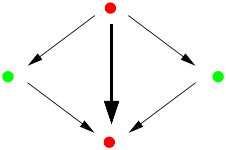
Indecomposable Representations in Physics



David Ridout

Department of Theoretical Physics
Australian National University

September 29, 2010

Introduction

- Symmetry is rather important in physics!
- Weyl and Wigner introduced group theory and representation theory into quantum physics. According to wikipedia:

> "*...the different quantum states of an elementary particle give rise to an irreducible representation of the Poincaré group.*"

- *ie*. quantum particle $\overset{\text{model}}{:=}$ irreducible representation for some symmetry group or algebra.
- In this way, the fundamental indivisibility of a particle is reflected in the mathematics.

Introduction (cont.)

- An irreducible representation is one whose only invariant subspaces are the zero vector and the whole thing.
- Completely reducible representations decompose as a direct sum of irreducibles: $V = \bigoplus_i V_i$.
- Mathematicians have long recognised that this is unusual — one may instead have indecomposable representations.
- These can have non-trivial invariant subspaces, $0 \subset W \subset V$, but $V \neq W \oplus W'$ for any W'!
- Physically, this suggests an indivisible particle (represented by V) which nevertheless has internal structure (represented by W).
- Alternatively, such reps could model unstable particles.

Ex. 1: Lie Superalgebras

- Generalisations of simple Lie algebras.
- *eg*. $\mathfrak{gl}(1|1)$ generated by E, N, ψ^+, ψ^- subject to

$$[N, \psi^\pm] = \pm\psi^\pm, \quad \{\psi^+, \psi^-\} = E \quad \text{(other brackets zero).}$$

- The adjoint representation is indecomposable:

$$\begin{tikzcd}
 & N \arrow[dl] \arrow[dr] & \\
\psi^+ \arrow[dr] & & \psi^- \arrow[dl] \\
 & E &
\end{tikzcd}$$

- Indecomposability common to most superalgebras.
- Superalgebra symmetries are used in AdS/CFT, *eg*.

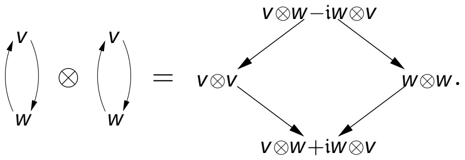
$$\mathrm{AdS}_5 \times S^5 \sim \frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(1,4) \times \mathrm{SO}(5)}.$$

Ex. 2: Quantum Groups

- q-deformations of Lie (super)algebras with tensor product.
- *eg*. $\mathscr{U}_q(\mathfrak{sl}(2))$ generated by E, K, F subject to

$$KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

- $q^N \neq 1$: Completely reducible.
  $q^N = 1$: Indecomposables arise, *eg*. for $q = \mathfrak{i}$,



- Quantum groups govern quantum integrable systems. TT, TQ and QQ relations reflect indecomposable structure.

Ex. 3: Temperley-Lieb Algebras

- Workhorse of statistical model physics.
- $\mathrm{TL}_n(q)$ generated by $\mathbf{1}, u_1, \dots u_{n-1}$ subject to

$$
\begin{aligned}
u_i^2 &= (q+q^{-1})u_i, && \text{for all } i, \\
u_i u_{i\pm1} u_i &= u_i, && \text{for all } i, \\
u_i u_j &= u_j u_i, && \text{when } |i-j| \geqslant 2.
\end{aligned}
$$

- $\color{red}{q^N = 1}$: Indecomposables arise, *eg*. for $q = \mathfrak{i}$ and $n = 6$,

$$
\mathrm{TL}_6(\mathfrak{i}) = 5\left(\begin{tikzcd} 5 \arrow[dr] & \\ & 4 \arrow[dl] \\ 5 & \end{tikzcd}\right) \oplus 4\left(\begin{tikzcd} & 4 \arrow[dl] \arrow[dr] & \\ 5 \arrow[dr] & & 1 \arrow[dl] \\ & 4 & \end{tikzcd}\right) \oplus \left(\begin{tikzcd} & 1 \arrow[dl] \\ 4 \arrow[dr] & \\ & 1 \end{tikzcd}\right).
$$

- Underlies many lattice models (Q-state Potts, RSOS). *eg*. $q = e^{\mathfrak{i}\pi/3}$ (percolation), $q = e^{\mathfrak{i}\pi/4}$ (Ising model).

Ex. 4: Virasoro Algebra

- Underlies 2D conformal field theories.
- $\mathfrak{Vir}$ generated by L_n, $n \in \mathbb{Z}$, subject to

$$\left[L_m, L_n\right] = (m-n)\, L_{m+n} + \frac{m\left(m^2-1\right)}{12}\,\delta_{m+n,0}\,c.$$

  $c \in \mathbb{R}$ is the central charge.
- Irreps $\mathscr{L}_h$ are infinite-dimensional, labelled by minimal conformal dimension (eigenvalue of L_0) h.
- Fusion of irreps can lead to indecomposables, *eg*.

$$\mathscr{L}_{1/3} \times \mathscr{L}_{1/3} = \mathscr{L}_{1/3} + \left(\begin{tikzcd} & \mathscr{L}_2 \arrow[dl] \arrow[dr] & \\ \mathscr{L}_0 \arrow[dr] & & \mathscr{L}_5 \arrow[dl] \\ & \mathscr{L}_2 & \end{tikzcd} \right) \qquad (c = 0).$$

- This indecomposable arises in critical percolation.

The Action of the Centre

- Diamond-shaped indecomposables crop up a lot!
- In each case, there is an element C which is

  self-adjoint, central and non-diagonalisable.

$$\begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] \arrow[dd, "C"'] & \\ \bullet \arrow[dr] & & \bullet \arrow[dl] \\ & \bullet & \end{tikzcd}$$

  These reps have no central character!
- For $\mathfrak{gl}(1|1)$, $C = \frac{1}{2}NE + \psi^- \psi^+$ is the Casimir.

  For $\mathscr{U}_q(\mathfrak{sl}(2))$, $C = EF + \frac{Kq^{-1} + K^{-1}q}{(q - q^{-1})^2}$ is also the Casimir.

  For $\mathrm{TL}_n(q)$, C is the "braid transfer matrix".

  For $\mathfrak{Vir}$, C is $\sin(2\pi L_0)$ (or $\cos(2\pi L_0)$).
- In each case, C has Jordan cells of rank 2.

Staggered Representations

- These algebras have a standard class of reps: Cell or link reps for $\mathrm{TL}_n(q)$, highest weight reps for the others.

Definition

A staggered rep $\mathscr{S}$ is an indecomposable rep for which we have a short exact sequence

$$0 \longrightarrow \mathscr{H}^L \xrightarrow{\iota} \mathscr{S} \xrightarrow{\pi} \mathscr{H}^R \longrightarrow 0,$$

in which it is understood that:

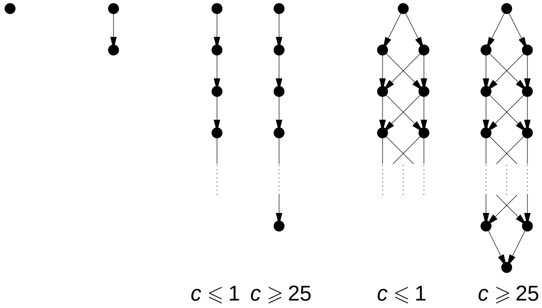
- $\mathscr{H}^L$ and $\mathscr{H}^R$ are standard reps,
- ι and π are homomorphisms, and
- the centre does not act diagonalisably on $\mathscr{S}$, possessing instead Jordan cells of rank at most 2.

Logarithmic Conformal Field Theory

- "Staggered" reps (for $\mathfrak{Vir}$) were so named by Rohsiepe (*Gestufte Moduln*) in studying logarithmic CFT (LCFT).
- When the Virasoro reps are staggered, there are correlation functions with logarithmic singularities.
- There are many theories described by LCFTs, *eg*.
  - Dense Polymers ($c = -2$)
  - Abelian Sandpiles ($c = -2$)
  - Percolation ($c = 0$)
  - Ising Model ($c = \frac{1}{2}$)
  - Symplectic Fermions ($c = -2$)
  - $\mathrm{GL}(1|1)$ WZW Model ($c = 0$)
  - $\beta\gamma$ Ghosts ($c = -1$)
- Some of these theories are also described by SLE (Schramm-Loewner Evolution)!

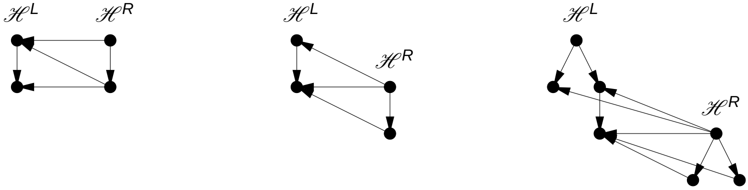
Highest Weight Virasoro Representations

- Highest weight reps are those generated by an eigenstate of L_0 which is annihilated by the L_n with $n > 0$.
- Universal highest weight reps (Verma reps) $\mathscr{V}_h$ are characterised by their minimal conformal dimension h. Every irrep $\mathscr{L}_h$ is the unique irreducible quotient of $\mathscr{V}_h$.
- Submodule structure of $\mathscr{V}_h$ can be complicated:



Staggered Virasoro Representations

- Staggered $\iff$ L_0 has rank 2 Jordan cells.
- Highest weight reps are never staggered. The latter are obtained by gluing highest weight reps together, *eg*.



Questions:

1. Do such staggered reps exist?
2. If so, how many (isomorphism classes of) such staggered reps are there?

Simple Observations

Proposition

Let $\mathscr{S}$ be staggered and h^R be the minimal conformal dim of $\mathscr{H}^R$. Then, $\mathscr{H}^L$ has a singular vector of conformal dim h^R.

Simple Observations

Proposition

Let $\mathscr{S}$ be staggered and h^R be the minimal conformal dim of $\mathscr{H}^R$. Then, $\mathscr{H}^L$ has a singular vector of conformal dim h^R.

Proposition

If h^L is the minimal conformal dim of $\mathscr{H}^L$, then staggered reps can only exist when $h^R - h^L \in \mathbb{N}$. Moreover, if $h^R > h^L$, then

$$h^L = \frac{r^2-1}{4}t - \frac{rs-1}{2} + \frac{s^2-1}{4}t^{-1},$$

where $c = 13 - 6\left(t + t^{-1}\right)$, for some $r, s \in \mathbb{Z}_+$.

Simple Observations

Proposition

Let $\mathscr{S}$ be staggered and h^R be the minimal conformal dim of $\mathscr{H}^R$. Then, $\mathscr{H}^L$ has a singular vector of conformal dim h^R.

Proposition

If h^L is the minimal conformal dim of $\mathscr{H}^L$, then staggered reps can only exist when $h^R - h^L \in \mathbb{N}$. Moreover, if $h^R > h^L$, then

$$h^L = \frac{r^2-1}{4}t - \frac{rs-1}{2} + \frac{s^2-1}{4}t^{-1},$$

where $c = 13 - 6\left(t + t^{-1}\right)$, for some $r, s \in \mathbb{Z}_+$.

Proposition

If $\mathscr{S}$ is staggered and $\mathscr{H}^L$ has a singular vector of dim h, then $\mathscr{H}^R$ also has a singular vector of dim h.

Not-so-Obvious Results

Proposition

If $\mathscr{S}$ is staggered, then one may construct another staggered rep $\check{\mathscr{S}}$ which differs only in that $\mathscr{H}^{\mathrm{R}}$ is replaced by $\check{\mathscr{H}}^{\mathrm{R}} = \mathscr{V}_{h^{\mathrm{R}}}$. Moreover, $\mathscr{S}$ may be identified with a quotient of $\check{\mathscr{S}}$.

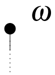
Not-so-Obvious Results

Proposition

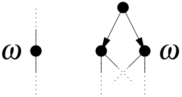
If $\mathscr{S}$ is staggered, then one may construct another staggered rep $\check{\mathscr{S}}$ which differs only in that $\mathscr{H}^R$ is replaced by $\check{\mathscr{H}}^R = \mathscr{V}_{h^R}$. Moreover, $\mathscr{S}$ may be identified with a quotient of $\check{\mathscr{S}}$.

Theorem

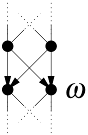
If $\mathscr{H}^L$ and $\mathscr{H}^R = \mathscr{V}_{h^R}$ pass the previous tests, then the space $\mathbb{S}$ of isomorphism classes of staggered reps is a (non-empty) vector space of dimension 0, 1 or 2. If ω denotes the singular vector of $\mathscr{H}^L$ of conformal dim h^R, then $\dim \mathbb{S}$ is as follows:



$\dim \mathbb{S} = 0$



$\dim \mathbb{S} = 1$



$\dim \mathbb{S} = 2.$

Not-so-Obvious Results (cont.)

Theorem

In general, when $\mathscr{H}^{\mathsf{L}}$ and $\mathscr{H}^{\mathsf{R}}$ pass the previous tests, then the space $\mathbb{S}$ of isomorphism classes of staggered reps is either empty or it is an affine space of dimension 0, 1 *or* 2*.*

Not-so-Obvious Results (cont.)

Theorem

In general, when $\mathscr{H}^{\mathsf{L}}$ and $\mathscr{H}^{\mathsf{R}}$ pass the previous tests, then the space $\mathbb{S}$ of isomorphism classes of staggered reps is either empty or it is an affine space of dimension 0, 1 or 2.

Theorem

The space $\mathbb{S}$ is parametrised by the logarithmic coupling / beta-invariant (DR & Mathieu) and its generalisations (DR & Kytölä). Identification of a staggered rep thereby reduces to the identification of $\mathscr{H}^{\mathsf{L}}$ and $\mathscr{H}^{\mathsf{R}}$ and the computation of at most two numbers.

[All this and more may be found in arXiv:0905.0108 [math-ph].]

Discussion and Outlook

- Indecomposable representations are notorious difficult to classify (tame vs. wild). Classifying physically relevant indecomposables might be easier?
- Staggered reps seem to form a tractable class, relevant to CFT, statistical physics, string theory, SLE, black hole holography (?), ...
- May clarify Temperley-Lieb/Virasoro correspondence.
- $\mathfrak{Vir}$ admits physically relevant indecomposables (?) in which L_0 has Jordan cells of rank 3. Comparable theory is completely unexplored.
- Staggered affine (super)algebra reps have not been classified, but are relevant to fractional level WZW models, supergroup string theories, strings on non-compact target spaces and AdS/CFT.