Non-negative integer Verlinde coefficients for fractional level WZW models

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- DR: arXiv:0810.3532, arXiv:1001.3960, arXiv:1012.2905 [hep-th].
- T Creutzig and DR: arXiv:1205.6513, arXiv:1306.4388 [hep-th]

The affine Kac-Moody algebra $\widehat{\mathfrak{sl}}(2)$

Recall that the complex semisimple Lie algebra $\mathfrak{sl}(2)$ with Killing form $\kappa(\cdot,\cdot)$ yields an (untwisted) affine Kac-Moody algebra:

$$\widehat{\mathfrak{sl}}(2) = \mathfrak{sl}(2) \otimes \mathbb{C}\left[t;t^{-1}\right] \oplus \mathbb{C}K,$$
$$\left[x \otimes t^m, y \otimes t^n\right]_{\widehat{\mathfrak{sl}}(2)} = \left[x,y\right]_{\mathfrak{sl}(2)} \otimes t^{m+n} + m\kappa(x,y)\delta_{m+n,0}K.$$

Define the vacuum module $\mathcal{V}$ to be the irreducible highest weight $\widehat{\mathfrak{sl}}(2)$-module of highest weight 0.

The central element K acts on $\mathcal{V}$ as a multiple k of the identity, called the level.

$\mathcal{V}$ carries the structure of a vertex operator algebra, denoted by $\widehat{\mathfrak{sl}}(2)_k$. K acts as $k\,\mathrm{id}$ on all $\widehat{\mathfrak{sl}}(2)_k$-modules.

Wess-Zumino-Witten models

The case where the level k is a non-negative integer describes strings propagating on $\mathsf{SU}(2)$ (or $\mathsf{SO}(3)$):

- The vertex algebra is rational: $\widehat{\mathfrak{sl}}(2)_k$-modules are integrable, unitary and semisimple; there are finitely many irreducibles.
- There is a tensor product $\times$ for vertex algebras (called fusion) which closes on the spectrum.
- Every irreducible $\widehat{\mathfrak{sl}}(2)_k$-module $\mathcal{M}$ has a unique conjugate $\widehat{\mathfrak{sl}}(2)_k$-module $\mathcal{M}^*$ for which

$$\mathcal{M} \times \mathcal{M}^* = \mathcal{V} \oplus \cdots.$$

- The $\widehat{\mathfrak{sl}}(2)_k$-module characters

$$\mathrm{ch}\big[\mathcal{M}\big] = \mathrm{tr}_{\mathcal{M}}\, z^{h_0} q^{L_0 - c/24}, \qquad c = \frac{3k}{k+2},$$

span a unitary representation of the modular group $\mathsf{SL}(2;\mathbb{Z})$.

- The S-transformation ($z = \mathrm{e}^{2\pi \mathrm{i} \zeta}$, $q = \mathrm{e}^{2\pi \mathrm{i} \tau}$)

$$\mathsf{S} \colon\ (\zeta, \tau) \mapsto (\zeta/\tau, -1/\tau), \quad \mathrm{ch}\big[\mathcal{M}_i\big] \mapsto \sum_j \mathsf{S}_{i \to j} \mathrm{ch}\big[\mathcal{M}_j\big]$$

is intimately related to fusion via the Verlinde formula:

$$\mathcal{M}_i \times \mathcal{M}_j = \bigoplus_k \mathbf{N}_{ij}{}^k \mathcal{M}_k, \quad \mathbf{N}_{ij}{}^k = \sum_\ell \frac{\mathsf{S}_{i \to \ell} \mathsf{S}_{j \to \ell} \mathsf{S}^*_{\ell \to k}}{\mathsf{S}_{\mathcal{V} \to \ell}} \in \mathbb{N}.$$

- Moreover, S^2 is a permutation matrix realising conjugation.

Beyond non-negative integer levels...

When the level k is not a non-negative integer, $\widehat{\mathfrak{sl}}(2)_k$ is still a vertex operator algebra ($k \neq -2$).

Such vertex algebras are relevant for the description of strings propagating on $\mathsf{SL}(2;\mathbb{R})$ (and its covers/quotients).

The spectrum of $\widehat{\mathfrak{sl}}(2)_k$-modules is generically uncountable. But, there is a countable collection of (admissible) levels k (including the non-negative integers) for which the irreducible vacuum module is not universal. This constrains the spectrum of the vertex operator algebra.

The constrained levels which are not non-negative integers are known as the fractional levels. We focus on them in what follows...

Fractional motivation: GKO cosets

Kent (86) proposed the existence of fractional level WZW models in order to extend the Goddard-Kent-Olive coset construction of the Virasoro minimal models to the non-unitary case:

$$\mathbf{M}(u,v) \stackrel{?}{=} \frac{\widehat{\mathfrak{sl}}(2)_k \oplus \widehat{\mathfrak{sl}}(2)_1}{\widehat{\mathfrak{sl}}(2)_{k+1}}, \qquad k = \frac{3u-2v}{v-u}.$$

Here, $u, v = 2, 3, 4, \ldots$ with $\gcd\{u,v\} = 1$.

The GKO construction requires $v - u = \pm 1$, hence $k \in \mathbb{N}$.

But, do such fractional level models exist?

Fractional evidence: Modular transformations

Consistency requires that the partition function (a sum of products of characters) should be invariant under modular transformations.

Kac and Wakimoto (88) found classes of irreducible highest-weight $\widehat{\mathfrak{sl}}(2)$-modules whose characters span a representation of the modular group $\mathsf{SL}(2;\mathbb{Z})$.

This class is non-empty if and only if k is of the form

$$k = \frac{3u - 2v}{v - u}, \quad u, v = 2, 3, 4, \ldots \text{ with } \gcd\{u, v\} = 1.$$

ie. that which is required to get a minimal model as a coset.

Adamović and Milas later proved (95) that these were precisely the category $\mathcal{O}$ irreducibles for the vertex algebra $\widehat{\mathfrak{sl}}(2)_k$.

Trouble in paradise

Modular invariant partition functions were constructed, but Koh and Sorba (88) found that Verlinde's formula for fusion did not quite work!

- The Verlinde formula gave negative (integer) fusion coefficients.
- Computing fusion rules via singular vector decoupling gave different fusion coefficients (with their own problems).
- The conjugation matrix S^2 also contained negative (integer) entries (so not a permutation).

Many *ad hoc* "solutions" proclaimed — but none were universally agreed upon. Di Francesco, Mathieu & Sénéchal wrote (97) that the fractional level theories may possess an "intrinsic sickness".

A new hope

The problem is categorical!

Category $\mathcal{O}$ is not closed under conjugation. Gaberdiel (01) investigated the closure under fusion for $k = -\frac{4}{3}$:

- Fusion may result in irreducible non-highest weight modules whose conformal dimensions are not bounded below.
- Fusion also generates new irreducible non-highest weight modules whose conformal dimensions are bounded below.
- The fusion of these new representations can result in indecomposable modules of logarithmic type.

Adamović & Milas (95) knew about some of these new irreducibles; Feigin, Semikhatov & Tipunin (97) and Maldacena & Ooguri (00) knew about the rest. The indecomposables were new.

Curing the sickness

We therefore reject category $\mathcal{O}$, instead enlarging the category to encompass all twisted, relaxed highest-weight $\widehat{\mathfrak{sl}}(2)_k$-modules.

"Relaxed" refers to dropping the requirement that a (relaxed) highest weight state is annihilated by e_0.

"Twisted" refers to composing with spectral flow automorphisms:

$$\sigma^{\ell}(e_n) = e_{n-\ell}, \quad \sigma^{\ell}(h_n) = h_n - \delta_{n,0}\ell k, \quad \sigma^{\ell}(f_n) = f_{n+\ell},$$
$$\sigma^{\ell}(L_0) = L_0 - \tfrac{1}{2}\ell h_0 + \tfrac{1}{4}\ell^2 k.$$

The action is given by ($J \in \widehat{\mathfrak{sl}}(2)_k$, $v \in \mathcal{M}$, $\sigma^{\ell}(v) \in \sigma^{\ell}(\mathcal{M})$):

$$J \cdot \sigma^{\ell}(v) = \sigma^{\ell}\left(\sigma^{-\ell}(J)\, v\right).$$

A complete (irreducible) spectrum

The irreducible spectrum in this category is:

- $\sigma^\ell(\mathcal{L}_{r,s})$ where $\mathcal{L}_{r,s}$ has highest weight $\lambda_{r,s} = r - 1 - \frac{u}{v}s$,
- $\sigma^\ell(\mathcal{E}_{\lambda;\Delta_{r,s}})$ where $\mathcal{E}_{\lambda;\Delta_{r,s}}$ has relaxed highest weight $\lambda \in \mathbb{R}/2\mathbb{Z}$.

Here, $r = 1, 2, \ldots, u-1$, $s = 1, 2, \ldots, v-1$ and

$$k = -2 + \frac{u}{v}, \text{ with } u, v = 2, 3, 4, \ldots \text{ and } \gcd\{u, v\} = 1.$$

The minimal conformal dimension of states in $\mathcal{L}_{r,s}$ or $\mathcal{E}_{\lambda;\Delta_{r,s}}$ is

$$\Delta_{r,s} = \frac{(vr - us)^2 - v^2}{4uv}.$$

(This almost looks like a Virasoro minimal model...)

Results

Theorem

If $\chi_{r,s}^{\mathrm{Vir}}(q)$ *denotes the Virasoro minimal model character of the irreducible Virasoro module of central charge and highest weight*

$$c = 1 - \frac{6(v-u)^2}{uv}, \quad h_{r,s} = \frac{(vr-us)^2 - (v-u)^2}{4uv},$$

then

$$\begin{aligned} \mathrm{ch}\left[\mathcal{E}_{\lambda;\Delta_{r,s}}\right](z;q) &= \frac{z^{\lambda}\chi_{r,s}^{\mathrm{Vir}}(q)}{\eta(q)^2} \sum_{n\in\mathbb{Z}} z^{2n} \\ &= \frac{\chi_{r,s}^{\mathrm{Vir}}(\tau)}{\eta(\tau)^2} \sum_{m\in\mathbb{Z}} \mathsf{e}^{\mathrm{i}\pi m\lambda}\delta(2\zeta - m). \end{aligned}$$

Theorem

The characters of the $\sigma^{\ell}\left(\mathcal{E}_{\lambda;\Delta_{r,s}}\right)$ *carry a projective representation of* $\mathsf{SL}\left(2;\mathbb{Z}\right)$ *of uncountably-infinite dimension. The S-matrix is*

$$\mathsf{S}_{\sigma^{\ell}(\mathcal{E}_{\lambda;\Delta_{r,s}})\to\sigma^{\ell'}(\mathcal{E}_{\lambda';\Delta_{r',s'}})} = \frac{1}{2}\frac{\left|\tau\right|}{-\mathfrak{i}\tau}\mathsf{e}^{-\mathrm{i}\pi(k\ell\ell'+\ell\lambda'+\ell'\lambda)}\mathsf{S}^{\mathrm{Vir}}_{(r,s)\to(r',s')},$$

where the $\mathbf{M}\left(u,v\right)$ *S-matrix is given by*

$$\mathsf{S}^{\mathrm{Vir}}_{(r,s)\to(r',s')} = -2\sqrt{\frac{2}{uv}}\left(-1\right)^{rs'+r's}\sin\frac{v\pi rr'}{u}\sin\frac{u\pi ss'}{v}.$$

Note that this S-matrix is unitary and symmetric. Moreover, S^2 is, up to the phase $-\mathsf{e}^{-2\mathrm{i}\arg\tau}$, the permutation describing conjugation.

The $\mathcal{E}_{\lambda_{r,s};\Delta_{r,s}}$ are reducible but indecomposable:

$$0 \longrightarrow \mathcal{L}_{r,s} \longrightarrow \mathcal{E}_{\lambda_{r,s};\Delta_{r,s}} \longrightarrow \mathcal{L}^*_{u-r,v-s} \longrightarrow 0.$$

This leads to resolutions for the highest weight modules $\mathcal{L}_{r,s}$, hence characters, hence S-transformation formulae.

Theorem

The S-matrix entries for the vacuum module $\mathcal{V} = \sigma^{-1}\big(\mathcal{L}_{u-1,v-1}\big)$ are

$$\mathsf{S}_{\mathcal{V} \to \sigma^{\ell'}(\mathcal{E}_{\lambda';\Delta_{r',s'}})} = \frac{1}{2} \frac{|\tau|}{-\mathfrak{i}\tau} \frac{\mathsf{S}^{\mathrm{Vir}}_{(1,1)\to(r',s')}}{2\cos\left(\pi\lambda'\right) + (-1)^{r'}\, 2\cos\left(k\pi s'\right)}.$$

We can now substitute into the Verlinde formula.

Theorem

The *Verlinde products* *of the* $\mathcal{E}_{\lambda_{r,s};\Delta_{r,s}}$ *are*

$$\mathcal{E}_{\lambda;\Delta_{r,s}} \times \mathcal{E}_{\lambda';\Delta_{r',s'}} = \sum_{r'',s''} \mathsf{N}^{\mathrm{Vir}}_{(r,s)(r',s')}{}^{(r'',s'')} \cdot \left(\sigma\left(\mathcal{E}_{\lambda+\lambda'-k;\Delta_{r'',s''}}\right) + \sigma^{-1}\left(\mathcal{E}_{\lambda+\lambda'+k;\Delta_{r'',s''}}\right)\right) + \sum_{r'',s''} \left(\mathsf{N}^{\mathrm{Vir}}_{(r,s)(r',s'-1)}{}^{(r'',s'')} + \mathsf{N}^{\mathrm{Vir}}_{(r,s)(r',s'+1)}{}^{(r'',s'')} \right) \mathcal{E}_{\lambda+\lambda';\Delta_{r'',s''}},$$

where $\mathsf{N}^{\mathrm{Vir}}_{(r,s)(r',s')}{}^{(r'',s'')}$ *denotes the* $\mathbf{M}(u,v)$ *fusion coefficients.*

The right-hand side need not be a direct sum — the Verlinde formula can only detect the Grothendieck ring of fusion.

Theorem

$$\mathcal{E}_{\lambda;\Delta_{r,s}} \times \mathcal{L}_{r',s'} = \sum_{r'',s''} {\mathsf{N}^{\mathrm{Vir}}_{(r,s)(r',s'+1)}}^{(r'',s'')} \mathcal{E}_{\lambda+\lambda_{r',s'};\Delta_{r'',s''}} + \sum_{r'',s''} {\mathsf{N}^{\mathrm{Vir}}_{(r,s)(r',s')}}^{(r'',s'')} \sigma\left(\mathcal{E}_{\lambda+\lambda_{r',s'+1};\Delta_{r'',s''}}\right),$$

$$\mathcal{L}_{r,s} \times \mathcal{L}_{r',s'} = \begin{cases} \sum_{r'',s''} {\mathsf{N}^{\mathrm{Vir}}_{(r,s)(r',s')}}^{(r'',s'')} \sigma\left(\mathcal{E}_{\lambda_{r'',s+s'+1};\Delta_{r'',s''}}\right) \\ \quad + \sum_{r''} {\mathsf{N}^{\mathrm{Vir}}_{(r,1)(r',1)}}^{(r'',1)} \mathcal{L}_{r'',s+s'}, & \textit{if } s+s'<v, \\ \sum_{r'',s''} {\mathsf{N}^{\mathrm{Vir}}_{(r,s+1)(r',s'+1)}}^{(r'',s'')} \sigma\left(\mathcal{E}_{\lambda_{r'',s+s'+1};\Delta_{r'',s''}}\right) \\ \quad + \sum_{r''} {\mathsf{N}^{\mathrm{Vir}}_{(r,1)(r',1)}}^{(r'',1)} \sigma\left(\mathcal{L}_{u-r'',s+s'-v+1}\right), & \textit{if } s+s' \geqslant v. \end{cases}$$

Note all coefficients are manifestly non-negative!

Future Directions

- We can check that quotienting by the $\sigma^{\ell}\left(\mathcal{E}_{\lambda;\Delta_{r,s}}\right)$ recovers the formulae of Koh and Sorba (including negative coefficients).
- Like to understand the indecomposable structures appearing in the fusion products, *eg*. the projective covers of the $\sigma^{\ell}\left(\mathcal{L}_{r,s}\right)$.
- Can fractional level modular invariants for $\widehat{\mathfrak{sl}}\left(2\right)_k$ be classified? Old results are incorrect as spectrum was wrong!
- Suggests that fusion for the $\mathsf{SL}\left(2;\mathbb{R}\right)$ (AdS_3) WZW model may finally be computed (*cf*. Maldacena & Ooguri).
- Generalise to higher rank fractional level WZW models?
- One instance of our programme to understand modular transformations and Verlinde formulae in logarithmic conformal field theory (*cf*. success with $W\left(p_+,p_-\right)$, $\widehat{\mathfrak{gl}}\left(1|1\right)$).