Lattice Discretisations of Integrable Sigma Models

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Motivation

The biggest buzzword in theoretical physics at the moment is:

Duality.

This refers to a collection of non-perturbative phenomena which describe (some sector of) a strongly-coupled quantum field theory in terms of some other (sector of a) weakly-coupled quantum field theory, and vice-versa.

ie. one can try to answer inaccessible questions about the former in the latter theory, and vice-versa.

An example of this is the famous AdS/CFT correspondence wherein the planar $N = 4$ super Yang-Mills theory and free IIB superstring theory on $\mathrm{AdS}_5 \times S^5$ are claimed to be dual.

Establishing such dualities requires substantial control over the relevant theories. Such control may be furnished by *eg*. quantum-integrability.

A quantum-integrable system admits an infinite number of commuting conserved charges. In favourable cases, this symmetry allows one to compute many important features of the theory, in particular, the spectrum.

Unfortunately, it is usually very hard to establish quantum-integrability. One of the main difficulties is the need for regularisation of ultraviolet divergences.

One is therefore led to ask for a regularised theory over which we still have substantial control, *ie*. we want an integrable regularisation.

One successful regularisation scheme involves discretisation.

eg. spin chains are often stated to provide integrable lattice regularisations of QFTs. However, the question of how to choose a spin chain to regularise a given QFT remains obscure.

We believe that we should prefer a lattice regularisation for which:

1. The local degrees of freedom may be related to those of the continuum theory.
2. The quantum groups underlying the integrability of the continuum and regularised theories coincide.

The latter point also facilitates the eventual solution of the theory (*eg*. in deriving TQ-relations, *etc*...).

A Proposal

We have therefore proposed a framework for studying the integrability of a certain class of non-linear sigma models and constructing integrable lattice discretisations for them.

This involves:

1. Identifying the relevant quantum group from the quantisation of the chiral halves of the interaction terms.
2. Constructing R- and L-matrices from the quantum group for the continuum and lattice theories (following the quantum inverse scattering method).

The models we consider involve bosons (compact, non-compact) and fermions. The interaction terms are restricted to being exponential. *eg*. affine Toda theories.

Outline of Talk

Quantum Affine Superalgebras

Integrability usually arises from the action of a quasitriangular Hopf (super)algebra, *eg*. a quantum affine superalgebra $\mathcal{U}_q(\widehat{\mathfrak{g}})$.

When all the real roots of $\widehat{\mathfrak{g}}$ have the same length, $\mathcal{U}_q(\widehat{\mathfrak{g}})$ is a Hopf algebra generated by E_i, F_i, $K_i = q^{H_i}$, and q^D. The relations are

$$K_i E_j = q^{A_{ij}} E_j K_i, \qquad K_i F_j = q^{-A_{ij}} F_j K_i, \qquad K_i K_j = K_j K_i,$$
$$q^D E_i = q^{\delta_{i0}} E_i q^D, \qquad q^D F_i = q^{-\delta_{i0}} F_i q^D, \qquad q^D K_i = K_i q^D,$$
$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}},$$

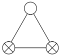
and the Serre relations. Here, A is the Cartan matrix of $\widehat{\mathfrak{g}}$.

The Serre relations for Lie algebras take the form

$$\sum_{n=0}^{1-A_{ij}} (-1)^n \begin{bmatrix} 1-A_{ij} \\ n \end{bmatrix}_q E_i^n E_j E_i^{1-A_{ij}-n} = 0,$$

and similarly for $E_i \to F_i$.

For Lie superalgebras, the Serre relations are far less regular. They also depend upon the choice of Dynkin diagram.

eg. for $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}(2|1)$,  , the Serre relations are

$$E_0^2 = E_2^2 = 0,$$
$$\big[\big[E_0, E_1\big]_{q^{-1}}, E_1\big]_q = \big[E_1, \big[E_1, E_2\big]_{q^{-1}}\big]_q = 0,$$
$$\big[E_0, \big[E_2, \big[E_0, \big[E_2, E_1\big]_{q^{-1}}\big]\big]\big]_q = \big[E_2, \big[E_0, \big[E_2, \big[E_0, E_1\big]_{q^{-1}}\big]\big]\big]_q,$$

and the same with $E_i \to F_i$. Here, $\big[A, B\big]_q = AB \mp q\, BA$.

Quantum affine superalgebras also come with a coproduct:

$$\Delta\left(E_i\right) = E_i \otimes K_i + \mathbf{1} \otimes E_i, \qquad \Delta\left(K_i\right) = K_i \otimes K_i,$$
$$\Delta\left(F_i\right) = F_i \otimes \mathbf{1} + K_i^{-1} \otimes F_i, \qquad \Delta\left(q^D\right) = q^D \otimes q^D.$$

The unit, counit and antipode are not useful for us here.

The coproduct allows us to define the tensor product of two $\mathcal{U}_q\left(\widehat{\mathfrak{g}}\right)$-modules V and W:

$$x \cdot (v \otimes w) = \Delta\left(x\right)\left(v \otimes w\right), \qquad \left(x \in \mathcal{U}_q\left(\widehat{\mathfrak{g}}\right),\ v \in V,\ w \in W\right).$$

Such tensor products are graded by the parity $p\left(\cdot\right) \in \mathbb{Z}_2$ in the super case:

$$\left(x_1 \otimes y_1\right)\left(x_2 \otimes y_2\right) = \left(-1\right)^{p(y_1)p(x_2)}\left(x_1 x_2 \otimes y_1 y_2\right).$$

R-matrices

Quantum affine superalgebras have a universal R-matrix $\mathcal{R}^+$ in (some completion of) $\mathcal{U}_q(\widehat{\mathfrak{g}}) \otimes \mathcal{U}_q(\widehat{\mathfrak{g}})$. $\mathcal{R}^+$ is (formally) invertible and satisfies the intertwining axiom

$$\mathcal{R}^+ \Delta(x) = \sigma(\Delta(x)) \mathcal{R}^+ \qquad \forall x \in \mathcal{U}_q(\widehat{\mathfrak{g}}),$$

where $\sigma(x \otimes y) = (-1)^{p(x)p(y)} y \otimes x$.

$\mathcal{R}^+$ also satisfies the abstract Yang-Baxter equation:

$$\mathcal{R}^+_{12}\mathcal{R}^+_{13}\mathcal{R}^+_{23} = \mathcal{R}^+_{23}\mathcal{R}^+_{13}\mathcal{R}^+_{12}.$$

Note that one can define a second universal R-matrix

$$\mathcal{R}^- = \sigma\left(\mathcal{R}^+\right)^{-1}$$

which also satisfies these relations!

Identifying the Quantum Group

In conformal field theory, there is a hidden quantum group symmetry that controls, *eg*. fusing and braiding relations. This symmetry can be made explicit in a free field realisation by computing the algebra generated by the screening charges.

We propose that this generalises to our integrable models: The screening charges are derived from the vertex operators representing the exponential interaction terms.

The screening charge algebra is deduced from the braiding of the vertex operators. The relations of this algebra are interpreted as Serre relations for some quantum affine superalgebra.

Unlike in CFT, this only extends to a Borel subalgebra $\mathcal{B}^+$ of the quantum group (using the zero modes). Note that superalgebras may have inequivalent Borel subalgebras.

Example: Sinh-Gordon

The sinh-Gordon model has classical action

$$S = \int \left[\frac{1}{4\pi} \partial_\mu \phi \partial^\mu \phi + \nu_+ \mathrm{e}^{+2b\phi} + \nu_- \mathrm{e}^{-2b\phi} \right] \mathrm{d}^2 z,$$

so the interaction terms give the chiral vertex operators

$$V_0^+ = \, : \mathrm{e}^{+2b\phi^+} : \quad \text{and} \quad V_1^+ = \, : \mathrm{e}^{-2b\phi^+} : .$$

From the usual vertex operator locality relation

$$: \mathrm{e}^{a\phi(z)} : \; : \mathrm{e}^{b\phi(w)} : \; = \mathrm{e}^{\mathrm{i}\pi ab} : \mathrm{e}^{b\phi(w)} : \; : \mathrm{e}^{a\phi(z)} : ,$$

we obtain the braiding relations

$$V_i^+ V_j^- = q^{A_{ij}} V_j^- V_i^+, \qquad A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},$$

where $q = \mathrm{e}^{2\pi \mathrm{i} b^2}$ and A is the Cartan matrix of $\mathcal{U}_q\left(\widehat{\mathfrak{sl}}(2)\right)$.

The screening charges are now defined by

$$Q_i^+ = \frac{1}{q - q^{-1}} \oint V_i^+(z) \frac{\mathrm{d}z}{2\pi\mathfrak{i}}.$$

Using the braiding relations for the V_i^+ and with due consideration of the contours of integration, one can search for independent relations. We find two which hold for all q:

$$\left(Q_i^+\right)^3 Q_j^+ - [3]_q \left(Q_i^+\right)^2 Q_j^+ Q_i^+ \\ + [3]_q\, Q_i^+ Q_j^+ \left(Q_i^+\right)^2 - Q_j^+ \left(Q_i^+\right)^3 = 0,$$

These are the quantum Serre relations of $\mathcal{U}_q\left(\widehat{\mathfrak{sl}}(2)\right)$.

With $H_0 = +\mathrm{p}/b$ and $H_1 = -\mathrm{p}/b$, where p is the momentum mode of ϕ^+, we conclude that the quantum symmetry algebra underlying the integrability of sinh-Gordon is the Borel subalgebra of $\mathcal{U}_q\left(\widehat{\mathfrak{sl}}(2)_0\right)$ (at level 0).

Example: $N = 2$ Super Sine-Gordon

The $N = 2$ super sine-Gordon model has action

$$S' = \int \Big[\tfrac{1}{4\pi}(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2) + \tfrac{1}{2\pi}\left(\bar{\psi}^+\partial_-\psi^+ + \bar{\psi}^-\partial_+\psi^-\right)$$
$$-b\nu_+ \mathrm{e}^{+b\phi_1}\left(\bar{\psi}^+\bar{\psi}^-\mathrm{e}^{+\mathrm{i}b\phi_2} + \psi^+\psi^-\mathrm{e}^{-\mathrm{i}b\phi_2}\right) - b\nu_-\mathrm{e}^{-b\phi_1}\left(\psi^+\psi^-\mathrm{e}^{+\mathrm{i}b\phi_2} + \bar{\psi}^+\bar{\psi}^-\mathrm{e}^{-\mathrm{i}b\phi_2}\right)$$
$$+4\pi\nu_-^2\,\mathrm{e}^{+2b\phi_1} - 8\pi\nu_-\nu_+\cos(2b\phi_2) + 4\pi\nu_+^2\,\mathrm{e}^{-2b\phi_1}\Big]\,\mathrm{d}^2z.$$

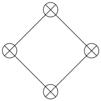
Treating the terms on the second line as interactions and those on the third line as counterterms generated by renormalisation as $b \to 0$, we obtain four chiral vertex operators:

$$V_0^+ = \bar{\psi}^+ : e^{-b\left(\phi_1^+ - \mathrm{i}\phi_2^+\right)} : , \qquad V_1^+ = \psi^+ : e^{+b\left(\phi_1^+ + \mathrm{i}\phi_2^+\right)} : ,$$
$$V_2^+ = \bar{\psi}^+ : e^{+b\left(\phi_1^+ - \mathrm{i}\phi_2^+\right)} : , \qquad V_3^+ = \psi^+ : e^{-b\left(\phi_1^+ + \mathrm{i}\phi_2^+\right)} : .$$

The screening charges $Q_i^+ = \oint V_i^+(z)\,\mathrm{d}z$ satisfy

$$\left(Q_i^+\right)^2 = 0, \qquad Q_i^+ Q_{i+2}^+ + Q_{i+2}^+ Q_i^+ = 0,$$
$$Q_{i-1}^+ Q_i^+ Q_{i+1}^+ Q_i^+ - Q_{i+1}^+ Q_i^+ Q_{i-1}^+ Q_i^+$$
$$+ [2]_q Q_i^+ Q_{i-1}^+ Q_{i+1}^+ Q_i^+ - Q_i^+ Q_{i+1}^+ Q_i^+ Q_{i-1}^+ + Q_i^+ Q_{i-1}^+ Q_i^+ Q_{i+1}^+ = 0,$$

where $q = \mathrm{e}^{2\pi \mathrm{i} b^2}$ and i is taken $\mathrm{mod}\ 4$. These are (some of) the quantum Serre relations of $\mathcal{U}_q\left(\widehat{\mathfrak{sl}}(2|2)\right)$ (there are others with degrees in $4\mathbb{Z}_+ + 2$) with the Dynkin diagram

 .

ie. the quantum symmetry algebra underlying the integrability of $N = 2$ super sine-Gordon is a Borel subalgebra of $\mathcal{U}_q\left(\widehat{\mathfrak{sl}}(2|2)_0\right)$.

Classical Considerations

To analyse a classical integrable system, one first looks for a zero-curvature condition

$$\left[\partial_t - U_t(\lambda), \partial_x - U_x(\lambda)\right] = 0,$$

which implies the equations of motion. This gives rise to a monodromy matrix $M(\lambda)$ as a path-ordered integral around the cylinder:

$$M(\lambda) = \mathcal{P} \exp \int_0^R U_x(\lambda)\, dx.$$

The transfer matrix is then

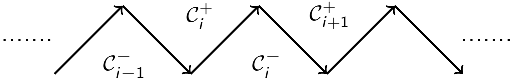
$$T(\lambda) = \operatorname{tr} M(\lambda),$$

and the infinitely many integrals of motion are now obtained through asymptotic expansion in the spectral parameter λ.

It is useful to pass to light-cone coordinates where we have

$$\left[\partial_+ - U_+(\lambda), \partial_- - U_-(\lambda)\right] = 0.$$

We deform the contour around the cylinder:



The monodromy matrix then becomes

$$M(\lambda) = L_N^-(\lambda) L_N^+(\lambda) \cdots L_1^-(\lambda) L_1^+(\lambda),$$
$$L_i^\pm(\lambda) = \mathcal{P} \exp \int_{\mathcal{C}_i^\pm} U_\pm(\lambda) \, \mathrm{d}x_\pm.$$

In the continuum, we typically take $N = 1$. $N > 1$ is useful to understand the idea behind discretisation!

Quantisation

If one can fix a gauge so that the $U_{\pm}(\lambda)$ contain only the exponential interaction terms, we may quantise naturally by replacing these terms by their normally-ordered equivalents.

If one knows the quantum group responsible for the integrability, quantum inverse scattering lets us construct the Lax matrices $L^{\pm}(\lambda)$ directly from the universal R-matrices:

$$L^{\pm}(\lambda) = \left(\pi_q^{\pm} \otimes \pi_a^{\lambda}\right)\left(\mathcal{R}^{\pm}\right) \qquad (N = 1, \text{ say}).$$

Here, π_a^{λ} is a finite-dimensional evaluation rep of the quantum affine superalgebra and the $\pi_q^{\pm}$ are certain infinite-dimensional reps of the Borel subalgebras $\mathcal{B}^{\pm}$.

The R-matrix itself is given (up to normalisation) by

$$R(\lambda, \mu) = \left(\pi_a^{\lambda} \otimes \pi_a^{\mu}\right)\left(\mathcal{R}^{\pm}\right).$$

Example: Sinh-Gordon

If we choose the $\mathcal{U}_q\left(\widehat{\mathfrak{sl}}(2)_0\right)$-representation

$$\pi_a^\lambda(E_0)=\begin{pmatrix}0&0\\\lambda^{-1}&0\end{pmatrix},\quad \pi_a^\lambda(F_0)=\begin{pmatrix}0&\lambda\\0&0\end{pmatrix},\quad \pi_a^\lambda(H_0)=\begin{pmatrix}-1&0\\0&1\end{pmatrix},$$
$$\pi_a^\lambda(E_1)=\begin{pmatrix}0&\lambda^{-1}\\0&0\end{pmatrix},\quad \pi_a^\lambda(F_1)=\begin{pmatrix}0&0\\\lambda&0\end{pmatrix},\quad \pi_a^\lambda(H_1)=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

and the $\mathcal{B}^\pm$-representations

$$\pi_q^\pm(H_0)=\pm\mathrm{p}^\pm/b,\qquad \pi_q^\pm(H_1)=\mp\mathrm{p}^\pm/b$$

with either

$$\pi_q^+(E_0)=Q_0^+,\quad \pi_q^+(E_1)=Q_1^+,\quad \text{or}\quad \pi_q^-(F_0)=Q_0^-,\quad \pi_q^-(F_1)=Q_1^-,$$
$$Q_0^\pm=\pm\frac{1}{q-q^{-1}}\oint :\mathrm{e}^{+2b\phi^\pm(z)}:\frac{\mathrm{d}z}{2\pi\mathrm{i}},\qquad Q_1^\pm=\pm\frac{1}{q-q^{-1}}\oint :\mathrm{e}^{-2b\phi^\pm(z)}:\frac{\mathrm{d}z}{2\pi\mathrm{i}},$$

then...

... we reproduce the monodromy matrix of BLZ,

$$L^+(\lambda_+) = \begin{pmatrix} \mathrm{e}^{-\pi b p^+} & 0 \\ 0 & \mathrm{e}^{+\pi b p^+} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \lambda_+ Q_0^+ \\ \lambda_+ Q_1^+ & \mathbf{1} \end{pmatrix},$$

$$L^-(\lambda_-) = \begin{pmatrix} \mathbf{1} & \lambda_-^{-1} Q_1^- \\ \lambda_-^{-1} Q_0^- & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\pi b p^-} & 0 \\ 0 & \mathrm{e}^{+\pi b p^-} \end{pmatrix},$$

though in a different "gauge".

This is, of course, formal and when $b \in \mathbb{R}$, there are ultraviolet divergence issues. For $b = \mathrm{i}\beta$, $\beta \in \mathbb{R}$ (sine-Gordon), these may be controllable.

The representation-theoretic approach is not needed in this case, but it generalises readily. More importantly, regularisation will be needed for sinh-Gordon and our approach can be readily adapted for when we discretise on a lattice.

Example: $N=2$ Super Sine-Gordon

For $\mathcal{U}_q\left(\widehat{\mathfrak{sl}}(2|2)_0\right)$, let π_a^λ be the "defining" evaluation representation and π_q^+ be again given by the screening charges, supplemented by

$$\pi_q^+(H_0)=+\left(\mathrm{p}_1^+-\mathrm{i}\mathrm{p}_2^+\right)/2b, \qquad \pi_q^+(H_1)=-\left(\mathrm{p}_1^++\mathrm{i}\mathrm{p}_2^+\right)/2b,$$
$$\pi_q^+(H_2)=-\left(\mathrm{p}_1^+-\mathrm{i}\mathrm{p}_2^+\right)/2b, \qquad \pi_q^+(H_3)=+\left(\mathrm{p}_1^++\mathrm{i}\mathrm{p}_2^+\right)/2b.$$

The resulting monodromy matrix is

$$L^+(\lambda_+)=\mathrm{e}^{2\pi b^2\rho^+ Z}e^{\mathrm{i}\pi b\left(\mathrm{p}^+H+\bar{\mathrm{p}}^+\bar{H}\right)}\mathcal{P}\exp\left(\lambda_+\oint U_+(z)\,\mathrm{d}z\right),$$

where ρ^+ is the fermion number operator, $H=\pi_a^\lambda(H_2-H_0)$, $\bar{H}=\pi_a^\lambda(H_1-H_3)$, $Z=\pi_a^\lambda(H_2+H_0)$, $\mathrm{p}^+=\mathrm{p}_1^++\mathrm{i}\mathrm{p}_2^+$ and

$$U_+(z)=\sum_{i=0}^{3}V_i^+(z)\,\pi_a^1(F_i).$$

$L^-(\lambda_-)$ may be computed similarly.

Sinh-Gordon on the Lattice

There is a well-known lattice discretisation of the sinh-Gordon model. One introduces a minimal distance (ultraviolet cutoff) Λ and convert fields to operators by averaging over intervals of length Λ.

For this to be a lattice discretisation, we need a Lax matrix $L_n(\lambda;\Lambda)$ for which:

1. The continuum matrix $U_x(\lambda)$ is recovered as $\Lambda \to 0$:

$$L_n(\lambda;\Lambda) = \mathbf{1} + \Lambda\, U_x(\lambda) + \mathcal{O}\left(\Lambda^2\right).$$

2. $L_n(\lambda;\Lambda)$ satisfies the Yang-Baxter equation:

$$R^{12}(\lambda/\mu)\, L_n^{13}(\lambda;\Lambda)\, L_n^{23}(\mu;\Lambda) = L_n^{23}(\mu;\Lambda)\, L_n^{13}(\lambda;\Lambda)\, R^{12}(\lambda/\mu).$$

Such a $L_n(\lambda;\Lambda)$ is known!

We can easily recover this known lattice discretisation, up to gauge choices, using our representation-theoretic method. The procedure is as follows:

1. Keep the auxiliary rep π_a^λ from the continuum.
2. Modify the momenta p appearing in $\pi_q^\pm(H_i)$ by affixing a lattice site label n.
3. Replace the vertex operators appearing in $\pi_q^+(E_i)$ or $\pi_q^-(F_i)$ by their position modes q, affixing a lattice label n, and removing the contour integral.

For $\mathcal{U}_q(\widehat{\mathfrak{sl}}(2)_0)$, the discretised quantum reps become

$$\pi_{q,n}^+(H_0) = \frac{+\mathrm{p}_n^+}{b}, \qquad \pi_{q,n}^+(E_0) = \frac{e^{+2b\mathrm{q}_n^+}}{q-q^{-1}},$$

$$\pi_{q,n}^+(H_1) = \frac{-\mathrm{p}_n^+}{b}, \qquad \pi_{q,n}^+(E_1) = \frac{e^{-2b\mathrm{q}_n^+}}{q-q^{-1}},$$

and similarly for $\pi_{q,n}^-$.

This procedure guarantees that the $\pi^{\pm}_{q,n}(H_i)$ will all commute and that the commutation relations of $\pi^{\pm}_{q,n}(H_i)$ with $\pi^{+}_{q,n}(E_i)$ or $\pi^{-}_{q,n}(F_i)$ will be correct.

For $\mathcal{U}_q(\widehat{\mathfrak{sl}}(2)_0)$, $\pi^{+}_{q,n}(E_0)$ commutes with $\pi^{+}_{q,n}(E_1)$, hence the Serre relations are satisfied. $\pi^{+}_{q,n}$ then defines a rep (and the same is true for $\pi^{-}_{q,n}$). The $\pi^{\pm}_{q,n}$ are the lattice-discretised quantum reps.

In general, this procedure will not give operators satisfying the Serre relations. One must then modify "by hand" the definition of the $\pi^{+}_{q,n}(E_i)$ or $\pi^{-}_{q,n}(F_i)$ by exponentials of the momenta — discretisation is not "algorithmic"!

Once the discretised reps are known, the lattice Lax matrix $L_n^{\pm}(\lambda)$ can be computed recursively using the intertwining axiom

$$\mathcal{R}^{\pm}\Delta(x) = \sigma(\Delta(x))\,\mathcal{R}^{\pm}, \qquad \forall x \in \mathcal{U}_q(\widehat{\mathfrak{g}}),$$

and the expansions

$$\mathcal{R}^{+} = q^{T}\left[\mathbf{1}\otimes\mathbf{1} + (q-q^{-1})\sum_i (-1)^{p(i)}\, E_i \otimes F_i + \dots\right],$$

$$\mathcal{R}^{-} = \left[\mathbf{1}\otimes\mathbf{1} - (q-q^{-1})\sum_i F_i \otimes E_i + \dots\right] q^{-T},$$

where T is a (known) tensor bilinear in the H_i.

For $\mathcal{U}_q\left(\widehat{\mathfrak{sl}}(2)_0\right)$, applying π_a^λ to F_0, F_1, F_0F_1 and F_1F_0 gives a basis of the 2×2 matrices. So, only $\pi_{q,n}^+(E_0)$, $\pi_{q,n}^+(E_1)$ and $\pi_{q,n}^+(E_0E_1)=\pi_{q,n}^+(E_1E_0)=\left(q-q^{-1}\right)^{-2}$ appear by weight considerations.

We may therefore truncate the expansion of $\mathcal{R}^+$ to first order when applying $\pi_a^\lambda\otimes\pi_{q,n}^+$ (and similarly for $\mathcal{R}^-$ and $\pi_a^\lambda\otimes\pi_{q,n}^-$).

Thus,

$$L_n^+(\lambda_+)=\begin{pmatrix} e^{-2\pi i b p_n^+} & \lambda_+ e^{-2\pi i b p_n^+}e^{2bq_n^+} \\ \lambda_+ e^{2\pi i b p_n^+}e^{-2bq_n^+} & e^{2\pi i b p_n^+}\end{pmatrix},$$
$$L_n^-(\lambda_-)=\begin{pmatrix} e^{-2\pi i b p_n^+} & \lambda_-^{-1} e^{2bq_n^+}e^{2\pi i b p_n^+} \\ \lambda_-^{-1} e^{-2bq_n^+}e^{-2\pi i b p_n^+} & e^{2\pi i b p_n^+}\end{pmatrix},$$

which indeed agrees with the known lattice discretisation. Note that the Yang-Baxter equation is satisfied by construction.

$N = 2$ Super Sine-Gordon on the Lattice

Repeating the above procedure for super sine-Gordon and $\mathcal{U}_q\left(\widehat{\mathfrak{sl}}(2|2)_0\right)$ is now straight-forward, if a little more involved.

We construct $\pi^{\pm}_{q,n}$ as before, modifying the discretised screening charges by a factor of $e^{2\pi b^2 \rho^{\pm}_n}$ (recall that ρ is the fermion number) in order to satisfy the quantum Serre relations.

Since $\pi^{\mu_\pm}_a$ is four-dimensional, we only need the expansions of $\mathcal{R}^{\pm}$ to third order. In fact, we can determine these expansions at the level of the Lax matrices, deriving recursion relations for the unknown coefficients.

The result is an integrable lattice discretisation which regularises the $N = 2$ super sine-Gordon model. The actual formulae for $L^{\pm}_n(\lambda)$ are not particularly exciting — see Sec. 7.5 of arXiv:1102.5716 for the gory detail.

Conclusions and Outlook

We have specified an approach to deducing the integrable structure for a class of non-linear sigma models. Quantum integrability may then be proven à la Feigin and Frenkel.

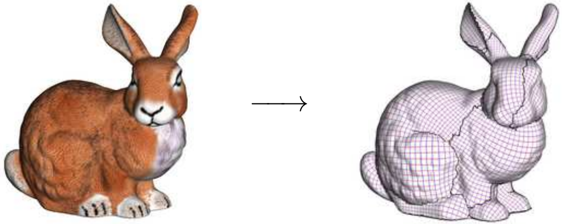
We have shown how to construct integrable lattice versions of these theories, thereby regularising ultraviolet divergences.

This has been tested on four distinct models with agreement between the classical, quantum-continuum and quantum-lattice results. One may now try to solve these models.

What other models can we analyse this way?

Our methods work for integrable perturbations of free field theories. Can one characterise the sigma models which have dual descriptions of this type? Is there a "free field" description of integrable sigma models generalising the conformal case?

Thankyou!



[Now we can discretise anything![1]]

[1] Subject to ethics approval...