Lattice Discretisations of Integrable Sigma Models

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Motivation

Lots of QFTs are believed to be quantum-integrable. If so, one gains some control over non-perturbative phenomena, *eg*. dualities.

Usually very hard to establish quantum-integrability. Often, regularisation is required. Then, want an integrable regularisation.

Better (?), we want a regularisation for which:

1. The local degrees of freedom may be related with the continuum fields.
2. The quantum groups underlying the integrability of the continuum and regularised theories coincide.

The latter point also facilitates the eventual solution of the theory (*eg*. in deriving TQ-relations, *etc*...).

A Proposal

We propose a framework for studying the integrability of a certain class of non-linear sigma models and constructing integrable lattice discretisations for them. This involves:

1. Identifying the relevant quantum group using the chiral quantisation of the interaction terms.
2. Constructing R- and L-matrices from the quantum group for the continuum and lattice theories (quantum inverse scattering).

The models we consider involve bosons (compact, non-compact) and fermions. The interaction terms are restricted to being exponential. *eg*. affine Toda theories.

Quantum Affine Superalgebras

Integrability usually arises from the action of a quasitriangular Hopf (super)algebra, *eg*. a quantum affine superalgebra $\mathcal{U}_q(\widehat{\mathfrak{g}})$.

These have a universal R-matrix $\mathcal{R}^+$ in (a completion of) the tensor square of the superalgebra which is (formally) invertible and satisfies the abstract Yang-Baxter equation.

They also possess a second universal R-matrix $\mathcal{R}^- = \sigma\left(\mathcal{R}^+\right)^{-1}$, where $\sigma\left(x \otimes y\right) = (-1)^{\bar{x}\bar{y}}\, y \otimes x$. This too satisfies the abstract Yang-Baxter equation.

For simple Lie algebras, $\mathcal{U}_q(\widehat{\mathfrak{g}})$ may be presented in the Chevalley-Serre manner. In particular, the quantum Serre relations are uniformly expressible in terms of the Cartan matrix. For superalgebras, the quantum Serre relations are considerably more involved [Yamane, Zhang].

Identifying the Quantum Group

It is well known that in CFT, the quantum group structure may be revealed by constructing a free field realisation and computing the algebra generated by the screening charges.

We propose that this generalises to our models: The screening charges are derived from the vertex operators representing the exponential interaction terms.

The screening charge algebra is deduced from the braiding of the vertex operators as in [BLZ3, App. A]. The relations of this algebra are interpreted as Serre relations for a quantum affine superalgebra.

Unlike in CFT, the symmetry may only be extended to a Borel subalgebra $\mathcal{B}^+$ of the quantum affine superalgebra. Note that superalgebras may have inequivalent Borel subalgebras.

Example: Sinh-Gordon

The sinh-Gordon model has classical action

$$S = \int \left[\frac{1}{4\pi} \partial_\mu \phi \partial^\mu \phi + \nu_+ \mathrm{e}^{+2b\phi} + \nu_- \mathrm{e}^{-2b\phi} \right] \mathrm{d}^2 z,$$

so the interaction terms give the chiral vertex operators

$$V_0^+ = \, :\mathrm{e}^{+2b\phi^+}: \quad \text{and} \quad V_1^+ = \, :\mathrm{e}^{-2b\phi^+}: .$$

Defining screening charges by $Q_i^+ = \oint V_i^+(z)\,\mathrm{d}z$, we find that

$$\left(Q_i^+\right)^3 Q_j^+ - [3]_q \left(Q_i^+\right)^2 Q_j^+ Q_i^+ + [3]_q Q_i^+ Q_j^+ \left(Q_i^+\right)^2 - Q_j^+ \left(Q_i^+\right)^3 = 0,$$

where $q = \mathrm{e}^{-\mathrm{i}\pi b^2}$. This is the quantum Serre relation of $\mathcal{U}_q\left(\widehat{\mathfrak{sl}}(2)\right)$.

ie. the quantum symmetry algebra underlying the integrability of sinh-Gordon is the Borel subalgebra of $\mathcal{U}_q\left(\widehat{\mathfrak{sl}}(2)\right)$.

Example: $N = 2$ Super Sine-Gordon

The $N = 2$ super sine-Gordon model has action

$$S' = \int \Big[\tfrac{1}{4\pi}(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2) + \tfrac{1}{2\pi}\left(\bar\psi^+\partial_-\psi^+ + \bar\psi^-\partial_+\psi^-\right)$$
$$- b\nu_+ e^{+b\phi_1}\left(\bar\psi^+\bar\psi^- e^{+ib\phi_2} + \psi^+\psi^- e^{-ib\phi_2}\right) - b\nu_- e^{-b\phi_1}\left(\psi^+\psi^- e^{+ib\phi_2} + \bar\psi^+\bar\psi^- e^{-ib\phi_2}\right)$$
$$+ 4\pi\nu_-^2 e^{+2b\phi_1} - 8\pi\nu_-\nu_+ \cos(2b\phi_2) + 4\pi\nu_+^2 e^{-2b\phi_1} \Big] d^2z.$$

Treating the terms on the second line as interactions and those on the third line as counterterms generated by renormalisation as $b \to 0$, we obtain four chiral vertex operators:

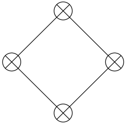
$$V_0^+ = \bar\psi^+ : e^{-b\left(\phi_1^+ - i\phi_2^+\right)} : , \qquad V_1^+ = \psi^+ : e^{+b\left(\phi_1^+ + i\phi_2^+\right)} : ,$$
$$V_2^+ = \bar\psi^+ : e^{+b\left(\phi_1^+ - i\phi_2^+\right)} : , \qquad V_3^+ = \psi^+ : e^{-b\left(\phi_1^+ + i\phi_2^+\right)} : .$$

The screening charges $Q_i^+ = \oint V_i^+(z)\,\mathrm{d}z$ satisfy

$$\left(Q_i^+\right)^2=0, \qquad Q_i^+ Q_{i+2}^+ + Q_{i+2}^+ Q_i^+ = 0,$$

$$Q_{i-1}^+ Q_i^+ Q_{i+1}^+ Q_i^+ - Q_{i+1}^+ Q_i^+ Q_{i-1}^+ Q_i^+ + [2]_q Q_i^+ Q_{i-1}^+ Q_{i+1}^+ Q_i^+$$
$$- Q_i^+ Q_{i+1}^+ Q_i^+ Q_{i-1}^+ + Q_i^+ Q_{i-1}^+ Q_i^+ Q_{i+1}^+ = 0,$$

where $q = \mathrm{e}^{-\mathrm{i}\pi b^2}$ and i is taken $\mathrm{mod}\ 4$. These are (some of) the quantum Serre relations of $\mathcal{U}_q\left(\widehat{\mathfrak{sl}}(2|2)\right)$ (there are others with degrees in $4\mathbb{Z}_+ + 2$) with the Dynkin diagram



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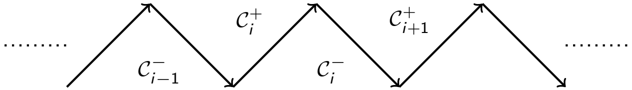
ie. the quantum symmetry algebra underlying the integrability of $N = 2$ super sine-Gordon is a Borel subalgebra of $\mathcal{U}_q\left(\widehat{\mathfrak{sl}}(2|2)\right)$.

Classical Considerations

Classically, one has a zero-curvature condition

$$\left[\partial_+ - U_+(\lambda), \partial_- - U_-(\lambda)\right],$$

giving rise to a monodromy matrix $M(\lambda)$ as a path-ordered integral around the cylinder. Choosing



for the integration contour gives

$$M(\lambda) = L_N^-(\lambda) L_N^+(\lambda) \cdots L_1^-(\lambda) L_1^+(\lambda),$$
$$L_i^\pm(\lambda) = \mathcal{P}\exp\int_{\mathcal{C}_i^\pm} U_\pm(\lambda)\, \mathrm{d}x_\pm.$$

Quantisation

If one can fix a gauge so that the $U_{\pm}(\lambda)$ contain only the exponential interaction terms, we may quantise by replacing these terms by their normally-ordered equivalents.

Alternatively, quantum inverse scattering lets us construct the Lax matrices directly from the universal R-matrices:

$$L_i^{\pm}(\lambda) = \left(\pi_q^{\pm} \otimes \pi_a^{\lambda}\right)\left(\mathcal{R}^{\pm}\right).$$

Here, π_a^{λ} is a finite-dimensional evaluation representation of the quantum affine superalgebra and the $\pi_q^{\pm}$ are certain infinite-dimensional representations of the Borel subalgebras.

The R-matrix itself is given (up to normalisation) by

$$R(\lambda, \mu) = \left(\pi_a^{\lambda} \otimes \pi_a^{\mu}\right)\left(\mathcal{R}^{\pm}\right).$$

Example: Sinh-Gordon

If we choose the $\mathcal{U}_q\big(\widehat{\mathfrak{sl}}(2)\big)$-representation

$$\pi_a^\lambda(e_0)=\begin{pmatrix}0&0\\ \lambda^{-1}&0\end{pmatrix},\quad \pi_a^\lambda(f_0)=\begin{pmatrix}0&\lambda\\ 0&0\end{pmatrix},\quad \pi_a^\lambda(h_0)=\begin{pmatrix}-1&0\\ 0&1\end{pmatrix},$$
$$\pi_a^\lambda(e_1)=\begin{pmatrix}0&\lambda^{-1}\\ 0&0\end{pmatrix},\quad \pi_a^\lambda(f_1)=\begin{pmatrix}0&0\\ \lambda&0\end{pmatrix},\quad \pi_a^\lambda(h_1)=\begin{pmatrix}1&0\\ 0&-1\end{pmatrix}$$

and the $\mathcal{B}^\pm$-representations

$$\pi_q^\pm(h_0)=\pm 2\mathrm{i}p/b,\qquad \pi_q^\pm(h_1)=\mp 2\mathrm{i}p/b$$

with either $\left(\tau_q=\left(q-q^{-1}\right)^{-1}\right)$

$$\pi_q^+(e_0)=\tau_q Q_0^+,\quad \pi_q^+(e_1)=\tau_q Q_1^+,\quad \text{or}\quad \pi_q^-(f_0)=\tau_q Q_0^-,\quad \pi_q^-(f_1)=\tau_q Q_1^-,$$
$$Q_0^\pm=\pm\oint :\mathrm{e}^{+2b\phi^\pm(z)}:\mathrm{d}z,\qquad Q_1^\pm=\pm\oint :\mathrm{e}^{-2b\phi^\pm(z)}:\mathrm{d}z,$$

then...

... we reproduce the monodromy matrix of BLZ,

$$L^{+}(\lambda_{+}) = \begin{pmatrix} e^{-\pi b p} & 0 \\ 0 & e^{+\pi b p} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \lambda_{+} Q_0^{+} \\ \lambda_{+} Q_1^{+} & \mathbf{1} \end{pmatrix},$$

$$L^{-}(\lambda_{-}) = \begin{pmatrix} \mathbf{1} & \lambda_{-}^{-1} Q_1^{-} \\ \lambda_{-}^{-1} Q_0^{-} & \mathbf{1} \end{pmatrix} \begin{pmatrix} e^{-\pi b p} & 0 \\ 0 & e^{+\pi b p} \end{pmatrix},$$

though in a different "gauge".

This is, of course, formal and when $b \in \mathbb{R}$, there are ultraviolet divergence issues. For $b = \mathrm{i}\beta$, $\beta \in \mathbb{R}$ (sine-Gordon), these may be controllable.

The representation-theoretic approach is not needed in this case, but it generalises readily. More importantly, regularisation will be needed for sinh-Gordon and our approach can be readily adapted to the lattice.

Example: $N = 2$ Super Sine-Gordon

For $\mathcal{U}_q\left(\widehat{\mathfrak{sl}}(2|2)\right)$, let π_a^λ be the "defining" evaluation representation and π_q^+ be again given by the screening charges, supplemented by

$$\pi_q^+(H_0) = +\mathrm{i}\left(\mathrm{p}_1^+ - \mathrm{i}\mathrm{p}_2^+\right)/b, \qquad \pi_q^+(H_1) = -\mathrm{i}\left(\mathrm{p}_1^+ + \mathrm{i}\mathrm{p}_2^+\right)/b,$$
$$\pi_q^+(H_2) = -\mathrm{i}\left(\mathrm{p}_1^+ - \mathrm{i}\mathrm{p}_2^+\right)/b, \qquad \pi_q^+(H_3) = +\mathrm{i}\left(\mathrm{p}_1^+ + \mathrm{i}\mathrm{p}_2^+\right)/b.$$

The resulting monodromy matrix is

$$L^+(\lambda_+) = q^{-\rho^+ Z/2} e^{\pi b\left(\mathrm{p}^+ H + \bar{\mathrm{p}}^+ \bar{H}\right)/2} \mathcal{P}\exp\left(\lambda_+ \oint U_+(z)\,\mathrm{d}z\right),$$

where ρ^+ is the fermion number operator, $H = \pi_a^\lambda(h_2 - h_0)$, $\bar{H} = \pi_a^\lambda(h_1 - h_3)$, $Z = \pi_a^\lambda(h_2 + h_0)$, $\mathrm{p}^+ = \mathrm{p}_1^+ + \mathrm{i}\mathrm{p}_2^+$ and

$$U_+ = \sum_{i=0}^{3} V_i^+ \pi_a^1(f_i).$$

$L^-(\lambda_-)$ may be computed similarly.

Sinh-Gordon on the Lattice

There is a well-known lattice discretisation of the sinh-Gordon model. In this picture, the chiral limits (KdV) of the monodromy matrix take the form

$$L_n^+(\mu_+) = \begin{pmatrix} u_n & \mu_+ v_n \\ \mu_+ v_n^{-1} & u_n^{-1} \end{pmatrix}, \quad L_n^-(\mu_-) = \begin{pmatrix} u_n & \mu_-^{-1} v_n^{-1} \\ \mu_-^{-1} v_n & u_n^{-1} \end{pmatrix},$$

where the u_n and v_n are operators chosen to satisfy

$$u_m v_n = q^{-\delta_{mn}} v_n u_m, \qquad q = \mathrm{e}^{-\mathrm{i}\pi b^2}.$$

In our picture, such operators are easily constructed by "discretising" the position q and momentum p of the fields:

$$u_m = \mathrm{e}^{2\pi b \mathrm{p}_m}, \qquad v_n = \mathrm{e}^{-b \mathrm{q}_n}.$$

To generalise beyond sinh-Gordon, we interpret this representation-theoretically. This involves "discretising" $\pi_q^{\pm}$, *eg*.

$$\pi_{q,n}^{+}(k_0) = u_n^{+2}, \qquad \pi_{q,n}^{+}(e_0) = \tau_q u_n^{+1} v_n^{-1},$$
$$\pi_{q,n}^{+}(k_1) = u_n^{-2}, \qquad \pi_{q,n}^{+}(e_1) = \tau_q u_n^{-1} v_n^{+1}.$$

It is easy to now check that

$$L_n^{+}(\mu_+) = \left(\pi_{q,n}^{+} \otimes \pi_a^{\mu_+}\right)\left(\mathcal{R}^{+}\right).$$

Similarly, one can obtain $L_n^{-}(\mu_-)$ from $\mathcal{R}^{-}$.

The construction guarantees that the lattice monodromy matrix

$$M(\lambda) = L_N^{-}(\lambda_-)\, L_N^{+}(\lambda_+) \cdots L_1^{-}(\lambda_-)\, L_1^{+}(\lambda_+)$$

satisfies a Yang-Baxter equation of the form $RLL = LLR$.

$N = 2$ Super Sine-Gordon on the Lattice

Repeating the above procedure for super sine-Gordon is now straight-forward, if rather intricate. We construct $\pi^{\pm}_{q,n}$ by discretising the momenta $\mathrm{p}^{\pm}_{n}$ of the continuum Cartan generators and replacing the screening operators of the nilpotent generators by the corresponding position operators $\mathrm{q}^{\pm}_{n}$.

The latter generally need modification in order to satisfy the quantum Serre relations. In this case, we multiply by $q^{-\rho^{\pm}_{n}/2}$.

It remains then to compute the L-matrices. This uses the intertwining axiom of the universal R-matrix to derive recursion relations that efficiently compute $L^{\pm}_{n}(\mu_{\pm})$ order-by-order. The finite-dimensionality of $\pi^{\mu_{\pm}}_{a}$ leads to a periodicity in the recursion which makes the computation terminate in finite time.

Conclusions and Outlook

We have specified an approach to deducing the integrable structure for a class of non-linear sigma models. Quantum integrability may then be proven à la [FF].

We have shown how to construct integrable lattice versions of these theories, thereby regularising ultraviolet divergences.

This has been tested on four distinct models with agreement between the classical, quantum-continuum and quantum-lattice structures. One may now try to solve these models.

What other models can we analyse this way?

Our methods work for integrable perturbations of free field theories. Can one characterise the sigma models which have dual descriptions of this type?