The Wess-Zumino-Witten Model on $\mathsf{SL}(2;\mathbb{R})$

David Ridout

Department of Theoretical Physics & Mathematical Sciences Institute,
Australian National University

June 25, 2013

Ancient History (pre-2000)

The WZW Model on SU (2)

Modern History (post-2000)

Strings on AdS_3

Breaking News (post-2010)

Fractional Level WZW Models

To the future...

The WZW Model on $\mathrm{SU}(2)$

Witten noted that adding a Wess-Zumino term to a non-linear sigma model restores conformal invariance.

The Wess-Zumino-Witten action is

$$S[g] = \frac{k}{8\pi}\int_{\Sigma} \kappa\left(g^*\vartheta, \star g^*\vartheta\right) + 2\pi\mathrm{i}\int_{\Gamma} \tilde{g}^* H,$$

where:

- g maps a Riemann surface Σ into $\mathrm{SU}(2)$.
- $\tilde{g}$ extends g to Γ with $\partial\Gamma = \Sigma$ (note $\mathrm{H}_2(\mathrm{SU}(2);\mathbb{R}) = 0$).
- ϑ is the canonical 1-form and κ the Killing form of $\mathrm{SU}(2)$.
- $\star$ is the Hodge star on Σ,and $k \in \mathbb{R}$ is the level.
- $H = \dfrac{k}{24\pi^2}\kappa\left(\vartheta, \mathrm{d}\vartheta\right)$ represents k in $\mathrm{H}^3(\mathrm{SU}(2);\mathbb{R}) = \mathbb{R}$.

Quantisation

The Feynman amplitudes $\mathrm{e}^{-S[g]}$ do not depend on the choice of Γ and $\tilde{g}$ if $k \in \mathbb{Z}$ (so $[H] \in \mathrm{H}^3(\mathrm{SU}(2);\mathbb{Z}) = \mathbb{Z}$).

Changing the sign of k reverses the orientation, so take $k \in \mathbb{N}$.

Standard quantisation gives the symmetry algebra $\mathcal{U}_k \otimes \mathcal{U}_k$, where:

- $\widehat{\mathfrak{sl}}(2) = \mathfrak{sl}(2;\mathbb{C}) \otimes \mathbb{C}[t;t^{-1}] \oplus \mathbb{C}K$.
- $\left[J \otimes t^m, J' \otimes t^n\right] = \left[J, J'\right] \otimes t^{m+n} + m\kappa\left(J, J'\right)\delta_{m+n=0}K$.
- $\mathcal{U}$ is the universal enveloping algebra of $\widehat{\mathfrak{sl}}(2)$.
- $\mathcal{U}_k = \dfrac{\mathcal{U}}{\langle K - k\mathbf{1}\rangle}$.

The quantum state space is therefore built from level k modules of the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}(2)$.

Spectrum

The vacuum module (highest weight 0) carries the structure of a vertex algebra.

Assuming irreducibility, the relation ($k \in \mathbb{N}$)

$$E_{-1}^{k+1}|0\rangle = 0$$

restricts the vertex algebra modules to the irreducible, integrable $\widehat{\mathfrak{sl}}(2)_k$-modules:

$$\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_k.$$

We obtain a rational CFT with quantum state space

$$\mathcal{H} = \bigoplus_{\lambda=0}^{k} (\mathcal{L}_\lambda \otimes \mathcal{L}_\lambda).$$

The WZW Model on $\mathsf{SL}(2;\mathbb{R})$

The action is again

$$S[g] = \frac{k}{8\pi} \int_{\Sigma} \kappa\left(g^* \vartheta, \star g^* \vartheta\right) + 2\pi \mathfrak{i} \int_{\Gamma} \tilde{g}^* H$$

with $\mathsf{H}_2(\mathsf{SL}(2;\mathbb{R});\mathbb{R}) = 0$, but $\mathsf{H}^3(\mathsf{SL}(2;\mathbb{R});\mathbb{R}) = 0$.

The level k is therefore not quantised.

The symmetry algebra is again $\mathcal{U}_k \otimes \mathcal{U}_k$ and the vacuum module again carries the structure of a vertex algebra.

But, imposing irreducibility of the vacuum module gives no vertex-algebraic constraints on the spectrum (for generic k).

How then should we proceed?

Strings on AdS$_3$

In hep-th/0001053, Maldacena and Ooguri proposed a spectrum for the WZW model on AdS$_3$, the universal cover of $\mathsf{SL}(2;\mathbb{R})$.

More precisely, they proposed a spectrum for $k < -2$, motivated by generalising a no-ghost theorem of Hwang (and others).

Recall that the *unitary* representations of AdS$_3$ include the:

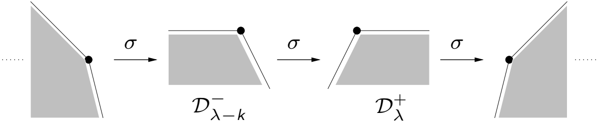
- Highest weight discrete series $\overline{\mathcal{D}}_\lambda^+$ with $\lambda < 0$.
- Lowest weight discrete series $\overline{\mathcal{D}}_\lambda^-$ with $\lambda > 0$.
- Principal continuous series $\overline{\mathcal{E}}_{\lambda;\Delta}$ with $\lambda \in \mathbb{R}/2\mathbb{Z}$ and $\Delta \leqslant -\frac{1}{2}$.

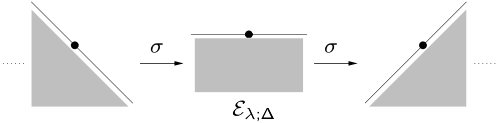
Here, λ parametrises the weight (mod 2) and Δ is the eigenvalue of the quadratic Casimir.

These also define unitary representations of $\mathfrak{sl}(2;\mathbb{C})$ which we induce to get $\mathcal{U}_k$-modules $\mathcal{D}_\lambda^{\pm}$ and $\mathcal{E}_{\lambda;\Delta}$ (which are not unitary).

The $\mathcal{U}_k$-modules may then be twisted by spectral flow automorphisms σ^ℓ with $\ell \in \mathbb{Z}$:

$$\sigma^\ell(E_n) = E_{n-\ell}, \qquad \sigma^\ell(H_n) = H_n - \delta_{n=0}\ell K,$$
$$\sigma^\ell(F_n) = F_{n+\ell}, \qquad \sigma^\ell(K) = K.$$





The proposal of Maldacena and Ooguri is that the spectrum is:

- $\sigma^\ell(\mathcal{D}^+_\lambda)$, with $k+1 < \lambda < -1$ and $\ell \in \mathbb{Z}$.
- $\sigma^\ell(\mathcal{E}_{\lambda;\Delta})$ with $\lambda \in \mathbb{R}/2\mathbb{Z}$, $\Delta \leqslant -\frac{1}{2}$ and $\ell \in \mathbb{Z}$.

The quantum state space is then

$$\mathcal{H} = \bigoplus_{\ell \in \mathbb{Z}} \left[\int_{k+1}^{-1} \sigma^\ell(\mathcal{D}^+_\lambda) \otimes \sigma^\ell(\mathcal{D}^+_\lambda) \, \mathrm{d}\lambda \right. \\ \left. \oplus \int_{-\infty}^{-1/2} \int_{\mathbb{R}/2\mathbb{Z}} \sigma^\ell(\mathcal{E}_{\lambda;\Delta}) \otimes \sigma^\ell(\mathcal{E}_{\lambda;\Delta}) \, \mathrm{d}\lambda \mathrm{d}\Delta \right].$$

They performed many string-theoretic checks of this proposal and it is generally accepted as a good one (for $k < -2$).

Questions one should ask

- Where is the vacuum (where is the vertex algebra)?

  String theorists conjecture that as $k \to -\infty$, the spectrum goes into that of the laplacian on $L^2(\mathrm{AdS}_3)$. The vacuum does not appear because the identity function is not normalisable.

  But, the identity is not normalisable in $L^2(\mathbb{R}^n)$, yet the vacuum appears in the corresponding WZW model.

- Are the induced $\mathcal{U}_k$-modules irreducible?

  The $\sigma^\ell(\mathcal{D}_\lambda^+)$ are, but this is not clear for the $\sigma^\ell(\mathcal{E}_{\lambda;\Delta})$.

- Why start with unitary $\mathfrak{sl}(2;\mathbb{R})$-modules?

  The string theory should be unitary, but the CFT is not.

- Do the fusion rules close on the proposed spectrum?

  Not known, though a proposal was made by Baron and Núñez.

Things to tantalise the brain...

Maldacena and Ooguri have proposed a spectrum whose stringy reduction has passed many consistency checks, *eg*. unitarity. Checking the proposal for the CFT is more delicate.

In 1102.4196 [hep-th], Fjelstad addresses this:

- The two- and three-point functions computed by Maldacena and Ooguri are consistent with the existence of conjugate fields.
- The tensor product rules of unitary $\mathfrak{sl}(2;\mathbb{C})$-modules appear to depend upon the choice of topology.
- If one allows "non-normalisable states", then tensor products of unitary modules need not be completely reducible, suggesting that the same is true for fusion.

ie. the AdS_3 WZW model may be a logarithmic CFT.

Fractional Level WZW Models

In 1986, Kent suggested that nice CFTs generalising the $\mathrm{SU}(2)$ WZW model should exist for fractional levels $k = -2 + \frac{u}{v}$, where $u, v \in \mathbb{Z}_{\geqslant 2}$ and $\gcd\{u, v\} = 1$.

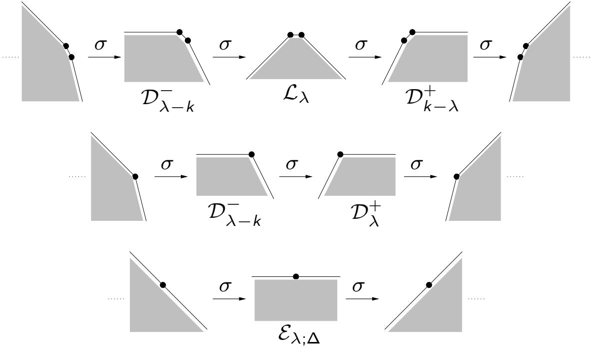
In 1988, Kac and Wakimoto announced that these were precisely the levels where there were a finite set of modules whose characters carried a representation of $\mathrm{SL}(2; \mathbb{Z})$.

At the same time, Verlinde published his famous formula giving fusion coefficients in terms of the modular S-transformation. Koh and Sorba immediately applied it to the modules of Kac and Wakimoto with peculiar results.

These results have only been fully explained recently (Creutzig and DR 1306.4388 [hep-th]).

These fractional level models have similar algebraic properties to the AdS_3 WZW model, though $k < -2$ is no longer satisfied.

However, the spectrum is constrained (which makes them easier to study). It includes spectral flows of discrete and continuous series modules, but there are more ($\mathcal{L}_\lambda$) including the vacuum module.



Fractional level models may tell us what to expect from the $\mathsf{SL}(2;\mathbb{R})$ and AdS_3 CFTs! For example...

There is an infinite series of orbifold modular invariants suggesting the covering

$$\mathsf{AdS}_3 \longrightarrow \cdots \longrightarrow \mathsf{SL}(2;\mathbb{R}) \longrightarrow \mathsf{PSL}(2;\mathbb{R}).$$

The diagonal modular invariant takes the form

$$Z = \sum_{\ell \in \mathbb{Z}} \sum_{\Delta} \int_{\mathbb{R}/2\mathbb{Z}} \left| \mathrm{ch}\left[\sigma^{\ell}(\mathcal{E}_{\lambda;\Delta})\right] \right|^2 \, \mathrm{d}\lambda,$$

where the sum over Δ is constrained to a finite set.

Modular invariance depends upon whether characters are treated as meromorphic functions (wrong) or distributions (right).

The quantum state space has a much more intricate structure. In particular, it does not factorise into left- and right-movers.

For each Δ, there are two choices of λ for which the $\sigma^{\ell}(\mathcal{E}_{\lambda;\Delta})$ are indecomposable sums of the $\sigma^{\ell'}(\mathcal{D}_{\lambda}^{\pm})$ and/or $\sigma^{\ell''}(\mathcal{L}_{\lambda})$.

These combine to form non-chiral indecomposable modules that are built from infinitely many irreducibles. The vacuum module $\mathcal{L}_0 \otimes \mathcal{L}_0$ is absorbed into one of these conglomerations:

$$
\begin{tikzcd}
 & \sigma^2(\mathcal{L}_1)\otimes\sigma^2(\mathcal{L}_1) \arrow[dl] \arrow[dr] \arrow[drr, dotted] & & & \mathcal{L}_0\otimes\mathcal{L}_0 \arrow[dl] \arrow[dll, dotted] \arrow[dr] \arrow[drr, dotted] & & & \sigma^{-2}(\mathcal{L}_{-1})\otimes\sigma^{-2}(\mathcal{L}_{-1}) \arrow[dll, dotted] \arrow[dl] \arrow[dr] & \\
\cdots \arrow[dr] & & \sigma^2(\mathcal{L}_1)\otimes\mathcal{L}_0 \arrow[dl] \arrow[drr, dotted] & \mathcal{L}_0\otimes\sigma^2(\mathcal{L}_1) \arrow[dr] \arrow[dll, dotted] & & \mathcal{L}_0\otimes\sigma^{-2}(\mathcal{L}_{-1}) \arrow[dl] \arrow[drr, dotted] & \sigma^{-2}(\mathcal{L}_{-1})\otimes\mathcal{L}_0 \arrow[dr] \arrow[dll, dotted] & & \cdots \arrow[dl] \\
 & \sigma^2(\mathcal{L}_1)\otimes\sigma^2(\mathcal{L}_1) & & & {\color{blue}\mathcal{L}_0\otimes\mathcal{L}_0} & & & \sigma^{-2}(\mathcal{L}_{-1})\otimes\sigma^{-2}(\mathcal{L}_{-1}) &
\end{tikzcd}
$$

The theory is logarithmic, meaning that the Virasoro zero-modes L_0 and $\bar{L}_0$ act non-semisimply.

Future Directions

Maldacena and Ooguri conclude hep-th/0111180 with

> *The* $\mathsf{SL}(2;\mathbb{R})$ *WZW model has an interesting algebraic structure which should be explored further.*

They are right!

- We need to honestly compute the fusion rules to see if the proposed spectrum is indeed closed.
- We expect that it is not and that fusion will generate reducible yet indecomposable chiral modules.
- We can be guided by our fractional level results to propose a consistent CFT spectrum.
- Presumably, this spectrum will lead to the same stringy reduction... or maybe there are new sectors???