Fractional Level Wess-Zumino-Witten Models, Modular Transformations and Verlinde Formulae

David Ridout

Department of Theoretical Physics & Mathematical Sciences Institute,
Australian National University

December 10, 2012

- DR: arXiv:0810.3532, arXiv:1001.3960, arXiv:1012.2905 [hep-th].
- T Creutzig and DR: arXiv:1205.6513 [hep-th], sequel in preparation...

Background

Ancient History (pre-2000)

Fractional level WZW modules

A few teething problems...

Modern History (post-2000)

Avoiding the mistakes of the past!

The case when $k = -\frac{1}{2}$

Breaking News (circa 2012)

$k = -\frac{1}{2}$ solved!

Future Directions

Affine Kac-Moody Algebras

Recall that a complex semisimple Lie algebra $\mathfrak{g}$ with Killing form $\kappa(\cdot,\cdot)$ yields an (untwisted) affine Kac-Moody algebra:

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}\left[t; t^{-1}\right] \oplus \mathbb{C}K, \qquad \left[x \otimes t^m, K\right] = 0,$$
$$\left[x \otimes t^m, y \otimes t^n\right]_{\widehat{\mathfrak{g}}} = \left[x, y\right]_{\mathfrak{g}} \otimes t^{m+n} + m\kappa(x, y)\delta_{m+n,0}K.$$

Define the vacuum module $\mathcal{L}_0$ to be the irreducible highest weight $\widehat{\mathfrak{g}}$-module of highest weight 0.

The central element K acts on the vacuum module $\mathcal{L}_0$ as a multiple k of the identity, called the level.

The vacuum module $\mathcal{L}_0$ carries the structure of a vertex operator algebra, denoted by $\widehat{\mathfrak{g}}_k$. K acts as $k\,\mathrm{id}$ on all $\widehat{\mathfrak{g}}_k$-modules.

Wess-Zumino-Witten Models

The case where the level k is a non-negative integer describes strings propagating on the connected, simply-connected, compact Lie group whose complexified Lie algebra is $\mathfrak{g}$:

- The vertex algebra is rational in category $\mathcal{O}$: $\widehat{\mathfrak{g}}_k$-modules are semisimple and there are finitely many irreducibles.
- The spectrum of $\widehat{\mathfrak{g}}_k$-modules may be identified with direct sums of integrable $\widehat{\mathfrak{g}}$-modules. These modules are unitary.
- There is a tensor product $\times$ for vertex algebras (called fusion) which closes on the spectrum.
- Every irreducible $\widehat{\mathfrak{g}}_k$-module $\mathcal{M}$ has a unique conjugate $\widehat{\mathfrak{g}}_k$-module $\mathsf{w}(\mathcal{M})$ for which

$$\mathcal{M} \times \mathsf{w}(\mathcal{M}) = \mathcal{L}_0 \oplus \cdots$$

- The $\widehat{\mathfrak{g}}_k$-module characters

$$\mathrm{ch}\big[\mathcal{M}\big] = \mathrm{tr}_{\mathcal{M}}\, z_1^{H_0^{(1)}} \cdots z_r^{H_0^{(r)}} q^{L_0 - c/24}, \qquad c = \frac{k \dim \mathfrak{g}}{k + \mathsf{h}^{\vee}},$$

  span a unitary representation of the modular group $\mathsf{SL}(2;\mathbb{Z})$.

- The S-transformation ($z = \mathsf{e}^{2\pi\mathrm{i}\zeta}$, $q = \mathsf{e}^{2\pi\mathrm{i}\tau}$)

$$\mathsf{S} \colon \ (\zeta,\tau) \mapsto (\zeta/\tau, -1/\tau), \quad \mathrm{ch}\big[\mathcal{M}_i\big] \mapsto \sum_j \mathsf{S}_{ij}\mathrm{ch}\big[\mathcal{M}_j\big]$$

  is intimately related to fusion via the Verlinde formula:

$$\mathcal{M}_i \times \mathcal{M}_j = \bigoplus_k \mathbf{N}_{ij}{}^k \mathcal{M}_k, \quad \mathbf{N}_{ij}{}^k = \sum_\ell \frac{\mathsf{S}_{i\ell}\mathsf{S}_{j\ell}\mathsf{S}_{\ell k}^*}{\mathsf{S}_{0\ell}}.$$

- Moreover, S^2 is a permutation matrix realising conjugation.

Beyond non-negative integer levels...

When the level k is not a non-negative integer, $\widehat{\mathfrak{g}}_k$ is still a vertex operator algebra ($k \neq -\mathsf{h}^\vee$).

Such vertex algebras may be relevant for the description of strings propagating on non-compact Lie groups.

The spectrum of $\widehat{\mathfrak{g}}_k$-modules is generically uncountable. One only expects constraints for a countable collection of levels k (including the non-negative integers).

The constrained levels which are not non-negative integers are known as the fractional levels. We focus on them in what follows...

Motivation: The Coset Construction

The coset construction of Goddard, Kent and Olive concretely realises the minimal models $\mathbf{M}\left(k+2,k+3\right)$ $(k=0,1,2,\ldots)$ in terms of unitary Wess-Zumino-Witten models:

$$\mathbf{M}\left(k+2,k+3\right)=\frac{\widehat{\mathfrak{sl}}\left(2\right)_k\otimes\widehat{\mathfrak{sl}}\left(2\right)_1}{\widehat{\mathfrak{sl}}\left(2\right)_{k+1}}.$$

This proved that these minimal models were unitary and completed the classification of unitary highest weight representations for the Virasoro algebra.

More precisely, $\mathbf{M}\left(k+2,k+3\right)$ is realised as the commutant of $\widehat{\mathfrak{sl}}\left(2\right)_{k+1}$ in $\widehat{\mathfrak{sl}}\left(2\right)_k\otimes\widehat{\mathfrak{sl}}\left(2\right)_1$.

Fractional Genesis

The complete set of (Virasoro) minimal models is, however, parametrised by two integers:

$$\mathbf{M}(p,q), \quad p,q = 2,3,4,\ldots \text{ with } p < q \text{ and } \gcd\{p,q\} = 1.$$

These are non-unitary for $q - p > 1$.

Comparing central charges suggests that

$$\mathbf{M}(p,q) \overset{?}{=} \frac{\widehat{\mathfrak{sl}}(2)_k \oplus \widehat{\mathfrak{sl}}(2)_1}{\widehat{\mathfrak{sl}}(2)_{k+1}}, \qquad k = \frac{3p-2q}{q-p}.$$

Question: Are there fractional level Wess-Zumino-Witten models corresponding to these values of k?

Evidence for Fractional Level WZW Models

One consistency requirement for these models is that the partition function (a sum of products of characters) should be invariant under modular transformations.

Kac and Wakimoto found a class of admissible (irreducible, highest-weight) modules whose characters spanned a representation of the modular group $\mathsf{SL}(2;\mathbb{Z})$.

This class is non-empty if and only if k is of the form

$$k = \frac{3p-2q}{q-p}, \quad p,q = 2,3,4,\ldots \text{ with } p<q \text{ and } \gcd\{p,q\} = 1.$$

ie. that which is required to get a minimal model as a coset.

Adamović and Milas later proved that these admissible modules were precisely the category $\mathcal{O}$ objects for the vertex algebra $\widehat{\mathfrak{sl}}(2)_k$.

Beware of overconfidence...

This led to many attempts to investigate fractional level models constructed from these admissible modules:

- Koh & Sorba (1988),
- Bernard & Felder (1990),
- Mathieu & Walton (1990),
- Awata & Yamada (1992),
- Ramgoolam arXiv:hep-th/9301121,
- Feigin & Malikov arXiv:hep-th/9310004,
- Andreev arXiv:hep-th/9504082,
- Petersen, Rasmussen & Yu arXiv:hep-th/9607129,
- Furlan, Ganchev & Petkova arXiv:hep-th/9608018.

But, while modular invariant partition functions could be found, other consistency requirements were found to be problematic.

Trouble in Paradise

In particular, the relationship between the modular S-matrix S and fusion did not quite meet expectations:

- The Verlinde formula gave negative (integer) fusion coefficients.
- Computing fusion rules via singular vector decoupling gave different fusion coefficients (with their own problems).
- The conjugation matrix S^2 also contained negative (integer) entries.
- In general, the conjugate of an admissible module was not itself admissible.

Many *ad hoc* "solutions" proclaimed — but none were universally agreed upon. Di Francesco, Mathieu & Sénéchal wrote that the fractional level theories may possess an "intrinsic sickness".

A New Approach

The category of admissible highest weight modules is not even closed under conjugation. A larger category must be sought.

Gaberdiel investigated the closure under fusion for $\widehat{\mathfrak{sl}}(2)_{-4/3}$:

- The fusion of two admissibles can result in an irreducible non-highest weight module whose conformal dimensions are not bounded below.
- The fusion of an admissible and its conjugate can result in new irreducible non-highest weight modules whose conformal dimensions are bounded below.
- The fusion of these new representations can result in indecomposable modules of logarithmic type.

Adamović & Milas knew about some of these new irreducibles; Feigin, Semikhatov & Tipunin and Maldacena & Ooguri knew about the rest. The indecomposables were new.

An Older Approach

Shortly thereafter, Lesage, Mathieu, Rasmussen & Saleur studied $\widehat{\mathfrak{sl}}(2)_{-1/2}$ as a subalgebra of the product of a lorentzian boson (Heisenberg vertex algebra) and a pair of symplectic fermions.

$\widehat{\mathfrak{sl}}(2)_{-1/2}$ is also a subalgebra of the $\beta\gamma$ ghost vertex algebra.

They proposed distinct "theories" according as to which, if any, of the fermion fields have antiderivatives. All but two proposals are ruled out by degeneracy of their two-point functions (DR).

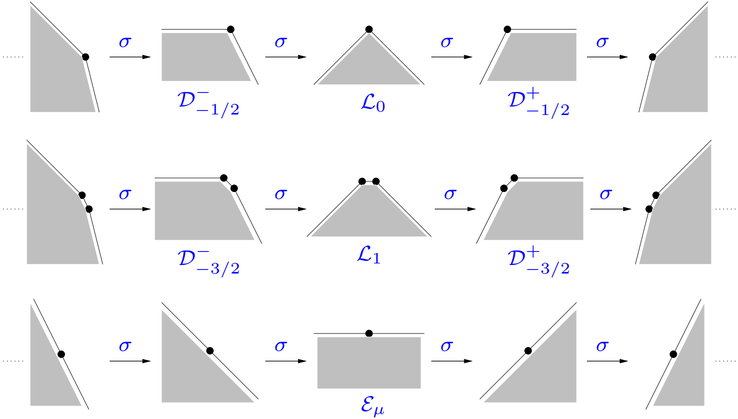
The results suggest logarithmic indecomposables are present in one theory (inherited from symplectic fermions), but not in the other.

Recent work (Creutzig & DR) casts doubt on the consistency of the non-logarithmic theory: $\widehat{\mathfrak{sl}}(2)_{-1/2}$ is logarithmic, like $\widehat{\mathfrak{sl}}(2)_{-4/3}$.

$\widehat{\mathfrak{sl}}(2)_{-1/2}$: The Spectrum

The irreducible spectrum is (Adamović–Milas + spectral flow σ):

$$\sigma^\ell(\mathcal{L}_0), \quad \sigma^\ell(\mathcal{L}_1), \quad \sigma^\ell(\mathcal{E}_\mu) \qquad (\ell \in \mathbb{Z},\ \mu \in \mathbb{R}/2\mathbb{Z}).$$



$\widehat{\mathfrak{sl}}(2)_{-1/2}$: Fusion Rules

The fusion rules are given by

$$\mathcal{L}_\lambda \times \mathcal{L}_{\lambda'} = \mathcal{L}_{\lambda+\lambda'}, \quad \mathcal{L}_\lambda \times \mathcal{E}_{\mu'} = \mathcal{E}_{\lambda+\mu'}, \quad \mathcal{L}_\lambda \times \mathcal{S}_{\lambda'} = \mathcal{S}_{\lambda+\lambda'},$$

$$\mathcal{E}_\mu \times \mathcal{E}_{\mu'} = \begin{cases} \mathcal{S}_{\mu+\mu'} & \text{if } \mu+\mu' \in \mathbb{Z}, \\ \sigma\left(\mathcal{E}_{\mu+\mu'+1/2}\right) \oplus \sigma^{-1}\left(\mathcal{E}_{\mu+\mu'-1/2}\right) & \text{otherwise.} \end{cases}$$

$$\mathcal{E}_\mu \times \mathcal{S}_{\lambda'} = \sigma^{-2}\left(\mathcal{E}_{\lambda'+\mu+1}\right) \oplus 2\,\mathcal{E}_{\lambda'+\mu} \oplus \sigma^{2}\left(\mathcal{E}_{\lambda'+\mu+1}\right),$$

$$\mathcal{S}_\lambda \times \mathcal{S}_{\lambda'} = \sigma^{-2}\left(\mathcal{S}_{\lambda+\lambda'+1}\right) \oplus 2\,\mathcal{S}_{\lambda+\lambda'} \oplus \sigma^{2}\left(\mathcal{S}_{\lambda+\lambda'+1}\right),$$

where $\lambda, \lambda' \in \mathbb{Z}/2\mathbb{Z}$ and $\mu, \mu' \in \mathbb{R}/2\mathbb{Z}$.

This is extended to the entire spectrum using the well-known conjecture that "fusion respects spectral flow":

$$\sigma^{\ell_1}\left(\mathcal{M}\right) \times \sigma^{\ell_2}\left(\mathcal{N}\right) = \sigma^{\ell_1+\ell_2}\left(\mathcal{M} \times \mathcal{N}\right).$$

$\widehat{\mathfrak{sl}}(2)_{-1/2}$: Logarithmic Indecomposables

The $\mathcal{S}_\lambda$ $(\lambda = 0, 1)$ generated by fusion are logarithmic indecomposables uniquely determined by their structure diagrams.

$$\begin{tikzcd}
& \mathcal{L}_0 \arrow[dl] \arrow[dr] & \\
\sigma^{-2}(\mathcal{L}_1) \arrow[dr] & \mathcal{S}_0 & \sigma^{2}(\mathcal{L}_1) \arrow[dl] \\
& \mathcal{L}_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{L}_1 \arrow[dl] \arrow[dr] & \\
\sigma^{-2}(\mathcal{L}_0) \arrow[dr] & \mathcal{S}_1 & \sigma^{2}(\mathcal{L}_0) \arrow[dl] \\
& \mathcal{L}_1 &
\end{tikzcd}$$

These modules are logarithmic because the Virasoro mode L_0 acts non-semisimply with rank 2 Jordan blocks. This leads to correlators with logarithmic singularities.

$\widehat{\mathfrak{sl}}(2)_{-1/2}$: Characters

The characters of the admissible irreducibles are

$$\operatorname{ch}[\mathcal{L}_0] = \frac{1}{2}\left[\frac{\eta(q)}{\vartheta_4(z;q)} + \frac{\eta(q)}{\vartheta_3(z;q)}\right], \qquad \operatorname{ch}[\mathcal{E}_\mu] = \frac{z^\mu}{\eta(q)^2}\sum_{n\in\mathbb{Z}} z^{2n},$$

$$\operatorname{ch}[\mathcal{L}_1] = \frac{1}{2}\left[\frac{\eta(q)}{\vartheta_4(z;q)} - \frac{\eta(q)}{\vartheta_3(z;q)}\right],$$

supplemented by

$$\operatorname{ch}[\sigma^\ell(\mathcal{M})](z;q) = z^{\ell k} q^{\ell^2 k/4} \operatorname{ch}[\mathcal{M}](zq^{\ell/2};q).$$

One therefore gets periodicities:

$$\operatorname{ch}[\sigma^{\ell-1}(\mathcal{L}_\lambda)] + \operatorname{ch}[\sigma^{\ell+1}(\mathcal{L}_{\lambda+1})] = 0.$$

Among the $\sigma^\ell(\mathcal{L}_\lambda)$, there are only four independent characters!

$\widehat{\mathfrak{sl}}(2)_{-1/2}$: A Quotient Ring

The modules

$$\sigma^{\ell-1}(\mathcal{L}_\lambda) \oplus \sigma^{\ell+1}(\mathcal{L}_{\lambda+1}), \quad \sigma^\ell(\mathcal{E}_\mu), \quad \sigma^\ell(\mathcal{S}_\lambda)$$

span an ideal of the fusion ring. The quotient Gr is free of rank 4 and is isomorphic to the character ring of the $\sigma^\ell(\mathcal{L}_\lambda)$.

A basis is given by the admissible highest weight representations:

$$[\mathcal{L}_0], \quad [\mathcal{L}_1], \quad [\mathcal{D}^+_{-1/2}], \quad [\mathcal{D}^+_{-3/2}].$$

The fusion product descends to Gr as:

$[\times]$	$[\mathcal{L}_0]$	$[\mathcal{L}_1]$	$[\mathcal{D}^+_{-1/2}]$	$[\mathcal{D}^+_{-3/2}]$
$[\mathcal{L}_0]$	$[\mathcal{L}_0]$	$[\mathcal{L}_1]$	$[\mathcal{D}^+_{-1/2}]$	$[\mathcal{D}^+_{-3/2}]$
$[\mathcal{L}_1]$	$[\mathcal{L}_1]$	$[\mathcal{L}_0]$	$[\mathcal{D}^+_{-3/2}]$	$[\mathcal{D}^+_{-1/2}]$
$[\mathcal{D}^+_{-1/2}]$	$[\mathcal{D}^+_{-1/2}]$	$[\mathcal{D}^+_{-3/2}]$	$-[\mathcal{L}_1]$	$-[\mathcal{L}_0]$
$[\mathcal{D}^+_{-3/2}]$	$[\mathcal{D}^+_{-3/2}]$	$[\mathcal{D}^+_{-1/2}]$	$-[\mathcal{L}_0]$	$-[\mathcal{L}_1]$

$\widehat{\mathfrak{sl}}(2)_{-1/2}$: Modular Transformations

The modular S-matrix and conjugation matrix are

$$\mathsf{S} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & \mathfrak{i} & \mathfrak{i} \\ -1 & -1 & \mathfrak{i} & \mathfrak{i} \end{pmatrix}, \quad \mathsf{S}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The negative coefficients in the conjugation matrix are natural because, *eg*. the conjugate of $\mathcal{D}^+_{-1/2}$ is $\mathcal{D}^-_{1/2} = \sigma^{-1}(\mathcal{L}_0)$ and

$$\operatorname{ch}[\sigma^{-1}(\mathcal{L}_0)] = -\operatorname{ch}[\sigma(\mathcal{L}_1)] = -\operatorname{ch}[\mathcal{D}^+_{-3/2}].$$

Similarly, the negative coefficients appearing in the product on Gr are precisely those obtained from the Verlinde formula.

Where are we now?

We have seen that the failure of the ancients to reconcile fusion via singular vector decoupling with the Verlinde formula has two main sources:

- An unsatisfactory choice of representation category.
- The map from the fusion ring to the character ring has a very large kernel.

We seem to have fixed the category problem, introducing uncountably many additional irreducibles. What can we do about the second?

The key is the observation (Feigin, Semikhatov & Tipunin; Lesage, Mathieu, Rasmussen & Saleur) that the characters of the $\sigma^{\ell}(\mathcal{L}_{\lambda})$ do not converge everywhere in the z-plane.

The $\mathrm{ch}\big[\sigma^\ell(\mathcal{L}_\lambda)\big]$ have poles which limit their convergence regions to annuli in z.

The periodicity $\mathrm{ch}\big[\sigma^{\ell-1}(\mathcal{L}_\lambda)\big] + \mathrm{ch}\big[\sigma^{\ell+1}(\mathcal{L}_{\lambda+1})\big] = 0$ relates characters with disjoint convergence regions, hence must be understood in the sense of meromorphic continuation.

As formal power series, such sums are not zero. In fact (DR),

$$\mathrm{ch}\big[\sigma^{\ell-1}(\mathcal{L}_\lambda)\big] + \mathrm{ch}\big[\sigma^{\ell+1}(\mathcal{L}_{\lambda+1})\big] = \mathrm{ch}\big[\sigma^{\ell}(\mathcal{E}_{\lambda+1/2})\big].$$

Moreover, the latter may be interpreted as a distribution supported at the poles of the $\mathrm{ch}\big[\sigma^\ell(\mathcal{L}_\lambda)\big]$.

In this distributional setting, the character ring is isomorphic to the (Grothendieck) fusion ring.

$\widehat{\mathfrak{sl}}(2)_{-1/2}$: Modular Transformations (Again)

The characters of the $\sigma^\ell(\mathcal{E}_\mu)$ become

$$\mathrm{ch}\left[\sigma^\ell(\mathcal{E}_\mu)\right] = \frac{\mathrm{e}^{\mathrm{i}\pi\ell^2\tau/4}}{\eta(\tau)^2} \sum_{m\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\pi m(\mu-\ell/2)} \delta(2\zeta+\ell\tau-m).$$

The S-transformation is

$$\mathrm{ch}\left[\sigma^\ell(\mathcal{E}_\mu)\right](\zeta/\tau, -1/\tau) = \sum_{\ell'\in\mathbb{Z}} \int_{-1}^{1} \mathsf{S}_{(\ell,\mu)(\ell',\mu')} \mathrm{ch}\left[\sigma^{\ell'}(\mathcal{E}_{\mu'})\right] \mathrm{d}\mu',$$

$$\mathsf{S}_{(\ell,\mu)(\ell',\mu')} = \frac{1}{2} \frac{|\tau|}{-\mathrm{i}\tau} \mathrm{e}^{\mathrm{i}\pi(\ell\ell'/2-\ell\mu'-\ell'\mu)}.$$

Because of the phases $|\tau|/-\mathrm{i}\tau$, the characters span a projective unitary representation of $\mathsf{SL}(2;\mathbb{Z})$.

$\widehat{\mathfrak{sl}}(2)_{-1/2}$: Conjugation

These phases will cancel when applying modular transformations to bulk partition functions.

S^2 only represents conjugation up to a phase:

$$\sum_{m \in \mathbb{Z}} \int_{-1}^{1} \mathsf{S}_{(\ell,\lambda)(m,\mu)} \mathsf{S}_{(m,\mu)(n,\nu)} \mathrm{d}\mu = -\frac{|\tau|^2}{\tau^2} \delta_{\ell+n=0} \delta(\lambda + \nu = 0).$$

But, the phase again cancels for bulk (non-chiral) characters.

Up to this phase, the conjugation "matrix" S^2 is a permutation (no negative entries)!

$\widehat{\mathfrak{sl}}(2)_{-1/2}$: Resolutions

The characters of the $\sigma^\ell(\mathcal{L}_\lambda)$ are found, as distributions, by constructing resolutions:

$$\cdots \longrightarrow \sigma^{\ell+9}(\mathcal{E}_{\lambda+1/2}) \longrightarrow \sigma^{\ell+7}(\mathcal{E}_{\lambda-1/2}) \longrightarrow \sigma^{\ell+5}(\mathcal{E}_{\lambda+1/2}) \longrightarrow \sigma^{\ell+3}(\mathcal{E}_{\lambda-1/2}) \longrightarrow \sigma^{\ell+1}(\mathcal{E}_{\lambda+1/2}) \longrightarrow \sigma^{\ell}(\mathcal{L}_\lambda) \longrightarrow 0$$

$$\Rightarrow \qquad \operatorname{ch}\left[\sigma^\ell(\mathcal{L}_\lambda)\right] = \sum_{j=0}^{\infty} (-1)^j \operatorname{ch}\left[\sigma^{\ell+2j+1}(\mathcal{E}_{\lambda+j+1/2})\right]$$

$$\Rightarrow \quad \operatorname{ch}\left[\sigma^\ell(\mathcal{L}_\lambda)\right](\zeta/\tau, -1/\tau) = \sum_{\ell' \in \mathbb{Z}} \int_{-1}^{1} \mathsf{S}_{\overline{(\ell,\lambda)}(\ell',\mu')} \operatorname{ch}\left[\sigma^{\ell'}(\mathcal{E}_{\mu'})\right] \mathrm{d}\mu',$$

$$\mathsf{S}_{\overline{(\ell,\lambda)}(\ell',\mu')} = \frac{1}{2} \frac{|\tau|}{-\mathrm{i}\tau} \frac{\mathrm{e}^{\mathrm{i}\pi(\ell\ell'/2 - \ell\mu' - \ell'\lambda)}}{2\cos(\pi\mu')}.$$

$\widehat{\mathfrak{sl}}(2)_{-1/2}$: The Verlinde Formula

The S-matrix phases cancel in the Verlinde formula:

$$\mathbf{N}_{(\ell,\lambda)(m,\mu)}{}^{(n,\nu)} = \sum_{\ell' \in \mathbb{Z}} \int_{-1}^{1} \frac{\mathsf{S}_{(\ell,\lambda)(\ell',\mu')} \mathsf{S}_{(m,\mu)(\ell',\mu')} \mathsf{S}^*_{(\ell',\mu')(n,\nu)}}{\mathsf{S}_{\overline{(0,0)}(\ell',\mu')}} \, \mathrm{d}\mu'$$

$$= \delta_{n=\ell+m+1} \delta\left(\nu = \lambda + \mu + \tfrac{1}{2}\right) + \delta_{n=\ell+m-1} \delta\left(\nu = \lambda + \mu - \tfrac{1}{2}\right).$$

The (Grothendieck) fusion coefficients are non-negative integers:

$$\left[\sigma^{\ell}(\mathcal{E}_{\lambda})\right] \times \left[\sigma^{m}(\mathcal{E}_{\mu})\right] = \sum_{n \in \mathbb{Z}} \int_{-1}^{1} \mathbf{N}_{(\ell,\lambda)(m,\mu)}{}^{(n,\nu)} \left[\sigma^{n}(\mathcal{E}_{\nu})\right] \mathrm{d}\nu$$

$$= \left[\sigma^{\ell+m+1}\left(\mathcal{E}_{\lambda+\mu+1/2}\right)\right] \oplus \left[\sigma^{\ell+m-1}\left(\mathcal{E}_{\lambda+\mu-1/2}\right)\right].$$

This is a (Grothendieck) fusion rule. The Verlinde formula also works for the fusion rules involving the $\sigma^{\ell}(\mathcal{L}_{\lambda})$.

Future Directions

- The Verlinde formula has also been "fixed" for $\widehat{\mathfrak{sl}}(2)_{-4/3}$ (Creutzig & DR). An article generalising this to all admissible k is in preparation.
- This new paradigm for fractional level WZW models is being checked (Canagasabey & DR) against the motivating coset story.
- Homological aspects of the new category need to be settled, *eg.* the $\mathcal{S}_\lambda$ and irreducible $\mathcal{E}_\mu$ should be projective and injective.
- Can fractional level modular invariants for $\widehat{\mathfrak{sl}}(2)_k$ be classified?
- Our success suggests that a detailed algebraic understanding of the $\mathrm{SL}(2;\mathbb{R})$ (AdS_3) WZW model may be in reach (*cf.* Maldacena & Ooguri).
- This is one instance of our programme (Creutzig & DR) to rewrite the role of modular transformations in logarithmic conformal field theory. Other instances include $\widehat{\mathfrak{gl}}(1|1)$ and the triplet algebras $W(1,p)$.

Danke!

(Little Salmon Bay, Rottnest Island)