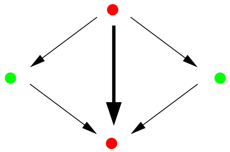
Whither Indecomposability?



David Ridout

CRM, Montréal

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Introduction

- Symmetry is rather important in physics!
- Weyl and Wigner introduced group theory and representation theory into quantum physics. According to wikipedia:

  > "*...the different quantum states of an elementary particle give rise to an irreducible representation of the Poincaré group.*"

- *ie*. quantum particle $\overset{\text{model}}{:=}$ irreducible representation for some symmetry group or algebra.
- In this way, the fundamental indivisibility of a particle is reflected in the mathematics.

Mathematical (*Un*)-Niceties

- An irreducible representation is one whose only invariant subspaces are the zero vector and the whole thing.
- Completely reducible representations decompose as a direct sum of irreducibles: $V = \bigoplus_i V_i$.
- But, there is a dark side to representation theory: Indecomposable representations.
- These can have non-trivial invariant subspaces, $0 \subset W \subset V$, but $V \neq W \oplus W'$ for any W'!
- Physically, this suggests an indivisible particle (represented by V) which nevertheless has internal structure (represented by W), much like quarks in a hadron.

Ex. 1: Lie Superalgebras

- Generalisations of simple Lie algebras.
- *eg*. $\mathfrak{gl}(1|1)$ generated by E, N (bosons), ψ^+, ψ^- (fermions):

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \psi^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \psi^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

- *eg*. the adjoint representation is indecomposable:

$$\begin{tikzcd} & N \arrow[dl] \arrow[dr] & \\ \psi^+ \arrow[dr] & & \psi^- \arrow[dl] \\ & E & \end{tikzcd}$$

- Indecomposability common to most superalgebras.
- Superalgebra symmetries are used in AdS/CFT, *eg*.

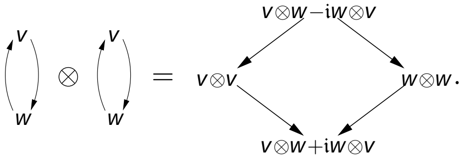
$$\mathrm{AdS}_5 \times S^5 \sim \frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(1,4) \times \mathrm{SO}(5)}.$$

Ex. 2: Quantum Groups

- q-deformations of Lie (super)algebras with tensor product.
- *eg*. $\mathscr{U}_q\big(\mathfrak{sl}(2)\big)$ generated by E, K, F subject to

$$KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

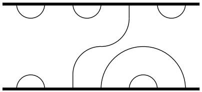
- $q^N \neq 1$: Completely reducible.
  $q^N = 1$: Indecomposables arise, *eg*. for $q = \mathrm{i}$,



- Quantum groups govern quantum integrable systems. TT, TQ and QQ relations reflect indecomposable structure.

Ex. 3: Temperley-Lieb Algebras

- Workhorse of statistical model physics.
- $\mathrm{TL}_n(q)$ spanned by diagrams like

 .

- Multiplication is vertical concatenation.
  Interior loop $\longrightarrow$ multiply by $q + q^{-1}$.
- $q^N = 1$: Indecomposables arise, *eg*. for $q = \mathrm{i}$ and $n = 6$,

$$\mathrm{TL}_6(\mathrm{i}) = 5\left(\begin{tikzcd} 5 \arrow[dr] & \\ & 4 \arrow[dl] \\ 5 & \end{tikzcd}\right) \oplus 4\left(\begin{tikzcd} & 4 \arrow[dl] \arrow[dr] & \\ 5 \arrow[dr] & & 1 \arrow[dl] \\ & 4 & \end{tikzcd}\right) \oplus \left(\begin{tikzcd} & 1 \arrow[dl] \\ 4 \arrow[dr] & \\ & 1 \end{tikzcd}\right).$$

- Underlies many lattice models (Q-state Potts, RSOS).
  eg. $q = e^{\mathrm{i}\pi/3}$ (percolation), $q = e^{\mathrm{i}\pi/4}$ (Ising model).

Staggered Representations

- Diamond-shaped indecomposables crop up a lot!
- In each case, there is an element C which is

  self-adjoint, central and non-diagonalisable.

$$
\begin{tikzcd}
& \bullet \arrow[dl] \arrow[dr] \arrow[dd, "C"'] & \\
\bullet \arrow[dr] & & \bullet \arrow[dl] \\
& \bullet &
\end{tikzcd}
$$

- For $\mathfrak{gl}(1|1)$, $C = \frac{1}{2}NE + \psi^- \psi^+$ is the Casimir.

  For $\mathscr{U}_q(\mathfrak{sl}(2))$, $C = EF + \frac{Kq^{-1}+K^{-1}q}{(q-q^{-1})^2}$ is also the Casimir.

  For $\mathrm{TL}_n(q)$, C is the "braid transfer matrix".
- Some folks call such representations staggered.

Logarithmic Conformal Field Theory

- "Staggered" comes from logarithmic CFT (LCFT).
- Critical lattice models have conformal limits (*cf*. SLE). String theories are conformal on the worldsheet.
- When the conformal (Virasoro) algebra reps are indecomposable (staggered), the CFT is logarithmic.
- Staggered structure $\longrightarrow$ correlators have log singularities.
- For lattice models, local observables (spin, energy) $\longrightarrow$ ordinary CFTs (minimal models).
  Non-local observables (crossing probabilities, fractal dimensions) $\longrightarrow$ LCFTs.
- Supergroup string theories $\longrightarrow$ LCFTs.
- Near-extremal black holes $\longrightarrow$ LCFTs via holographic duality (maybe).

Ex. 4a: Staggered Virasoro Representations

- Virasoro algebra is generated by L_n, $n \in \mathbb{Z}$, subject to

$$\left[L_m, L_n\right] = (m-n) L_{m+n} + \frac{m\left(m^2-1\right)}{12} \delta_{m+n,0} c.$$

  $c \in \mathbb{R}$ is the central charge.
  Reps are infinite-dimensional (think Fock spaces).
- Staggered representations mean LCFTs. *eg*.
  - Dense Polymers ($c = -2$)
  - Abelian Sandpiles ($c = -2$)
  - Percolation ($c = 0$)
  - Symplectic Fermions ($c = -2$)
  - GL$\left(1|1\right)$ WZW Model ($c = 0$)
- The non-diagonalisable self-adjoint element is L_0, or $\cos\left(2\pi L_0\right)$ if we want it to be central.

Ex. 4b: Staggered $\widehat{\mathfrak{sl}}(2)$-Representations

- Fractional-level WZW models are LCFTs. Virasoro symmetry is enhanced to affine Kac-Moody symmetry.
- *eg*. $\widehat{\mathfrak{sl}}(2)$ is generated by E_n, H_n, F_n, $n \in \mathbb{Z}$, subject to

$$\left[J_m, J'_n\right] = \left[J, J'\right]_{m+n} + m\kappa\left(J, J'\right)\delta_{m+n,0}k\,.$$

  $\kappa = \mathfrak{sl}(2)$ Killing form, $k \in \mathbb{R}$ is the level, $c = \frac{3k}{k+2}$.
- The non-diagonalisable self-adjoint element is

$$L_0 = \frac{1}{2\left(k+2\right)} \sum_{r \in \mathbb{Z}} : \frac{1}{2} H_r H_{-r} + E_r F_{-r} + F_r E_{-r} : .$$

- $k = -\frac{1}{2}$ $(c = -1) \longrightarrow$ (bosonic, $\Delta = \frac{1}{2}$) $\beta\gamma$ ghosts.

Discussion

- Indecomposable representations are notorious difficult to classify (tame vs. wild). Classifying physically relevant indecomposables might be easier?
- $\mathfrak{gl}(1|1)$-, $\mathscr{U}_q(\mathfrak{sl}(2))$-, $\mathrm{TL}_n(q)$-indecomposables have been classified. Most generalisations are wild.
- Staggered structure not widely appreciated.
- Staggered Virasoro reps recently classified (Kytölä, Ridout), though many other indecomposables exist!
- These reps are relevant to statistical physics, SLE and black hole holography.
- Staggered $\widehat{\mathfrak{sl}}(2)$-reps have not been classified. Next to nothing known about them.
- Staggered affine (super)algebra reps are relevant to supergroup strings, strings on non-compact target spaces and AdS/CFT.