A (Gentle) Introduction to Lie Superalgebras

David Ridout

Department of Theoretical Physics
& Mathematical Sciences Institute,
Australian National University

April 29, 2011

The Importance of Being Lie
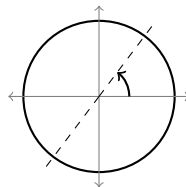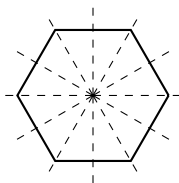
Discrete groups describe discrete symmetries.

Continuous symmetries are described by so-called Lie groups.

eg. a regular hexagon: 6 rotations and 6 reflections
$\rightarrow$ dihedral group D_{12} (discrete).

eg. a circle: Infinitely many rotations and reflections
$\rightarrow$ orthogonal group $O(2)$ or $U(1)$ (Lie).

Lie Algebras

Being continuous (better, smooth manifold!), Lie groups allow calculus. Differentiating smooth curves through 1 at 1 gives the Lie algebra: A vector space equipped with a bracket $[\cdot,\cdot]$ satisfying

- $[x,y] = -[y,x]$,
- $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$.

Canonical example:

$$\mathfrak{gl}(n) = \left\{ n \times n \text{ matrices } : \ [A,B] = AB - BA \right\}.$$

Lie algebras are the linearisations of Lie groups and characterise the latter modulo topology.

Classification

Lie algebras come in two flavours (Levi), solvable and semisimple.

Solvable Lie algebras are (probably) unclassifiable.

Semisimple Lie algebras $\mathfrak{g}$ were classified in 1894!

They decompose as direct sums of ideals $\mathfrak{i}$ — vector subspaces with $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}$ — which are simple in that they have no (non-trivial) ideals of their own.

There are precisely four infinite series and five exceptional examples of simple Lie algebras (over $\mathbb{C}$):

$$\mathfrak{a}_r, \quad \mathfrak{b}_r, \quad \mathfrak{c}_r, \quad \mathfrak{d}_r, \quad \mathfrak{e}_6, \quad \mathfrak{e}_7, \quad \mathfrak{e}_8, \quad \mathfrak{f}_4, \quad \mathfrak{g}_2.$$

Representations

Lie algebras $\mathfrak{g}$ (and Lie groups!) are typically encountered through a representation where they act on a vector space V: $x \in \mathfrak{g} \mapsto x \cdot \in \mathfrak{gl}(V)$. We need

- $(ax + by)\cdot = a(x\cdot) + b(y\cdot)$ for all $a, b \in \mathbb{C}$, $x, y \in \mathfrak{g}$,
- $[x, y]\cdot = x \cdot y \cdot - y \cdot x \cdot$ for all $x, y \in \mathfrak{g}$.

eg. the adjoint representation takes $V = \mathfrak{g}$ and $x \cdot y = [x, y]$.

A subrepresentation of V is a subspace W for which $\mathfrak{g} \cdot W \subseteq W$. If V has no (non-trivial) subrepresentations, it is irreducible. If V is the direct sum of irreducibles, it is completely reducible.

Theorem (Weyl): *Every finite-dimensional representation of a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ is completely reducible.*

Outline   Lie Algebras and Superalgebras   Examples: $\mathfrak{gl}(2)$ and $\mathfrak{gl}(1|1)$ Examples: $\mathfrak{sl}(3)$ and $\mathfrak{sl}(2|1)$ Discussion

○○○○●○   ○○○○○   ○○○○   ○○○○

Lie Superalgebras

A Lie superalgebra is a $\mathbb{Z}_2$-graded generalisation:

$$\mathfrak{g} = \underbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1}_{\substack{\text{as vector} \\ \text{spaces}}}, \qquad \big[\mathfrak{g}_i, \mathfrak{g}_j\big] \subseteq \mathfrak{g}_{i+j}, \quad (i, j \in \mathbb{Z}_2).$$

For $x \in \mathfrak{g}_{\bar{x}}$, $y \in \mathfrak{g}_{\bar{y}}$, $z \in \mathfrak{g}_{\bar{z}}$, the bracket satisfies

- $\big[x, y\big] = -(-1)^{\bar{x}\bar{y}} \big[y, x\big]$,
- $\big[[x, y], z\big] + (-1)^{\bar{x}(\bar{y}+\bar{z})} \big[[y, z], x\big] + (-1)^{(\bar{x}+\bar{y})\bar{z}} \big[[z, x], y\big] = 0$.

In physics, Lie algebras describe bosonic degrees of freedom.
Lie superalgebras allow fermionic degrees of freedom as well.

Canonical Example

Declare an $(m+n) \times (m+n)$ block matrix $\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$ to be even if $A_{01} = A_{10} = 0$ and odd if $A_{00} = A_{11} = 0$.

The space of $(m+n) \times (m+n)$ block matrices is the Lie superalgebra $\mathfrak{gl}(m|n)$ with:

- $\mathfrak{gl}(m|n)_0 = \{\text{even matrices}\}$, $\mathfrak{gl}(m|n)_1 = \{\text{odd matrices}\}$.

- $[A, B] = \begin{cases} AB - BA & \text{if } A \text{ or } B \text{ is even,} \\ AB + BA & \text{if } A \text{ and } B \text{ is odd.} \end{cases}$

Example: $\mathfrak{gl}(2)$

Recall $\mathfrak{gl}(2) = \big\{ 2 \times 2 \text{ matrices } : [A, B] = AB - BA \big\}$.

$I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$ spans an ideal of $\mathfrak{gl}(2)$: $[A, I] = 0$ for all $A \in \mathfrak{gl}(2)$.

$\mathfrak{sl}(2) = \{ A \in \mathfrak{gl}(2) : \operatorname{tr} A = 0 \}$ is an ideal of $\mathfrak{gl}(2)$:

$$\operatorname{tr}[A, B] = \operatorname{tr} AB - \operatorname{tr} BA = 0 \qquad \text{for all } A \in \mathfrak{gl}(2),\ B \in \mathfrak{sl}(2).$$

Since $A = \big(A - \tfrac{1}{2} \operatorname{tr} A\, I \big) + \tfrac{1}{2} \operatorname{tr} A\, I \in \mathfrak{sl}(2) + \mathbb{C} I$, for all $A \in \mathfrak{gl}(2)$, and $\mathfrak{sl}(2) \cap \mathbb{C} I = \{0\}$,

$$\mathfrak{gl}(2) = \mathfrak{sl}(2) \oplus \mathbb{C} I \cong \mathfrak{sl}(2) \oplus \mathfrak{u}(1) \,.$$

Further decomposition is not possible.

Representations of $\mathfrak{gl}(2)$

The defining representation is given by

$$E = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \quad H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \quad F = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \quad I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

It is irreducible. The non-zero commutators are

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

The adjoint representation is then

$$E\cdot = \left(\begin{smallmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right), \quad H\cdot = \left(\begin{smallmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right), \quad F\cdot = \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right), \quad I\cdot = 0.$$

It is completely reducible ($\mathfrak{gl}(2) = \mathfrak{sl}(2) \oplus \mathfrak{u}(1)$).

Example: $\mathfrak{gl}(1|1)$

$\mathfrak{gl}(1|1)$ has defining representation

$$\underbrace{Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},}_{\text{even}} \quad \underbrace{\psi^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \psi^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{\text{odd}}$$

and non-zero (anti)commutators

$$\left[N, \psi^+ \right] = 2\psi^+, \quad \left[N, \psi^- \right] = -2\psi^-, \quad \left\{ \psi^+, \psi^- \right\} = Z.$$

It is irreducible. The adjoint representation is $Z \cdot = 0$,

$$N \cdot = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad \psi^+ \cdot = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \psi^- \cdot = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$$

It is not even completely reducible...

Outline   Lie Algebras and Superalgebras   **Examples: $\mathfrak{gl}(2)$ and $\mathfrak{gl}(1|1)$**   Examples: $\mathfrak{sl}(3)$ and $\mathfrak{sl}(2|1)$ Discussion

ooooo   ○○○●○   ○○○○   ○○○○

What does that mean, anyway?

$Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ spans an ideal.

The traceless matrices do not: $\{\psi^+, \psi^-\} = Z$.

Define the supertrace by $\operatorname{str} \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} = \operatorname{tr} A_{00} - \operatorname{tr} A_{11}$. Then, $\mathfrak{sl}(1|1) = \{A \in \mathfrak{gl}(1|1) \ : \ \operatorname{str} A = 0\}$ is an ideal.
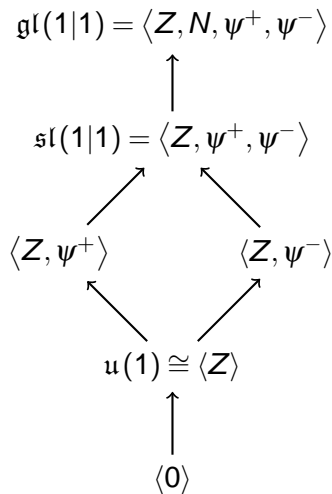
But, $\mathfrak{gl}(1|1) \neq \mathfrak{sl}(1|1) \oplus \mathbb{C}Z$. Indeed, $Z \in \mathfrak{sl}(1|1)$.

And $\mathfrak{gl}(1|1) \neq \mathfrak{sl}(1|1) \oplus \mathbb{C}N$, as $\mathbb{C}N$ is not an ideal: $[N, \psi^\pm] = \pm 2\psi^\pm$.

In fact, *no* non-trivial ideal $\mathfrak{i}$ of $\mathfrak{gl}(1|1)$ has a complement $\mathfrak{j}$:

$$\mathfrak{gl}(1|1) \neq \mathfrak{i} \oplus \mathfrak{j}.$$

Ideal Structure of $\mathfrak{gl}(1|1)$

$$\mathfrak{gl}(1|1) = \left\langle Z, N, \psi^+, \psi^- \right\rangle$$

$$\mathfrak{sl}(1|1) = \left\langle Z, \psi^+, \psi^- \right\rangle$$

$$\left\langle Z, \psi^+ \right\rangle \qquad \left\langle Z, \psi^- \right\rangle$$

$$\mathfrak{u}(1) \cong \left\langle Z \right\rangle$$

$$\left\langle 0 \right\rangle$$

Example: $\mathfrak{sl}(3)$

$\mathfrak{sl}(3) = \{A \in \mathfrak{gl}(3) \ : \ \operatorname{tr} A = 0\}$ is simple (no non-trivial ideals).

It has basis $\{H_1 = E_{11} - E_{22}, H_2 = E_{22} - E_{33}\} \cup \{E_{jk} \ : \ j \neq k\}$.

The roots α_{jk} of $\mathfrak{sl}(3)$ are defined by

$$\left[H_i, E_{jk}\right] = \underbrace{\alpha_{jk}\left(H_i\right)}_{\in \mathbb{C}} E_{jk}.$$

(j,k)	$(1,2)$	$(2,3)$	$(1,3)$	$(2,1)$	$(3,2)$	$(3,1)$
$\alpha_{jk}\left(H_1\right)$	2	-1	1	-2	1	-1
$\alpha_{jk}\left(H_2\right)$	-1	2	1	1	-2	-1

Roots are the fundamental geometric data of semisimple Lie algebras.

Outline   Lie Algebras and Superalgebras   Examples: $\mathfrak{gl}(2)$ and $\mathfrak{gl}(1|1)$ Examples: $\mathfrak{sl}(3)$ and $\mathfrak{sl}(2|1)$ Discussion

oooooo   ooooo   o●oo   oooo

Root Diagrams

Roots may be partitioned as positive and negative by a (general) hyperplane through 0.

The two (in this case) positive roots closest to the hyperplane are termed simple. The classification of semisimple Lie algebras derives from the geometry of the simple roots.
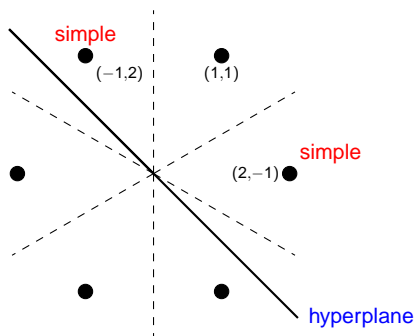
The choice of hyperplane and simple roots is *immaterial* to the classification!

Example: $\mathfrak{sl}(2|1)$

$\mathfrak{sl}(2|1) = \{A \in \mathfrak{gl}(2|1) \; : \; \operatorname{str} A = 0\}$ is a simple Lie superalgebra.

A convenient basis is
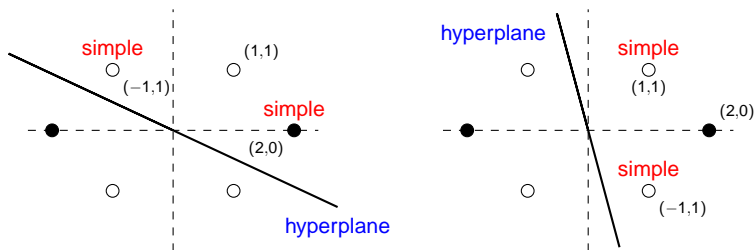
$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

$$\psi^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \bar{\psi}^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \psi^- = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{\psi}^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

and the action of $\big[H, \cdot\big]$ and $\big[I, \cdot\big]$ give the roots α:

	E	F	ψ^+	$\bar{\psi}^+$	ψ^-	$\bar{\psi}^-$
$\alpha(H)$	2	-2	1	-1	1	-1
$\alpha(I)$	0	0	1	1	-1	-1

Root System

We draw even roots in black and odd roots in white:

There are now two distinct (equivalence classes of) choices for the hyperplane and simple roots. Two distinct objects in the classification correspond to the same Lie superalgebra.

This is a consequence of the reduced *symmetry* of the roots.

Outline   Lie Algebras and Superalgebras   Examples: $\mathfrak{gl}(2)$ and $\mathfrak{gl}(1|1)$ Examples: $\mathfrak{sl}(3)$ and $\mathfrak{sl}(2|1)$ Discussion

○○○○○○   ○○○○○   ○○○○   ●○○○

Discussion

We have seen two examples of ways in which the theory of Lie superalgebras differs from the theory of Lie algebras:

- Complete reducibility of representations may fail.
- There may be inequivalent choices for the simple roots.

To this we can add a further geometric complication:

- Whereas the lengths of the roots of a semisimple Lie algebra are positive, a Lie superalgebra may have odd roots of zero length. It is even possible that "length" makes no sense.

The first is by far the biggest problem, especially for applications to physics. The second is in more of a nuisance, though it affects generalisations directly (quantum groups).

Discussion (cont.)

Nevertheless, the finite-dimensional simple Lie superalgebras over $\mathbb{C}$ have been classified:

$$\mathfrak{a}(r,s), \quad \mathfrak{b}(r,s), \quad \mathfrak{c}(r), \quad \mathfrak{d}(r,s),$$
$$\mathfrak{f}(4), \quad \mathfrak{g}(3), \quad \underbrace{\mathfrak{d}(2,1;\alpha)}_{\alpha\in\mathbb{C}}, \quad \underbrace{\mathfrak{p}(r), \quad \mathfrak{q}(r)}_{\text{strange!}}.$$

eg. $\mathfrak{sl}(2|1) = \mathfrak{a}(1,0)$.

Theorem (Djokovic-Hochschild): *Every finite-dimensional representation of a finite-dimensional simple Lie superalgebra $\mathfrak{g}$ over $\mathbb{C}$ is completely reducible if and only if $\mathfrak{g}$ is a simple Lie algebra or $\mathfrak{g} = \mathfrak{b}(0,s) = \mathfrak{osp}(1|2s)$.*

Discussion (cont.)

This means that the fundamental building blocks of Lie superalgebra representation theory are not necessarily irreducible. Unfortunately:

Theorem (Germoni): *The finite-dimensional indecomposable representations of $\mathfrak{a}(r,s) = \mathfrak{sl}(r+1|s+1)$ are unclassifiable (wild) if $r, s \geqslant 1$.*

Even if we restrict to irreducible representations, the reduction of the symmetry of the root system raises obstacles.

Fact: *We still have no general formulae for the dimensions of the irreducible representations of a simple Lie superalgebra (maybe we never will).*

What this means for physics is unclear...

Outline   Lie Algebras and Superalgebras   Examples: $\mathfrak{gl}(2)$ and $\mathfrak{gl}(1|1)$ Examples: $\mathfrak{sl}(3)$ and $\mathfrak{sl}(2|1)$ **Discussion**

oooooo   ooooo   oooo   ooo●

Concluding Remarks

Simple Lie algebras and their representations are well understood. For Lie superalgebras, the situation is bleak.

We will never have a complete picture of the representation theory. Progress can only be made by limiting our ambition!

The many applications of Lie superalgebras in physics (statistical models, superstrings, conformal field theory) suggest a way: Understand the projective covers of the irreducible representations.

Advanced homological methods from the theory of associative algebras are proving extremely useful. Unfortunately, this is not a part of the standard toolkit for mathematical physicists.

In any case, there remains a lot of (fun) exploratory research to be tackled.