asymptotic form of the character (3.11) for $y_a = 1$ was derived in [44] (see also [45,43,46]). The present result is a straightforward generalization of these derivations.

To prove the conjecture beyond the comparison of thermodynamics requires an exact computation of the partition function starting from first principles, i.e., eqn. (2.1), as has been done for g-ons [32]. Alternatively, it has been argued that the analytic continuation of (2.12) to the covering space of $\mathbb{C}\backslash\{0,1\}$ contains information about *all* the excitations in the spectrum. This idea has been successfully applied to some minimal models of conformal field theory [17], (generalized) parafermions [47], $(\widehat{\mathfrak{sl}_n})_{k=1}$ WZW models [48] and $(\widehat{\mathfrak{sl}_2})_{k\geq 1}$ WZW models [49] and might be put on a more rigorous footing.

Exclusion statistics in conformal field theory can be studied by a method based on recursion relations for truncated characters [9]. Truncated characters $P_L^{(i)}(y;q)$ are defined by taking the partition function of those states (3.2) where all the modes satisfy $n_i + \Delta(i) \leq L$. Thus, for large L, we will have (cf. (3.3))

$$P_L^{(i)}(y;q) \;\sim\; \prod_{a\in\mathcal{S}^{\rm ps}} \lambda_a(y_a) \prod_{a\in\mathcal{S}^{\rm ph}} \; \prod_{0\leq l\leq L} \lambda_a(y_a q^l)\,, \tag{3.13}$$

where the products are subject to certain restrictions depending on the sector a. Thus

$$P_{L+1}^{(i)}(y;q)/P_L^{(i)}(y;q) \;\sim\; \prod_{a\in\mathcal{S}^{\rm ph}} \lambda_a(y_a q^L)\,, \qquad \text{as } L\to\infty\,. \tag{3.14}$$

More generally, if some of the physical particles $a \in \mathcal{S}^{\rm ph}$ are composites of ℓ_a more fundamental particles, then their modes will be cut off at $\ell_a L$ and we find

$$P_{L+1}^{(i)}(y;q)/P_L^{(i)}(y;q) \;\sim\; \prod_{a\in\mathcal{S}^{\rm ph}} \lambda_a(y_a q^L)^{\ell_a} \;=\; \lambda_{\rm tot}(y_a q^L)\,, \tag{3.15}$$

where $\lambda_{\rm tot}$ is defined in (3.9). Therefore, from recursion relations for the truncated characters $P_L^{(i)}(y;q)$ in the large L limit, one derives algebraic equations for the total 1-particle partitions functions $\lambda_{\rm tot}(y)$, which can be compared to the $\lambda_{\rm tot}(y)$ arising from a solution of (2.6), with a statistical interaction matrix as suggested by the UCPF formula (3.11). This procedure was carried out, and agreement was found, in several cases including g-ons (the one-component case of (3.11)) [32,29] and several multi-component cases [9,13,12,14]. All these cases involve only physical particles ($u_a = \infty$), i.e., correspond to intertwiners with a unique fusion path, and the corresponding statistics was therefore called 'abelian exclusion statistics'. From (2.6) it is clear that the absence of pseudo-particles always leads to small x expansions of the form $\lambda_a(x) = 1 + x_a + \mathcal{O}(x^2)$. In [10,14] it was observed, however, that generally we have expansions of the form $\lambda_a(x) = 1 + \alpha_a x_a + \mathcal{O}(x^2)$, where α_a is the largest eigenvalue of the fusion matrix of the quasi-particle a [14]. The exclusion statistics corresponding to the more general case, $\alpha_a \neq 1$, was named 'non-abelian exclusion statistics'. It was recently recognized that non-abelian exclusion statistics can be accounted for in the Haldane approach by incorporating pseudo-particles [15]. The UCPF-exclusion statistics conjecture was subsequently confirmed in various non-abelian cases, namely $(\widehat{\mathfrak{sl}_2})_{k>1}$ WZW-models and generalized fermions in minimal models of CFT [15].

The main purpose of the remainder of this paper is to verify the UCPF-exclusion statistics conjecture in various other, rather non-trivial, WZW-examples (both abelian and non-abelian) for which UCPF characters have been found recently [33]. In all examples we find complete agreement, thus supporting the conjecture above. To find agreement, however, we have had to slightly extend the definition of the UCPF form to account for certain null-particles, as remarked before.

4. Towards the UCPF for WZW models

The following questions naturally arise. Given a conformal field theory, how does one identify a set of quasi-particles in terms of which the characters take the UCPF form, and how is the statistical interaction matrix $\mathbf{G}$ determined in terms of the conformal field theory data (chiral algebra, modules, fusion rules, conformal dimensions, etc.)? In this section we will give a partial answer for WZW models.

Thus far, only isolated cases of UCPF characters for affine Lie algebras were known. These included $(\widehat{\mathfrak{sl}_2})_{k=1}$ [5,6], $(\widehat{\mathfrak{sl}_2})_{k>1}$ [7] and $(\widehat{\mathfrak{sl}_n})_{k=1}$ [28,50].[5] Recently an algorithm

[5] In principle one can get quasi-particle affine Lie algebra characters by taking limits of

was given which, in principle, can be used to obtain affine Lie algebra characters in UCPF form for any affine Lie algebra $\widehat{\mathfrak{g}}$ and at arbitrary level [33]. Here we will briefly explain the idea, we refer to [33] for the technical details. In the next section we discuss some examples, examine the exclusion statistics, and make a comparison to the results of [14].

Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra of rank ℓ, let $\{\Lambda_i\}_{i=1}^{\ell}$ be the set of fundamental weights and $L(\Lambda_i)$ the corresponding finite dimensional irreducible representations. As our set of quasi-particles we take the intertwiners[6]

$$\phi^a \begin{pmatrix} \Lambda_i \\ \Lambda' \; \Lambda \end{pmatrix} (z) , \qquad a = 1, \ldots, \dim L(\Lambda_i) , \tag{4.1}$$

corresponding to all fundamental representations $L(\Lambda_i)$ and between all possible $\widehat{\mathfrak{g}}$ modules (given by Λ and Λ') at level-k, as determined by the fusion rules $N_{\Lambda\Lambda_i}{}^{\Lambda'}$. For example, for $\widehat{\mathfrak{sl}}_3$ we take intertwiners transforming in both the $\mathbf{3}$ as well as the $\bar{\mathbf{3}}$ representation of $\mathfrak{sl}_3$. Next, we need to decouple the pseudo-particle excitations, representing the excited fusion paths with respect to a reference fusion path, from the physical excitations. The sums over pseudo-particle excitations are well known from RSOS and spin chain models and yield, up to a factor, level-k restricted modified Hall-Littlewood polynomials $M^{(k)}_{\lambda\mu}(y;q)$ (or Kostka-Foulkes polynomials in the case of $\mathfrak{sl}_n$) for which, in some cases, UCPF expressions are known (see, e.g., [51,44,46,52] for $y = 1$). On the other hand, the physical excitations would yield a (Fock space) contribution to the character given by

$$\sum_{\sum_a m_a^{(i)} = M_i} \left(\prod_i y_i^{M_i} \right) \frac{q^{\frac{1}{2}\mathbf{M}\cdot\mathbf{B}\cdot\mathbf{M}}}{\prod_i \prod_a (q)_{m_a^{(i)}}} , \tag{4.2}$$

where M_i is the total number of intertwiners in $L(\Lambda_i)$ and we have specialized to the case where all particles in $L(\Lambda_i)$ have the same fugacity y_i. The bilinear form matrix $\mathbf{B}$ is given, in the case of simply laced Lie algebras, by the inverse Cartan matrix $A^{-1}_{ij} = (\Lambda_i, \Lambda_j)$

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the $\mathcal{W}$-algebra minimal model characters (i.e., coset characters) of, e.g., [44]. These will however involve an infinite number of pseudo-particles and are not of the type considered here.

[6] While in some level-1 cases it is possible to give the character in a UCPF form using less quasi-particles than the ones discussed here, these formulas do not seem to generalize to level $k > 1$ just by the inclusion of additional pseudo-particles. We refer to Section 5.6 for an example.

of $\mathfrak{g}$ and arises from the mutual locality of the basic vertex operators $\exp(i\Lambda_i\cdot\phi)$. For non-simply laced Lie algebra these vertex operators have to be corrected by fermionic operators to account for the difference between $\frac{1}{2}|\Lambda_i|^2$ and the conformal dimension $\Delta_i = (\Lambda_i, \Lambda_i + 2\rho)/(\mathrm{h}^\vee + 1)$ of V_i at level $k=1$ – the corresponding $\mathbf{B}$ easily follows from [53]. Combining these two ingredients gives the required result for, e.g., $(\widehat{\mathfrak{sl}_2})_{k\geq 1}$ (see Section 5.1). Unfortunately, in general this is not the whole story as we have not yet incorporated the possible null-states in the physical quasi-particle Fock space. It turns out, as discussed in [33], that these can be accounted for in UCPF form by interpreting the Fock space modulo null-states as the coordinate ring of an affinized projective variety and applying standard techniques from algebraic geometry. Besides deforming the exponent $\frac{1}{2}\mathbf{M}\cdot\mathbf{B}\cdot\mathbf{M}$ in (4.2) by a term $\frac{1}{2}\mathbf{m}\cdot\mathbf{Q}\cdot\mathbf{m}$, this will in general involve the addition of null quasi-particles, corresponding to certain composites of the basic physical quasi-particles (hence their chemical potentials are fixed in terms of those of their constituents).

The final answer for the UCPF is then of the form

$$\mathrm{ch}_\lambda(y;q) \;=\; \sum_{\mu = M_1\Lambda_1+\ldots+M_\ell\Lambda_\ell} \frac{1}{\prod_{i=1}^{\ell}(q)_{M_i}}\; M^{(k)}_{\lambda\mu}(y;q) M_\mu(y;q)\,, \tag{4.3}$$

where

$$M_\mu(y;q) \;=\; \left(\prod_{i=1}^{\ell}(q)_{M_i}\right)\sum_{\mathbf{m}}\left(\prod_a y_a^{m_a}\right)\frac{q^{\frac{1}{2}\mathbf{m}\cdot\mathbf{Q}\cdot\mathbf{m}}}{\prod_a (q)_{m_a}}\,. \tag{4.4}$$

The factor containing $\mathbf{B}$ in (4.2) has been absorbed in $M^{(k)}_{\lambda\mu}(q)$. Indeed, it has been conjectured (and proven in some cases) [54,55,52] that the affine Lie algebra characters $\mathrm{ch}_\lambda(y;q)$ are indeed of the form (4.3), where $M^{(k)}_{\lambda\mu}(y;q)$ and $M_\mu(y;q)$ are, respectively, the level-k restricted and unrestricted Hall-Littlewood polynomials.

The procedure leading to (4.4) is not unique, however, and various equivalent UCPFs with different null quasi-particle contents may be given (see the examples in Section 5).[7]

The equality between the characters with the various null-state subtractions is based on the successive application of the following two identities (see, e.g., [50] for an elementary proof)

$$\frac{1}{(q)_M(q)_N} \;=\; \sum_{m\geq 0}\frac{q^{(M-m)(N-m)}}{(q)_m(q)_{M-m}(q)_{N-m}}\,, \tag{4.5}$$

[7] An interesting example in the context of minimal models of conformal field theory was recently discussed in [20].

and

$$\frac{q^{MN}}{(q)_M(q)_N} = \sum_{m\geq 0}(-1)^m \frac{q^{\frac{1}{2}m(m-1)}}{(q)_m(q)_{M-m}(q)_{N-m}}, \tag{4.6}$$

which are, in a sense, 'inverses' of each other. Both identities are specializations of the q-Chu-Vandermonde identity for the basic hypergeometric series ${}_2\phi_1$ (see, e.g., [56]).[8] Both are intimately related to, and in fact constitute a proof of, the five-term identity for Rogers' dilogarithm

$$L(x) + L(y) = L(xy) + L\left(\frac{x(1-y)}{1-xy}\right) + L\left(\frac{y(1-x)}{1-xy}\right). \tag{4.7}$$

Indeed, by comparing $c_{\rm eff}(y)$ for the asymptotics of two characters related by a single application of either (4.5) or (4.6) one discovers (4.7). Denoting the solutions of (2.13) for the corresponding variables on the left hand side of (4.5) and (4.6) by (ξ_1,ξ_2) and on the right hand side by (ξ_1',ξ_2',ξ_3') we find that for (4.5) they are related by

$$\xi_1' = \frac{\xi_1(1-\xi_2)}{1-\xi_1\xi_2}, \qquad \xi_2' = \frac{\xi_2(1-\xi_1)}{1-\xi_1\xi_2}, \qquad \xi_3' = \xi_1\xi_2, \tag{4.8}$$

while for (4.6) we find the inverse relations

$$\xi_1 = \frac{\xi_1'(1-\xi_2')}{1-\xi_1'\xi_2'}, \qquad \xi_2 = \frac{\xi_2'(1-\xi_1')}{1-\xi_1'\xi_2'}, \qquad \xi_3' = -\frac{\xi_1'\xi_2'}{1-\xi_1'\xi_2'}. \tag{4.9}$$

With the use of (2.12) and (3.10) both lead to (4.7). The fact that the characters of the various null-particle formulations always seem to be related by either (4.5) or (4.6) can be taken as further evidence for the conjecture that, loosely speaking, dilogarithm identities are always accessible by means of the five-term identity ('Goncharov's conjecture', see [46]).

Having obtained the WZW characters in a UCPF form we can read off the statistical interaction matrix $\mathbf{G}$ and verify whether the alleged exclusion statistics defined by (2.6) indeed agrees with the exclusion statistics derived by the recursion method in [14]. We will carry out this procedure in several non-trivial examples (Section 5) and find agreement in all cases.

In [14] one of the authors and K. Schoutens conjectured that the recursion relations for the truncated characters $P_L^{(i)}(q)$ of the level-1 WZW models were, in all cases, solved

[8] We are grateful to Ole Warnaar for pointing this out to us.

(upto an overall q-factor) by modified Hall-Littlewood polynomials $M_{\lambda(L,i)}(q)$, with argument $q \to q^{-1}$, and λ a function of both i and L. Thus, these modified Hall-Littlewood polynomials have to approach the WZW characters $\mathrm{ch}_i(q)$ in the limit $L \to \infty$. This observation leads to the conclusion that, asymptotically, $M_{\lambda(L,i)}(q)$ is given by an expression like (3.13). Extrapolating this reasoning a bit further, using the fact that $M_\lambda(q)$, for $\lambda = M_1\Lambda_1 + \ldots + M_\ell\Lambda_\ell$, is precisely the TBA-limit of the partition function of an integrable spin chain with M_i spins in the minimal $U_q(\widehat{\mathfrak{g}})$ affinization W_i of the fundamental representation $L(\Lambda_i)$ whose elementary excitations are precisely the quasi-particles (4.1) [57], we are led to the conjecture that, asymptotically,

$$M_\lambda(q) \;\sim\; q^{N(\lambda)} \prod_{i=1}^{\ell} \prod_{l_i=1}^{M_i} \zeta_i(q^{-l_i}) \,, \tag{4.10}$$

where $N(\lambda)$ is such that $M_\lambda(q) = \mathrm{const} + \mathcal{O}(q)$ and the ζ_i are expressed in terms of the λ_a according to the fusion rules, i.e., according to which composites of the quasi-particles $\phi^a \begin{pmatrix} \Lambda_i \\ \Lambda' \; \Lambda \end{pmatrix} (z)$ make up the sectors j. For $\mathfrak{sl}_n$ we would have, more explicitly,

$$\zeta_i(x) \;=\; \prod_j \left(\prod_a \widetilde{\lambda}_a^{(j)}(x) \right)^{B_{ij}} \,, \tag{4.11}$$

where B_{ij} is the inverse Cartan matrix of $\mathfrak{sl}_n$ and $\widetilde{\lambda}_a^{(j)}$ are the solutions $\lambda_a^{(j)}$ to (2.6) for the physical particles corresponding to (4.1) dressed with the λ's for the composite null-particles which contain that particle. Explicit formulas for ζ_i will be given in the examples of Section 5. Note however that the modified Hall-Littlewood polynomial is a q-deformation of the character of the tensor product of M_i-fold copies of W_i . Thus, a consistency check on the assertion (4.10) is that

$$\zeta_i(1) \;=\; \dim W_i \,. \tag{4.12}$$

We will verify this in the examples. In fact, the analogous statement seems to be true in higher level cases as well as suggested by the $(\widehat{\mathfrak{sl}_2})_k$ example in Section 5.1.

The fact that limits of modified Hall-Littlewood (or Kostka-Foulkes) polynomials lead to WZW characters for $(\widehat{\mathfrak{sl}_n})_k$ was first conjectured in [46] and subsequently proven, in special cases, in [58,52].

5. Examples

5.1. $\widehat{\mathfrak{sl}_2}$, level-$k$

The (non-abelian) exclusion statistics for the case $(\widehat{\mathfrak{sl}_2})_k$ has been extensively discussed in [59,11,14,15], where it was shown, among other things, that the solutions to the equation (2.6) indeed agree with the expressions obtained from the recursion approach (at least for small k). Here we suffice by making a few additional remarks.

At level-k there are $k+1$ integrable highest weight modules of $\widehat{\mathfrak{sl}_2}$ labeled by (twice) the $\mathfrak{sl}_2$ spin, $2j = 0, 1, \ldots, k+1$. The character of $(\widehat{\mathfrak{sl}_2})_{k\geq 1}$ can be written as [7,41,60]

$$\begin{aligned} \mathrm{ch}_j(z;q) \; = \; & \sum_{\substack{m_+, m_- \geq 0 \\ m_+ + m_- = m_1}} \; \sum_{\substack{m_2, \ldots, m_k \geq 0 \\ \text{restrictions}}} q^{-\frac{1}{4}m_1^2 + \frac{1}{4}\mathbf{m}\cdot\mathbf{A}_k\cdot\mathbf{m}} \prod_{a=2}^{k} \begin{bmatrix} ((\mathbf{1} - \frac{1}{2}\mathbf{A}_k)\cdot\mathbf{m} + \frac{\mathbf{u}_j}{2})_a \\ m_a \end{bmatrix} \\ & \times \frac{1}{(q)_{m_+}(q)_{m_-}} z^{\frac{1}{2}(m_+ - m_-)} \,, \end{aligned} \tag{5.1}$$

where $\mathbf{A}_k$ is the Cartan matrix of $A_k \cong \mathfrak{sl}_{k+1}$, and $(\mathbf{u}_j)_a = \delta_{a,2j}$. The restrictions are such that all entries in the q-binomials are integers. This character is obviously of the UCPF form (3.11) with

$$\mathbf{G} \; = \; \left(\begin{array}{cc|cccc} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & & & \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & & & \\ \hline -\frac{1}{2} & -\frac{1}{2} & & & & \\ & & & \frac{1}{2}\mathbf{A}_{k-1} & & \\ & & & & & \end{array} \right) , \tag{5.2}$$

where the entries of $\mathbf{G}$ correspond to the summation variables $\{m_+, m_-, m_2, \ldots, m_k\}$ in (5.1). In particular $u_+ = u_- = \infty$ while $u_a < \infty$ for $a = 2, \ldots, k$.

We find the following solution to (2.13) for $y_+ = y_- = 1$

$$\begin{aligned} \xi_+ \; = \; \xi_- \; = \; 1 - \frac{1}{k+1} \,, \qquad & \xi_a \; = \; 1 - \left(\frac{1}{k+2-a} \right)^2 \,, \quad a = 2, \ldots, k \,, \\ \eta_+ \; = \; \eta_- \; = \; 0 \,, \qquad & \eta_a \; = \; 1 - \left(\frac{\sin \frac{\pi}{k+2}}{\sin \frac{\pi a}{k+2}} \right)^2 \,, \quad a = 2, \ldots, k \,, \end{aligned} \tag{5.3}$$

leading to

$$\begin{aligned}\left(\frac{\pi^2}{6}\right) c &= \sum_a \left(L(\xi_a) - L(\eta_a)\right) \;=\; L(\xi_+) + L(\xi_-) + \sum_{a=2}^{k} \left(L(1-\eta_a) - L(1-\xi_a)\right) \\ &= 2L\left(\frac{k}{k+1}\right) - \sum_{a=1}^{k} L\left(\frac{1}{a^2}\right) + \sum_{a=1}^{k} L\left(\left(\frac{\sin\frac{\pi}{k+2}}{\sin\frac{\pi a}{k+2}}\right)^2\right) \;=\; \sum_{a=1}^{k} L\left(\left(\frac{\sin\frac{\pi}{k+2}}{\sin\frac{\pi a}{k+2}}\right)^2\right) \\ &= \frac{\pi^2}{6}\left(\frac{3k}{k+2}\right) ,\end{aligned} \tag{5.4}$$

as required. Moreover, as shown in [9] for $k=1$ and [15] for $k=2,\ldots,4$, the solution to (2.6) agrees with the one found by the recursion method [11,14]. Also note that from (5.3) it follows that the expression

$$\zeta \;=\; (\lambda_+\lambda_-)^{\frac{1}{2}} , \tag{5.5}$$

at $x=1$ is given by $\zeta = k+1$, which is consistent with the interpretation of the quasi-particles as the excitations of an $SU(2)$ spin chain with $2S=k$.

5.2. $\widehat{\mathfrak{sl}_3}$, level-1

The affine Lie algebra $\widehat{\mathfrak{sl}_3}$, at level-$k=1$, has three integrable representations corresponding to the singlet $\mathbf{1}$, the vector $\mathbf{3} = L(\Lambda_1)$ and the conjugate vector $\bar{\mathbf{3}} = L(\Lambda_2)$ of $\mathfrak{sl}_3$. As discussed in Section 2, for the quasi-particles we take the intertwiners $\phi^a(z)$ and $\phi^{\bar a}(z)$ transforming in, respectively, the $\mathbf{3}$ and $\bar{\mathbf{3}}$ representations. Since at level $k=1$ the fusion path is unique, there will be no pseudo-particles. However, the quasi-particle Fock space will contain null-states as a consequence of the null-field $\sum_a :\phi^a(z)\phi^{\bar a}(z):$. The most natural way of incorporating this null-field is by introducing one null-particle with $\tau=-1$. The following character formula for the integrable highest weight modules of $(\widehat{\mathfrak{sl}_3})_{k=1}$ was found in [28]:

$$\mathrm{ch}_i(y;q) \;=\; \sum_{\substack{M_1,M_2\geq 0\\ M_1+2M_2\equiv i \bmod 3}} y_1^{M_1} y_2^{M_2} q^{\frac{1}{2}\mathbf{M}\cdot\mathbf{B}\cdot\mathbf{M}} \sum_{\substack{m_a,m_{\bar a},m\\ \sum m_a+m=M_1\\ \sum m_{\bar a}+m=M_2}} (-1)^m \frac{q^{\frac{1}{2}m(m-1)}}{(q)_m} \frac{1}{\prod_a (q)_{m_a}(q)_{m_{\bar a}}} , \tag{5.6}$$

where

$$\mathbf{B} \;=\; \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} , \tag{5.7}$$

is the inverse Cartan matrix of $\mathfrak{sl}_3$. This is indeed of the UCPF form (3.11) with

$$\mathbf{G} = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \vdots & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \vdots & 1 \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \vdots & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \vdots & 1 \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \vdots & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \vdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \vdots & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \vdots & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \vdots & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \vdots & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \vdots & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \vdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \vdots & 1 & 1 & 1 & \vdots & 3 \end{pmatrix} . \tag{5.8}$$

and

$$\begin{aligned} \tau &= \{1,1,1\,|\,1,1,1\,|\,-1\}\,, \\ \mathbf{u} &= \{\infty,\infty,\infty\,|\,\infty,\infty,\infty\,|\,\infty\}\,. \end{aligned} \tag{5.9}$$

As remarked in Section 2, the fugacity of the null particle is given in terms of that of its constituents as $-y_1 y_2$. The central charge (2.12) works out correctly, as $c = 2$, with $\{\xi_a\} = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3} | \frac{1}{3}, \frac{1}{3}, \frac{1}{3} | -\frac{1}{8}\}$.

To compare the exclusion statistics based on the statistical interaction matrix (5.8) with the results of [9,14] we have to solve (2.6) with

$$\begin{aligned} \{\lambda_a\} &= \{\lambda_1, \lambda_2, \lambda_3\,|\,\lambda_{\bar{1}}, \lambda_{\bar{2}}, \lambda_{\bar{3}}\,|\,\mu\}\,, \\ \{z_a\} &= \{x, x, x | x^2, x^2, x^2 | -x^3\}\,. \end{aligned} \tag{5.10}$$

We find

$$\begin{aligned} \lambda \equiv \lambda_1 = \lambda_2 = \lambda_3 &= \frac{1+2x}{1+x} = 1 + \frac{x}{1+x}\,, \\ \bar{\lambda} \equiv \lambda_{\bar{1}} = \lambda_{\bar{2}} = \lambda_{\bar{3}} &= \frac{1+x+x^2}{1+x} = 1 + \frac{x^2}{1+x}\,, \\ \mu = \frac{(1+x)^3}{(1+2x)(1+x+x^2)} &= 1 - \frac{x^3}{(1+2x)(1+x+x^2)}\,. \end{aligned} \tag{5.11}$$

This indeed implies

$$\lambda_{\rm tot}(x) = \left(\prod \lambda_a\right) \left(\prod \lambda_{\bar{a}}\right)^2 \mu^3 = (1+x+x^2)^3\,, \tag{5.12}$$

in accordance with the results of [9,14].

Alternatively, we might also incorporate the effect of the null-field by slightly changing the statistics of the physical particles ϕ^1 and $\phi^{\bar{1}}$. In the characters this amounts to applying (4.6). This yields

$$\mathrm{ch}_i(y;q) \;=\; \sum_{\substack{M_1,M_2\geq 0\\ M_1+2M_2\equiv i \bmod 3}} y_1^{M_1} y_2^{M_2}\, q^{\frac{1}{2}\mathbf{M}\cdot\mathbf{B}\cdot\mathbf{M}} \sum_{\substack{m_a,m_{\bar a}\\ \sum m_a=M_1,\\ \sum m_{\bar a}=M_2}} \frac{q^{m_1 m_{\bar 1}}}{\prod_a (q)_{m_a}(q)_{m_{\bar a}}} \tag{5.13}$$

which is of the UCPF form with

$$\mathbf{G} \;=\; \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \vdots & \frac{4}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \vdots & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \vdots & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{4}{3} & \frac{1}{3} & \frac{1}{3} & \vdots & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \vdots & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \vdots & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} . \tag{5.14}$$

The corresponding solutions to (2.6) are now given by

$$\begin{aligned} \lambda'_1 \;&=\; \lambda\mu\,, \qquad \lambda'_2 \;=\; \lambda'_3 \;=\; \lambda\,, \\ \lambda'_{\bar 1} \;&=\; \bar\lambda\mu\,, \qquad \lambda'_{\bar 2} \;=\; \lambda'_{\bar 3} \;=\; \bar\lambda\,. \end{aligned} \tag{5.15}$$

where $\lambda, \bar\lambda$ and μ are as in (5.11). In other words, changing the statistics of the physical particles ϕ^1 and $\phi^{\bar 1}$ precisely corresponds to dressing these particles by the null-particle in the previous formulation. Again we find that $\lambda_{\rm tot} = (\prod \lambda'_a)(\prod \lambda'_{\bar a})$ is given by (5.12).

5.3. $\widehat{\mathfrak{sl}_4}$, level-1

The affine Lie algebra $\widehat{\mathfrak{sl}_4}$ at level $k=1$ has four integrable highest weight modules, corresponding to the singlet $\mathbf{1}$, the vector $L(\Lambda_1) = \mathbf{4}$, the rank-2 anti-symmetric tensor $L(\Lambda_2) = \mathbf{6}$ and the conjugate vector $L(\Lambda_3) = \bar{\mathbf{4}}$. The UCPF form of the characters, corresponding to quasi-particles (intertwiners) transforming in the $\mathbf{4}$, $\mathbf{6}$ and $\bar{\mathbf{4}}$, was obtained in [33]. To incorporate the null-states in the quasi-particle Fock space we need to deform both the inverse Cartan matrix of $\mathfrak{sl}_4$

$$\mathbf{B} \;=\; \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix} \tag{5.16}$$

as well as introduce one additional null-particle (corresponding to the composite of two **6** particles). The analogue of the $\mathfrak{sl}_3$ expression (5.14) is given by (3.11) with

$$
\mathbf{G} \;=\;
\left(\begin{array}{cccc|cccccc|cccc|c}
\frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & 2 \\
\frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 2 \\
\frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 1 \\
\frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 1 \\
\hline
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 1 & 2 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & 2 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 2 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & 2 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & 2 \\
\frac{3}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 2 \\
\frac{3}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 2 & 1 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 2 \\
\frac{3}{2} & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 2 & 1 & 1 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 2 \\
\hline
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & 1 \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & 1 \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & 2 \\
\frac{5}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & 2 \\
\hline
2 & 2 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 4
\end{array}\right)
\tag{5.17}
$$

and $A_a = 0$, $u_a = \infty$ and $\tau_a = 1$ for all a. The solution to (2.13) is given by

$$
\{\xi_a\} \;=\; \{\tfrac{1}{7}, \tfrac{2}{9}, \tfrac{1}{4}, \tfrac{1}{4} | \tfrac{1}{10}, \tfrac{1}{8}, \tfrac{1}{7}, \tfrac{1}{7}, \tfrac{1}{8}, \tfrac{1}{10} | \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{2}{9}, \tfrac{1}{7} | \tfrac{1}{81}\}\,, \tag{5.18}
$$

and leads to

$$
c \;=\; \left(\frac{6}{\pi^2}\right) \sum_a \; L(\xi_a) \;=\; 3\,, \tag{5.19}
$$

as it should. The solution of (2.6) with

$$
\begin{aligned}
\{\lambda_a\} \;&=\; \{\lambda_1, \lambda_2, \lambda_3, \lambda_4 | \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{23}, \lambda_{24}, \lambda_{34} | \lambda_{123}, \lambda_{124}, \lambda_{134}, \lambda_{234} | \mu\} \\
\{z_a\} \;&=\; \{\underbrace{x, \ldots, x}_{4} | \underbrace{x^2, \ldots, x^2}_{6} | \underbrace{x^3, \ldots, x^3}_{4} | x^4\}
\end{aligned}
\tag{5.20}
$$

is given in Appendix A, and leads to

$$
\begin{aligned}
\zeta_1 &\equiv \left(\prod \lambda_i\right)^{\frac{3}{4}} \left(\prod \lambda_{ij}\right)^{\frac{1}{2}} \left(\prod \lambda_{ijk}\right)^{\frac{1}{4}} \mu &&= 1+3x\,, \\
\zeta_2 &\equiv \left(\prod \lambda_i\right)^{\frac{1}{2}} \left(\prod \lambda_{ij}\right) \left(\prod \lambda_{ijk}\right)^{\frac{1}{2}} \mu^2 &&= 1+2x+3x^2\,, \\
\lambda_{\rm tot}^{\frac{1}{4}} = \zeta_3 &\equiv \left(\prod \lambda_i\right)^{\frac{1}{4}} \left(\prod \lambda_{ij}\right)^{\frac{1}{2}} \left(\prod \lambda_{ijk}\right)^{\frac{3}{4}} \mu &&= 1+x+x^2+x^3\,.
\end{aligned} \tag{5.21}
$$

Note that the expression for $\lambda_{\rm tot}$ is in complete agreement with the results of [9,14], confirming that the exclusion statistics of $(\widehat{\mathfrak{sl}_4})_{k=1}$ is indeed described by a statistical interaction matrix (5.17), while the $\zeta_i(x=1)$ are in agreement with (4.12).

5.4. $\widehat{\mathfrak{sl}_n}$, level $k \geq 1$

Obtaining results for for $\mathfrak{sl}_n$, $n \geq 5$, at level-1, using the algorithm described in [33] becomes extremely cumbersome. No complete results are known, but preliminary investigations suggest

$$
\zeta_i(x) \;=\; \sum_{k=0}^{i} \binom{n-i-1-k}{k} x^k\,, \qquad \lambda_{\rm tot} \;=\; \zeta_n^{n-1}\,, \tag{5.22}
$$

where ζ_i is defined in (4.11), such that indeed

$$
\zeta_i(1) \;=\; \sum_{k=0}^{i} \binom{n-i-1-k}{k} \;=\; \binom{n}{i} \;=\; \dim L(\Lambda_i)\,. \tag{5.23}
$$

As explained in Section 4, once the results for the level $k=1$ UCPF characters are known, one can immediately obtain the level $k>1$ characters by correcting for the level-k pseudo-particles as in (4.3).

5.5. $\widehat{\mathfrak{so}_5}$, level-1

The affine Lie algebra $\widehat{\mathfrak{so}_5}$, at level $k=1$, has three integrable highest weight representations, corresponding to the singlet $\mathbf{1}$, the vector $v=\mathbf{5}=L(\Lambda_1)$ and the spinor $s=\mathbf{4}=L(\Lambda_2)$ of $\mathfrak{so}_5$. The UCPF form of the $(\widehat{\mathfrak{so}_5})_1$ characters is obtained by combining the results of [55,14,33]. In [55] the character was given in terms of (restricted) $\mathfrak{so}_5$ Kostka polynomials and a recipe was given to compute the restricted Kostka polynomial. Explicit expressions for the restricted Kostka polynomial (corresponding to the pseudo-particle part of the character) were given in [14] while in [33] the UCPF form of the physical particles was found. The final result requires one pseudo-particle, physical particles transforming in the $\mathbf{5}$ and $\mathbf{4}$ of $\mathfrak{so}_5$ and one null-particle (corresponding to the composite of two $\mathbf{5}$ particles). The characters are given by (3.11) with

$$\mathbf{G} \;=\; \left(\begin{array}{c|ccccc|cccc|c}
1 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\
\hline
0 & 1 & 1 & 1 & 1 & 2 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & 2 \\
0 & 1 & 1 & 1 & 2 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & 2 \\
0 & 1 & 1 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & 2 \\
0 & 1 & 2 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 2 \\
0 & 2 & 1 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 2 \\
\hline
-\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & 1 \\
-\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & 1 \\
-\frac{1}{2} & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & 2 \\
-\frac{1}{2} & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & 2 \\
\hline
0 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 4
\end{array}\right), \tag{5.24}$$

which is a deformation of the matrix

$$\mathbf{B} \;=\; \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix}, \tag{5.25}$$

entering (4.2). Furthermore, $A_a = 0$ and $\tau_a = 1$ for all $a \in \mathcal{S}$, and

$$\mathbf{u} \;=\; \begin{cases} \{0|\underbrace{\infty,\ldots,\infty}_{5}|\underbrace{\infty,\ldots,\infty}_{4}|\infty\} & \text{for } 1 \text{ and } v\,, \\ \{1|\underbrace{\infty,\ldots,\infty}_{5}|\underbrace{\infty,\ldots,\infty}_{4}|\infty\} & \text{for } s\,, \end{cases} \tag{5.26}$$

while there are also some restrictions on the summation over the m_a's (see [14]). Note that this case corresponds to order $k = 2$ non-abelian exclusion statistics in the sense of [15] as far as the coupling of the pseudo-particle to the physical spinor-particles are concerned. The physical vector-particles have a unique fusion rule and therefore do not couple to the pseudo-particle.

The equations (2.13) have the solution

$$\begin{aligned}\{\xi_a\} &= \{\tfrac{11}{16}\,|\,\tfrac{1}{12},\tfrac{1}{8},\tfrac{5}{33},\tfrac{7}{40},\tfrac{11}{60}\,|\,\tfrac{4}{11},\tfrac{4}{11},\tfrac{16}{49},\tfrac{8}{33}\,|\,\tfrac{1}{49}\}\\ \{\eta_a\} &= \{\tfrac{1}{2}\,|\,0,0,0,0,0\,|\,0,0,0,0\,|\,0\}\end{aligned} \tag{5.27}$$

leading to

$$c \;=\; \left(\frac{6}{\pi^2}\right)\sum_a \big(L(\xi_a)-L(\eta_a)\big) \;=\; 3-\tfrac{1}{2} \;=\; \tfrac{5}{2}\,, \tag{5.28}$$

as it should. Moreover, we have verified that the total 1-particle partition function $\lambda_{\rm tot} = \left(\prod_i \lambda_i\right)^2 \left(\prod_\alpha \lambda_\alpha\right)\mu^4$, resulting from the solution of (2.6) with

$$\begin{aligned}\{\lambda_a\} &= \{\lambda|\lambda_1,\lambda_2,\lambda_0,\lambda_{\bar 2},\lambda_{\bar 1}|\lambda_{++},\lambda_{+-},\lambda_{-+},\lambda_{--}|\mu\}\,,\\ \{z_a\} &= \{1|\underbrace{x^2,\ldots,x^2}_{5}|\underbrace{x,\ldots,x}_{4}|x^4\}\,,\end{aligned} \tag{5.29}$$

satisfies, up to at least $\mathcal{O}(x^{11})$, the equation

$$\lambda_{\rm tot}^{\frac{3}{2}} - (2+3x^2)\lambda_{\rm tot} + (3x^2-1)(x^2-1)\lambda_{\rm tot}^{\frac{1}{2}} - x^2(x^2-1)^2 \;=\; 0\,, \tag{5.30}$$

derived in [61,14] from the recursion approach (see Appendix B for the explicit solution up to $\mathcal{O}(x^{11})$).

In addition, from (5.27), we obtain that the expressions for

$$\begin{aligned}\lambda_{\rm tot}^{\frac{1}{2}} \;=\; \zeta_1 &= \Big(\prod\lambda_i\Big)\Big(\prod\lambda_\alpha\Big)^{\frac{1}{2}}\mu^2\,,\\ \zeta_2 &= \lambda^{\frac{1}{2}}\Big(\prod\lambda_i\Big)^{\frac{1}{2}}\Big(\prod\lambda_\alpha\Big)^{\frac{1}{4}}\mu\,,\end{aligned} \tag{5.31}$$

at $x=1$ are given by, respectively, $\zeta_1 = 5$ and $\zeta_2 = 4$. Again this is in complete agreement with (4.12). The results in this section might prove to be useful with regards to certain quasi-particle excitations ('non-abelian electrons') in $SO(5)$ superspin regimes for strongly correlated electrons on a two-leg ladder [61].

The UCPF and corresponding exclusion statistics for higher level $\widehat{\mathfrak{so}}_5$ modules can be worked out using the results of [55].

5.6. $\widehat{\mathfrak{sl}}_n$, level-1, revisited

For $(\widehat{\mathfrak{sl}}_n)_{k=1}$ it is also possible to give a description purely in terms of quasi-particles ('spinons') ϕ^a transforming in the n-dimensional vector representation $\mathbf{n}$. In this case the null-field will be of the form $:\phi^1(z)\dots\phi^n(z):$. The corresponding character formula was found in [28]

$$\mathrm{ch}_i(y;q) \;=\; \sum_{\substack{m_a\geq 0\\ \sum m_a\equiv i \bmod n}} y^{\sum m_a} \sum_{m\geq 0} (-1)^m \frac{q^{\frac{1}{2}m(m-1)}}{(q)_m} \frac{q^{\frac{1}{2}(\sum m_a^2-\frac{1}{n}(\sum m_a)^2)}}{\prod_a (q)_{m_a-m}}\,. \tag{5.32}$$

It can be brought in the UCPF form by shifting the summation variables $m_a \to m_a + m$ (for $y \neq 1$). Then,

$$\mathbf{G} \;=\; \left(\begin{array}{cccc|c} & & & & \\ & \delta_{ab}-\frac{1}{n} & & & 0 \\ & & & & \\ \hline \dots & \dots & \dots & \dots & \dots \\ & 0 & & & 1 \end{array}\right)\,. \tag{5.33}$$

The corresponding equations (2.6) with $\{\lambda_a\} = \{\lambda_1,\dots,\lambda_n|\mu\}$, and $\{z_a\} = \{x,\dots,x|-x^n\}$ have the solution

$$\lambda_1 \;=\; \dots \;=\; \lambda_n \;=\; \frac{1}{1-x}\,, \qquad \mu \;=\; 1-x^n\,, \tag{5.34}$$

so that indeed

$$\lambda_{\mathrm{tot}} \;=\; (\lambda_1\dots\lambda_n)\mu^n \;=\; (1+x+\dots+x^{n-1})^n\,. \tag{5.35}$$

It is also possible to write (5.32) in terms of a non-alternating sum by repeated application of (4.6) and (4.5). Besides the n spinons this requires $n-2$ additional null-particles for $\mathfrak{sl}_n$. Here give the result for $\widehat{\mathfrak{sl}_3}$ (see [33] for the origin of this formula and the generalization to $\widehat{\mathfrak{sl}_n}$)

$$\mathrm{ch}_i(y;q) \;=\; \sum_{m_a,m\geq 0} y^{\sum m_a+2m} \frac{q^{\frac{1}{2}\mathbf{m}\cdot\mathbf{G}\cdot\mathbf{m}}}{(q)_{m_1}(q)_{m_2}(q)_{m_3}(q)_m}\,, \tag{5.36}$$

with

$$\mathbf{G} \;=\; \left(\begin{array}{ccc|c} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \hline \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{array}\right)\,, \tag{5.37}$$

leading to the solution

$$\lambda_1 \;=\; \lambda_2 \;=\; 1+x\,, \qquad \lambda_3 \;=\; 1+x+x^2\,, \qquad \mu \;=\; \frac{1+x+x^2}{1+x}\,, \tag{5.38}$$

and again confirming

$$\lambda_{\rm tot} \;=\; (\lambda_1\lambda_2\lambda_3)\mu^2 \;=\; (1+x+x^2)^3\,. \tag{5.39}$$

In contrast to the UCPF formulas in section 4.2 and 4.3, it does not appear that the formula (5.32) has a straightforward generalization to levels $k>1$.

6. Conclusions

In this work we have to tried to reconcile Haldane's notion of exclusion statistics [1] with the Stony Brook group's proposal of a Universal Chiral Partition Function form for all (chiral) characters of two-dimensional conformal field theories [29]. We have seen that besides the pseudo-particles of [15], in general, this requires yet another kind of particles, so-called null-particles. In support of the conjectured relation between Haldane statistics and the UCPF, we have shown that an ideal gas of physical, pseudo- and null-particles, with linear dispersion relations, in the thermodynamic limit exhibits the same effective central charge as the UCPF. It would of course be most desirable to extend this comparison to the different sectors of the UCPF and gain an understanding of the restrictions that enter the sum.

The UCPF was put forward to structuralize the form of the characters of CFT. By indicating how the characters of affine Lie algebras may be written in the UCPF form by introducing null-particles we have obtained further support for the alleged 'universality' of the UCPF.

To demonstrate this method we have discussed various examples of UCPFs for WZW-models and the associated exclusion statistics and found agreement with previous results, computed by the recursion method [9,14], in all cases.

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Appendix A. Solution for $\mathfrak{sl}_4$

The explicit solution of (2.6) with (5.20) is given by

$$
\begin{aligned}
\lambda_1 &= \frac{1+3x+3x^2}{1+2x+3x^2} = 1+\frac{x}{1+2x+3x^2}\,, \\
\lambda_2 &= \frac{(1+2x)^2}{1+3x+3x^2} = 1+\frac{x(1+x)}{1+3x+3x^2}\,, \\
\lambda_3 &= \frac{1+3x}{1+2x} = 1+\frac{x}{1+2x}\,, \\
\lambda_4 &= \frac{1+3x}{1+2x} = 1+\frac{x}{1+2x}\,, \\
\lambda_{12} &= \frac{(1+2x+2x^2)(1+3x+2x^2+2x^3)}{(1+2x)^2(1+x+x^2+x^3)} = \frac{x^2(1+3x)}{(1+2x)^2(1+x+x^2+x^3)}\,, \\
\lambda_{13} &= \frac{(1+x)^2(1+2x+3x^2)}{(1+x+x^2)(1+3x+3x^2)} = 1+\frac{x^2(1+2x)}{(1+x+x^2)(1+3x+3x^2)}\,, \\
\lambda_{14} &= \frac{1+2x+3x^2+x^3}{(1+x)(1+x+x^2)} = 1+\frac{x^2}{(1+x)(1+x+x^2)}\,, \\
\lambda_{23} &= \frac{1+3x+3x^2}{(1+x)(1+2x)} = 1+\frac{x^2}{(1+x)(1+2x)}\,, \\
\lambda_{24} &= \frac{(1+x)^2(1+2x+3x^2)}{(1+2x)(1+2x+3x^2+x^3)} = 1+\frac{x^2(1+x+x^2)}{(1+2x)(1+2x+3x^2+x^3)}\,, \\
\lambda_{34} &= \frac{(1+2x+2x^2)(1+3x+2x^2+2x^3)}{(1+3x)(1+x+x^2)^2} = 1+\frac{x^2(1+x+x^2+x^3)}{(1+3x)(1+x+x^2)^2}\,, \\
\lambda_{123} &= \frac{1+x+x^2+x^3}{1+x+x^2} = 1+\frac{x^3}{1+x+x^2}\,, \\
\lambda_{124} &= \frac{1+x+x^2+x^3}{1+x+x^2} = 1+\frac{x^3}{1+x+x^2}\,, \\
\lambda_{134} &= \frac{(1+x+x^2)^2}{1+2x+3x^2+x^3} = 1+\frac{x^3(1+x)}{1+2x+3x^2+x^3}\,, \\
\lambda_{234} &= \frac{1+2x+3x^2+x^3}{1+2x+3x^2} = 1+\frac{x^3}{1+2x+3x^2}\,, \\
\mu &= \frac{(1+2x)^2(1+x+x^2)^2}{(1+x)(1+2x+2x^2)(1+3x+2x^2+2x^3)} \\
&= 1+\frac{x^4}{(1+x)(1+2x+2x^2)(1+3x+2x^2+2x^3)}\,,
\end{aligned}
\tag{A.1}
$$

Appendix B. Approximate solution for so_5

Up to $\mathcal{O}(x^{11})$ the solution of (2.6) with (5.29) is given by

$$
\begin{aligned}
\lambda\ &= 2+4y-8y^2+18y^3-48y^4+\tfrac{303}{2}y^5-544y^6+\tfrac{8505}{4}y^7-8768y^8+\tfrac{1198427}{32}y^9\\
&\quad -163968y^{10}+\mathcal{O}(y^{11})\,,\\
\lambda_1\ &= 1+2y^2-16y^3+108y^4-696y^5+4408y^6-27702y^7+173424y^8-1083451y^9\\
&\quad +6760800y^{10}+\mathcal{O}(y^{11})\,,\\
\lambda_2\ &= 1+2y^2-12y^3+64y^4-334y^5+1736y^6-\tfrac{18053}{2}y^7+47008y^8-\tfrac{980991}{4}y^9\\
&\quad +1281696y^{10}+\mathcal{O}(y^{11})\,,\\
\lambda_0\ &= 1+2y^2-12y^3+68y^4-374y^5+2024y^6-\tfrac{21709}{2}y^7+57904y^8-\tfrac{1231627}{4}y^9\\
&\quad +1634080y^{10}+\mathcal{O}(y^{11})\,,\\
\lambda_{\bar 2}\ &= 1+2y^2-8y^3+28y^4-92y^5+272y^6-619y^7+160y^8+\tfrac{21685}{2}y^9-100320y^{10}\\
&\quad +\mathcal{O}(y^{11})\,,\\
\lambda_{\bar 1}\ &= 1+2y^2-8y^3+28y^4-84y^5+152y^6+569y^7-9616y^8+\tfrac{166483}{2}y^9-601248y^{10}\\
&\quad +\mathcal{O}(y^{11})\,,\\
\lambda_{++}\ &= 1+2y-6y^2+25y^3-116y^4+\tfrac{2255}{4}y^5-2808y^6+\tfrac{113577}{8}y^7-72496y^8\\
&\quad +\tfrac{23858843}{64}y^9-1926944y^{10}+\mathcal{O}(y^{11})\,,\\
\lambda_{+-}\ &= 1+2y-6y^2+25y^3-116y^4+\tfrac{2255}{4}y^5-2808y^6+\tfrac{113577}{8}y^7-72496y^8\\
&\quad +\tfrac{23858843}{64}y^9-1926944y^{10}+\mathcal{O}(y^{11})\,,\\
\lambda_{-+}\ &= 1+2y-6y^2+21y^3-80y^4+\tfrac{1255}{4}y^5-1240y^6+\tfrac{39117}{8}y^7-19104y^8+\tfrac{4700235}{64}y^9\\
&\quad -275296y^{10}+\mathcal{O}(y^{11})\,,\\
\lambda_{--}\ &= 1+2y-6y^2+13y^3-24y^4+\tfrac{151}{4}y^5-40y^6-\tfrac{363}{8}y^7+576y^8-\tfrac{203605}{64}y^9\\
&\quad +15264y^{10}+\mathcal{O}(y^{11})\,,\\
\mu\ &= 1+4y^4-48y^5+400y^6-2872y^7+19072y^8-120906y^9+743936y^{10}+\mathcal{O}(y^{11})\,,
\end{aligned}
\tag{B.1}
$$

where $y=x/\sqrt{2}$. This leads to

$$
\begin{aligned}
\lambda_{\rm tot}^{\frac12}\ &= \Big(\prod_i \lambda_i\Big)\Big(\prod_\alpha \lambda_\alpha\Big)^{\frac12}\mu\ =\ 1+4y+2y^2+2y^3-8y^4+\tfrac{63}{2}y^5-128y^6\\
&\quad +\tfrac{2145}{4}y^7-2304y^8+\tfrac{323323}{32}y^9-45056y^{10}+\mathcal{O}(y^{11})\,.
\end{aligned}
\tag{B.2}
$$

References

[1] D. Haldane, Phys. Rev. Lett. **67** (1991) 937.

[2] Y.-S. Wu, Phys. Rev. Lett. **73** (1994) 922;
C. Nayak and F. Wilczek, Phys. Rev. Lett. **73** (1994) 2740;
S.B. Isakov, Mod. Phys. Lett. **B8** (1994) 319;
A. Dasnières de Veigy and S. Ouvry, Phys. Rev. Lett. **72** (1994) 600; *ibid.*, Mod. Phys. Lett. **B9** (1995) 271, [`cond-mat/9411036`];
A.K. Rajagopal, Phys. Rev. Lett. **74** (1995) 1048;
Y.-S. Wu and Y. Yu, Phys. Rev. Lett. **75** (1995) 890;
S.B. Isakov, D.P. Arovas, J. Myrheim, A.P. Polychronakos, Phys. Lett. **A212** (1996) 299, [`cond-mat/9601108`].

[3] T. Fukui and N. Kawakami, Phys. Rev. **B51** (1995) 5239, [`cond-mat/9408015`].

[4] F. Haldane, Phys. Rev. Lett. **66** (1991) 1529;
F. Haldane, Z. Ha, J. Talstra, D. Bernard and V. Pasquier, Phys. Rev. Lett. **69** (1992) 2021.

[5] D. Bernard, V. Pasquier and D. Serban, Nucl. Phys. **B428** (1994) 612, [`hep-th/9404050`].

[6] P. Bouwknegt, A. Ludwig and K. Schoutens, Phys. Lett. **338B** (1994) 448, [`hep-th/9406020`].

[7] P. Bouwknegt, A. Ludwig and K. Schoutens, Phys. Lett. **359B** (1995) 304, [`hep-th/9412108`].

[8] L. Takhtajan, Phys. Lett. **A87** (1982) 479;
L. Faddeev and N. Reshetikhin, Ann. Phys. **167** (1986) 227;
I. Affleck and F. Haldane, Phys. Rev. **B36** (1987) 5291;
N. Reshetikhin, J. Phys. **A24** (1991) 3299;
P. Fendley, Phys. Rev. Lett. **71** (1993) 2845, [`cond-mat/9304031`].

[9] K. Schoutens, Phys. Rev. Lett. **79** (1997) 2608, [`cond-mat/9706166`].

[10] K. Schoutens, Phys. Rev. Lett. **81** (1998) 15704, [`cond-mat/9803169`].

[11] H. Frahm and M. Stahlsmeier, Phys. Lett. **A250** (1998) 293, [`cond-mat/9803381`].

[12] J. Gaite, Nucl. Phys. **B525** (1998) 627, [`hep-th/9804025`].

[13] R. van Elburg and K. Schoutens, Phys. Rev. **B58** (1998) 15704, [`cond-mat/9801272`].

[14] P. Bouwknegt and K. Schoutens, *Exclusion statistics in conformal field theory – generalized fermions and spinons for level-*1 *WZW theories*, Nucl. Phys. **B**, to appear, [`hep-th/9810113`].

[15] S. Guruswamy and K. Schoutens, *Non-abelian exclusion statistics*, [`cond-mat/9903045`].

[16] R. Kedem and B.M. McCoy, J. Stat. Phys. **71** (1994) 865, [`hep-th/9210146`];
G. Albertini, S. Dasmahapatra and B. McCoy, Int. J. Mod. Phys. **A7** Suppl. **1A** (1992) 1, ; *ibid.*, Phys. Lett. **A170** (1992) 397;

S. Dasmahapatra, R. Kedem, B.M. McCoy and E. Melzer, J. Stat. Phys. **74** (1994) 239, [hep-th/9304150].

[17] D. O'Brien, P. Pearce and S.O. Warnaar, Nucl. Phys. **B501** (1997) 773.

[18] V. Bazhanov, S. Lukyanov and A. Zamolodchikov, Comm. Math. Phys. **177** (1996) 381, [hep-th/9412229]; *ibid.*, Comm. Math. Phys. **190** (1997) 247, [hep-th/9604044].

[19] A.B. Zamolodchikov, Advanced Studies in Pure Math. **19** (1989) 641;
D. Bernard and A. Le Clair, Nucl. Phys. **B340** (1990) 721;
C. Ahn, D. Bernard and A. Le Clair, Nucl. Phys. **B346** (1990) 409;
F.A. Smirnov, Nucl. Phys. **B337** (1990) 156; *ibid.*, Int. J. Mod. Phys. **A6** (1991) 1407;
G. Mussardo, Phys. Rep. **218** (1992) 215.

[20] A. Berkovich, B. McCoy and P. Pearce, Nucl. Phys. **B519** (1998) 597;
A. Berkovich and B. McCoy, *The perturbation $\varphi_{2,1}$ of the $M(p,p+1)$ models of conformal field theory and related polynomial character identities*, [math.QA/9809066];
S.O. Warnaar, *q-Trinomial identities*, [math.QA/9810018].

[21] J. Lepowski and M. Primc, *Structure of the standard modules for the affine Lie algebra $A_1^{(1)}$*, Contemp. Math. **46**, (Amer. Math. Soc., Providence, 1985).

[22] R. Kedem, T.R. Klassen, B.M. McCoy and E. Melzer, Phys. Lett. **304B** (1993) 263, [hep-th/9211102]; *ibid.*, Phys. Lett. **307B** (1993) 68, [hep-th/9301046];
S. Dasmahapatra, R. Kedem, T.R. Klassen, B.M. McCoy and E. Melzer, Int. J. Mod. Phys. **B7** (1993) 3617, [hep-th/9303013].

[23] E. Melzer, Int. J. Mod. Phys. **A9** (1994) 1115, [hep-th/9305114];
A. Berkovich, Nucl. Phys. **B431** (1994) 315, [hep-th/9403073];
A. Berkovich and B.M. McCoy, Lett. Math. Phys. **37** (1996) 49, [hep-th/9412030];
S.O. Warnaar, J. Stat. Phys. **82** (1996) 657, [hep-th/9501134]; *ibid.*, J. Stat. Phys. **84** (1996) 49, [hep-th/9508079];
A. Berkovich, B.M. McCoy and A. Schilling, Comm. Math. Phys. **191** (1998) 325, [q-alg/9607020].

[24] O. Foda and Y.-H. Quano, Int. J. Mod. Phys. **A12** (1997) 1651, [hep-th/9408086];
O. Foda, K.S.M. Lee and T.A. Welsh, Int. J. Mod. Phys. **A13** (1998) 4967, [q-alg/9710025];
O. Foda and T. Welsh, *Melzer's identities revisited*, [math.QA/9811156].

[25] A. Schilling, Nucl. Phys. **B459** (1996) 393, [hep-th/9508050];
A. Schilling and S.O. Warnaar, The Ramanujan Journal **2** (1998) 459, [q-alg/9701007].

[26] B.L. Feigin and A.V. Stoyanovsky, *Quasi-particle models for the representations of Lie algebras and geometry of flag manifolds*, [hep-th/9308079]; *ibid.*, Funct. Anal. Appl. **28** (1994) 55.

[27] G. Georgiev, J. Pure Appl. Algebra **112** (1996) 247, [hep-th/9412054]; *ibid.*, *Combinatorial constructions of modules for infinite-dimensional Lie algebras, II. Parafermionic*

space, [q-alg/9504024].

[28] P. Bouwknegt and K. Schoutens, Nucl. Phys. **B482** (1996) 345, [hep-th/9607064].

[29] A. Berkovich and B. McCoy, *The universal chiral partition function for exclusion statistics*, [hep-th/9808013].

[30] R. Kedem, B. McCoy and E. Melzer, in “Recent progress in Statistical Mechanics and Quantum Field Theory”, eds. P. Bouwknegt et al., (World Scientific, Singapore, 1995), [hep-th/9304056].

[31] D. Bernard and Y.-S. Wu, *A note on statistical interactions and the thermodynamic Bethe Ansatz*, in “New developments in integrable systems and long-range interaction models”, Nankai Lecture Notes on Mathematical Physics, (World Scientific, Singapore, 1994), [cond-mat/9404025].

[32] K. Hikami, Phys. Lett. **A205** (1995) 364.

[33] P. Bouwknegt, *q-Identities and affinized projective varieties, I. Quadratic monomial ideals*, [math-ph/9902010];
P. Bouwknegt and N. Halmagyi, *q-Identities and affinized projective varieties, II. Flag varieties*, [math-ph/9903033].

[34] A. Bytsko and A. Fring, Nucl. Phys. **B532** (1998) 588, [hep-th/9803005].

[35] C.N. Yang and C.O. Yang, J. Math. Phys. **10** (1969) 1115;
B. Sutherland, Phys. Rev. Lett. **20** (1967) 98.

[36] Al.B. Zamolodchikov, Nucl. Phys. **B358** (1991) 497; *ibid.*, Nucl. Phys. **B358** (1991) 524; *ibid.*, Nucl. Phys. **B366** (1991) 122;
P. Fendley, H. Saleur and Al.B. Zamolodchikov, Int. J. Mod. Phys. **A8** (1993) 5751.

[37] F. Ravanini, Phys. Lett. **282B** (1992) 73, [hep-th/9202020];
F.R. Ravanini, R. Tateo and A. Valleriani, Int. J. Mod. Phys. **A8** (1993) 1707, [hep-th/9207040].

[38] V.V. Bazhanov and N.Y. Reshetikhin, Int. J. Mod. Phys. **A4** (1989) 115; *ibid.*, J. Phys. **A23** (1990) 1477; *ibid.*, Prog. Theor. Phys. Suppl. **102** (1990) 301.

[39] Al.B. Zamolodchikov, Nucl. Phys. **B342** (1990) 695.

[40] T. Klassen and E. Melzer, Nucl. Phys. **B338** (1990) 485; *ibid.*, Nucl. Phys. **B350** (1991) 635; *ibid.*, Nucl. Phys. **B370** (1992) 511.

[41] A. Nakayashiki and Y. Yamada, Comm. Math. Phys. **178** (1996) 179, [hep-th/9504052].

[42] A. Bytsko and A. Fring, *Factorized combinations of Virasoro characters*, [hep-th/9809001].

[43] W. Nahm, A. Recknagel and M. Terhoeven, Mod. Phys. Lett. **A8** (1993) 1835, [hep-th/9211034].

[44] S. Dasmahapatra, R. Kedem, T. Klassen, B. McCoy and E. Melzer, Int. J. Mod. Phys. **B7** (1993) 3617, [hep-th/9303013].

[45] B. Richmond and G. Szekeres, J. Austral. Math. Soc. (Series A) **31** (1981) 362.

[46] A. Kirillov, Prog. Theor. Phys. Suppl. **118** (1995) 61, [hep-th/9408113].

[47] A. Kuniba and T. Nakanishi, Mod. Phys. Lett. **A7** (1992) 3487, [hep-th/9206034]; A. Kuniba, T. Nakanishi and J. Suzuki, Mod. Phys. Lett. **A8** (1993) 1649, [hep-th/9301018].

[48] J. Suzuki, J. Phys. **A31** (1998) 6887, [cond-mat/9805242].

[49] J. Suzuki, *Spinons in magnetic chains of arbitrary spins at finite temperatures*, [cond-mat/9807076].

[50] P. Bouwknegt and K. Schoutens, *Spinon decomposition and Yangian structure of $\widehat{\mathfrak{sl}_n}$ modules*, in "Geometric Analysis and Lie Theory in Mathematics and Physics, Australian Mathematical Society Lecture Series **11**, *eds.* A.L. Carey and M.K. Murray, (Cambridge University Press, Cambridge, 1997), [q-alg/9703021].

[51] A. Berkovich, B. McCoy and A. Schilling, Comm. Math. Phys. **191** (1998) 325, [q-alg/9607020].

[52] G. Hatayama, A. Kirillov, A. Kuniba, M. Okado, T. Takagi and Y. Yamada, Nucl. Phys. **B536** (1999) 575, [math.QA/9802085];
G. Hatayama, A. Kuniba, M. Okado, T. Takagi and Y. Yamada, *Remarks on fermionic formula*, [math.QA/9812022].

[53] P. Goddard, W. Nahm, D. Olive and A. Schwimmer, Comm. Math. Phys. **107** (1986) 179;
D. Bernard and J. Thierry-Mieg, Comm. Math. Phys. **111** (1987) 181.

[54] A. Nakayashiki and Y. Yamada, *On spinon character formulas*, in "Frontiers in Quantum Field Theories", *eds.* H. Itoyama et al., (World Scientific, Singapore, 1996).

[55] Y. Yamada, *On q-Clebsch Gordan rules and the spinon character formulas for affine $C_2^{(1)}$ algebra*, [q-alg/9702019].

[56] G. Gasper and M. Rahman, *Basic hypergeometric series*, Encycl. of Math. and its Appl. **35** (Cambridge University Press, Cambridge, 1990).

[57] A. Kirillov and N. Reshetikhin, J. Sov. Math. **41** (1988) 925.

[58] A. Nakayashiki and Y. Yamada, Selecta Math. (N.S.) **3** (1997) 547, [q-alg/9512027].

[59] T. Fukui, N. Kawakami and S.-K. Yang, *Spin-S generalization of fractional exclusion statistics*, [cond-mat/9507143].

[60] T. Arakawa, T. Nakanishi, K. Oshima and A. Tsuchiya, Comm. Math. Phys. **181** (1996) 157, [q-alg/9507025].

[61] P. Bouwknegt and K. Schoutens, *Non-abelian electrons: $SO(5)$ superspin regimes for correlated electrons on a two-leg ladder*, Phys. Rev. Lett. , to appear, [cond-mat/9805232].