Quantum Symmetries and Lattice Regularisations

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June 3, 2009

Background

- What are we studying?
- What does this mean?
- Advantages

Results

- Past Work (Quantum Symmetries and R-Matrices)
- Present Work (L-matrices)
- Future Work (To be / Should be done)

Conclusions

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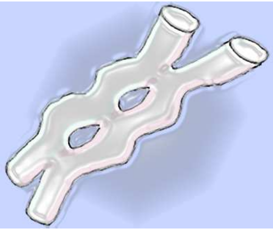
- Want to systematically solve quantised integrable sigma models like those relevant to AdS/CFT.



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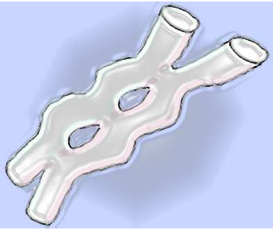

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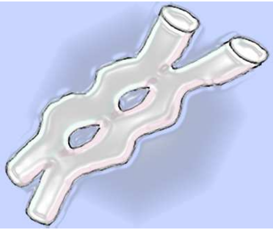

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- **Our approach**:
  Determine quantum symmetry of integrable sigma model, then construct a lattice regularisation.

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- Still infinitely many if perturbations generate (certain) quantum symmetry algebras.
- Derive and exploit these symmetries to compute the sigma model spectrum!

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- Once quantum symmetries are identified, compute:
  1. R-matrix (integrability of quantum algebra).
  2. L-matrix (integrability of quantum sigma model).
  3. T-matrix (generator of conserved quantities).
  4. Q-operators (generators of auxiliary conserved quantities).
  5. Spectrum (oscillation frequencies of T-matrix).

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- Programme called Quantum Inverse Scattering Method.
- But, suffers from usual infinities common to quantum field theory...

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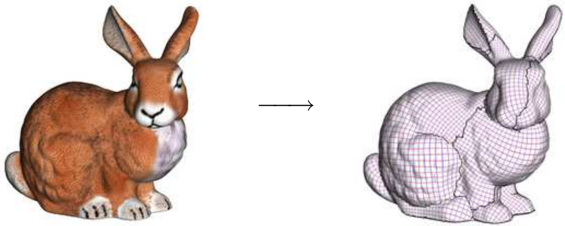

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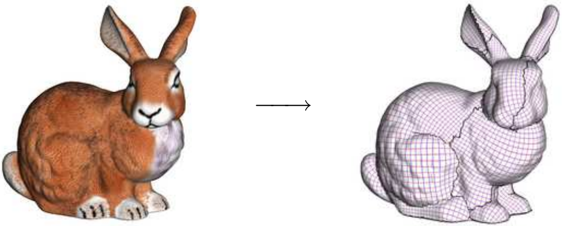

- Controls infinities — know how to deal with them in the continuum limit.

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3. Under full control — we develop formalism for explicit computation.
4. General — applies to other integrable sigma models.
5. Mathematically exciting — combines modern algebra with classical analysis and suggests new directions for mathematical research.

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- Straight-forward to compute defining equations of quantum symmetry algebra, the *q*-Serre relations, up to order 7.
- Reproduces (known) symmetry of sine-Gordon model,

$$\mathcal{U}_q\left(\widehat{\mathfrak{sl}}(2)\right),$$

among others.

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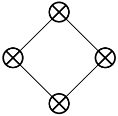

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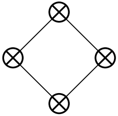

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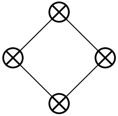

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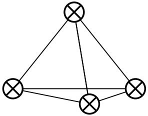
“  ”

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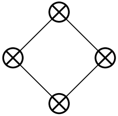

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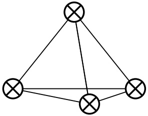
“  ”

- Have determined full (Hopf) algebraic structure.

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- Currently working on R-matrices for SS model symmetry algebra.

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5. Investigate and characterise further examples of quantum symmetry algebras.

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- Approach allows creation and study of many new families of integrable models and their lattice regularisations.