Module categories for affine VOAs at admissible levels

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The WZW Model on SU (2)

Witten noted that adding a Wess-Zumino term to the obvious non-linear sigma model restores conformal invariance.

The Wess-Zumino-Witten action is

$$S[g] = \frac{k}{8\pi} \int_{\Sigma} \kappa\big(g^* \vartheta, \star g^* \vartheta\big) + 2\pi \mathfrak{i} \int_{\Gamma} \tilde{g}^* H,$$

where:

- g maps a Riemann surface Σ into SU (2).
- $\tilde{g}$ extends g to Γ with $\partial\Gamma = \Sigma$ (note $\mathsf{H}_2\left(\mathsf{SU}(2);\mathbb{R}\right) = 0$).
- ϑ is the canonical 1-form and κ the Killing form of SU (2).
- $\star$ is the Hodge star on Σ and $k \in \mathbb{R}$ is the level.
- $H = \dfrac{k}{24\pi^2} \kappa\big(\vartheta, \mathrm{d}\vartheta\big)$ represents k in $\mathsf{H}^3\left(\mathsf{SU}(2);\mathbb{R}\right) = \mathbb{R}$.

Quantisation

The Feynman amplitudes $\mathrm{e}^{-S[g]}$ do not depend on the choice of Γ and $\tilde{g}$ if $k \in \mathbb{Z}$ (so $[H] \in \mathsf{H}^3(\mathsf{SU}(2);\mathbb{Z}) = \mathbb{Z}$).

Changing the sign of k reverses the orientation, so take $k \in \mathbb{Z}_{\geqslant 0}$.

Quantisation gives the symmetry algebra $\mathcal{U}_k \otimes \mathcal{U}_k$, where:

- $\widehat{\mathfrak{sl}}(2) = \mathfrak{sl}(2;\mathbb{C}) \otimes \mathbb{C}[t;t^{-1}] \oplus \mathbb{C}K$.
- $\left[J_m, J'_n\right] = \left[J, J'\right]_{m+n} + m\,\kappa\big(J, J'\big)\delta_{m+n=0}K$ ($J_n = J \otimes t^n$).
- $\mathcal{U}_k = \dfrac{\mathcal{U}\big(\widehat{\mathfrak{sl}}(2)\big)}{\langle K - k\mathbf{1}\rangle}$.

The quantum state space is therefore built from level k modules of the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}(2)$.

The Sugawara construction makes the quantum state space a Virasoro module of central charge $c = \dfrac{3k}{k+2}$.

Highest weight modules

Highest weight vectors $\lvert\lambda\rangle$ for $\mathfrak{U}_k$ are defined by

$$H_0\lvert\lambda\rangle = \lambda\lvert\lambda\rangle, \qquad E_{n-1}\lvert\lambda\rangle = H_n\lvert\lambda\rangle = F_n\lvert\lambda\rangle = 0 \quad (n \geqslant 1).$$

$\lvert\lambda\rangle$ spans a one-dimensional module of $\mathfrak{U}_k^{\geqslant 0}$ (generated by the E_n, H_n and F_{n+1}, with $n \geqslant 0$). Verma modules are then induced in the usual manner:

$$\mathcal{V}_\lambda = \mathfrak{U}_k \otimes_{\mathfrak{U}_k^{\geqslant 0}} \mathbb{C}\lvert\lambda\rangle.$$

They have a unique simple quotient $\mathcal{L}_\lambda$ whose highest weight vector has conformal weight

$$\Delta_\lambda = \frac{\lambda\,(\lambda+2)}{4\,(k+2)}.$$

The spectrum

The vacuum module $\mathcal{L}_0$ admits a vertex operator algebra structure.

Quotienting $\mathcal{V}_0$ amounts to making a singular vector null:

$$E_{-1}^{k+1}|0\rangle = 0 \qquad (k \in \mathbb{Z}_{\geqslant 0}).$$

This constrains the VOA modules to be integrable and highest weight. The simple VOA modules are precisely

$$\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_k.$$

The category of integrable highest weight VOA modules is semisimple.

We therefore obtain a rational CFT with quantum state space

$$\mathcal{H} = \bigoplus_{\lambda=0}^{k} (\mathcal{L}_\lambda \otimes \mathcal{L}_\lambda).$$

Generalisations

What can be said for more general levels k?

One can consider the Wess-Zumino-Witten model on the universal cover of $\mathsf{SL}(2;\mathbb{R})$. Physicists call this AdS_3. Because

$$\mathsf{H}_2(\mathsf{AdS}_3;\mathbb{R}) = \mathsf{H}^3(\mathsf{AdS}_3;\mathbb{R}) = 0,$$

the model is well-defined, but its level is not quantised.

Unfortunately, the generic k vacuum modules give no constraint upon the spectrum: All $\mathcal{U}_k$ modules are VOA modules. This is a very complicated situation!

An easier question to ask is for which levels are there constraints upon the spectrum coming from choosing a simple vacuum module? We call these levels admissible.

There seems to be no geometric formulation for admissible levels...

Admissible levels

The admissible levels are precisely those for which the maximal proper submodule of the $\widehat{\mathfrak{sl}}(2)$ Verma module $\mathcal{V}_0$ is generated by two singular vectors.

This follows easily from Kac-Kazhdan: The $\widehat{\mathfrak{sl}}(2)$ level k is admissible if and only if

$$t \equiv k + 2 = \frac{u}{v}, \text{ with } \gcd\{u, v\} = 1,\ u \in \mathbb{Z}_{\geqslant 2} \text{ and } v \in \mathbb{Z}_{\geqslant 1}.$$

Taking $v = 1$ precisely recovers the non-negative integer levels.

The singular vector relation for $v > 1$ is not of the form

$$\left(E_{-1}^{m} + \cdots\right) \left|0\right\rangle = 0.$$

Consequently, the admissible level VOAs are not C_2-cofinite for $v > 1$ (and that's a good thing).

Admissible modules

The admissible level $\mathcal{U}_k$ modules which define VOA modules are also said to be admissible.

The simple admissible highest weight modules were classified by Adamović-Milas: They are the $\mathcal{L}_{r,s} \equiv \mathcal{L}_{\lambda_{r,s}}$, where

- $\lambda_{r,s} = r - 1 - u\dfrac{s}{v}$ (recall $t = k + 2 = \dfrac{u}{v}$),
- $r = 1, 2, \ldots, u - 1$,
- $s = 0, 1, 2, \ldots, v - 1$.

The conformal weight of the highest weight vector $\left|\lambda_{r,s}\right\rangle$ is

$$\Delta_{r,s} \equiv \Delta_{\lambda_{r,s}} = \frac{(vr - us)^2 - v^2}{4uv}.$$

Aside from allowing $s = 0$, this is very reminiscent of the corresponding formula for the Virasoro minimal model $\mathbf{M}(u, v)$.

Admissible Kac tables

Following the Virasoro analogy, we can construct analogues of Kac tables summarising the $\mathfrak{sl}(2)$ and conformal weights of the simple admissible highest weight modules, *eg*.

$\lambda_{r,s}$

0	$\frac{-3}{2}$
1	$\frac{-1}{2}$

$\Delta_{r,s}$

0	$\frac{-1}{8}$
$\frac{1}{2}$	$\frac{-1}{8}$

$k = -\frac{1}{2},\ t = \frac{3}{2}$

$\lambda_{r,s}$

0	$\frac{-2}{3}$	$\frac{-4}{3}$

$\Delta_{r,s}$

0	$\frac{-1}{3}$	$\frac{-1}{3}$

$k = -\frac{4}{3},\ t = \frac{2}{3}$

$\lambda_{r,s}$

0	$\frac{-5}{2}$
1	$\frac{-3}{2}$
2	$\frac{-1}{2}$
3	$\frac{1}{2}$

$\Delta_{r,s}$

0	$\frac{1}{8}$
$\frac{3}{10}$	$\frac{-3}{40}$
$\frac{4}{5}$	$\frac{-3}{40}$
$\frac{3}{2}$	$\frac{1}{8}$

$k = \frac{1}{2},\ t = \frac{5}{2}$

$\lambda_{r,s}$

0	$\frac{-4}{3}$	$\frac{-8}{3}$
1	$\frac{-1}{3}$	$\frac{-5}{3}$
2	$\frac{2}{3}$	$\frac{-2}{3}$

$\Delta_{r,s}$

0	$\frac{-1}{6}$	$\frac{1}{3}$
$\frac{9}{16}$	$\frac{-5}{48}$	$\frac{-5}{48}$
$\frac{3}{2}$	$\frac{1}{3}$	$\frac{-1}{6}$

$k = -\frac{2}{3},\ t = \frac{4}{3}$

Note the analogue of the Kac symmetries:

$$\lambda_{u-r,v-s} = -\lambda_{r,s} - 2, \quad \Delta_{u-r,v-s} = \Delta_{r,s} \qquad (s \neq 0).$$

On matters categorical...

The category of admissible highest weight modules is also semisimple (Kac-Wakimoto), suggesting that the corresponding CFTs are rational.

This conclusion is premature when $v > 1$, *ie.* $k \notin \mathbb{Z}_{\geqslant 0}$.

Physical CFT spectra must always be closed under conjugation. For $\widehat{\mathfrak{sl}}(2)$, this is twisting by the (non-affine) Weyl reflection:

$$\mathsf{w}(E_n) = F_n, \qquad \mathsf{w}(H_n) = -H_n, \qquad \mathsf{w}(F_n) = E_n.$$

Unfortunately, the conjugate module of a simple admissible highest weight module is again highest weight if and only if $s = 0$. In fact,

$$\mathsf{w}\big(\mathcal{L}_{r,0}\big) = \mathcal{L}_{r,0}.$$

But, $\mathsf{w}\big(\mathcal{L}_{r,s}\big)$ is not even in category $\mathcal{O}$ for $s > 0$.

Why we should relax

The VOA module $\mathcal{L}_{r,s}$ is the simple quotient of the $\mathcal{U}_k$ module induced from the simple $\mathfrak{sl}(2)$ module with highest weight $\lambda_{r,s}$.

For $s = 0$, this simple $\mathfrak{sl}(2)$ module is finite-dimensional. For $s > 0$, it is infinite-dimensional.

The conjugate module $\mathsf{w}(\mathcal{L}_{r,s})$ can be similarly induced, but now from the simple $\mathfrak{sl}(2)$ module with lowest weight $-\lambda_{r,s}$.

We should extend the category of admissible modules to include these conjugates. However, doing so means we lose semisimplicity! There exist non-split extensions $\mathcal{E}^{\pm}_{r,s}$:

$$0 \longrightarrow \mathcal{L}_{r,s} \longrightarrow \mathcal{E}^{+}_{r,s} \longrightarrow \mathsf{w}(\mathcal{L}_{u-r,v-s}) \longrightarrow 0,$$
$$0 \longrightarrow \mathsf{w}(\mathcal{L}_{u-r,v-s}) \longrightarrow \mathcal{E}^{-}_{r,s} \longrightarrow \mathcal{L}_{r,s} \longrightarrow 0.$$

These extensions are examples of relaxed highest weight modules.

Relaxed highest weight modules

Relaxed highest weight vectors $|\lambda;\Delta\rangle$ for $\mathcal{U}_k$ are defined by

$$H_0|\lambda;\Delta\rangle = \lambda|\lambda;\Delta\rangle, \qquad L_0|\lambda;\Delta\rangle = \Delta|\lambda;\Delta\rangle,$$
$$E_n|\lambda;\Delta\rangle = H_n|\lambda;\Delta\rangle = F_n|\lambda;\Delta\rangle = 0 \quad (n \geqslant 1).$$

Because we no longer require $E_0|\lambda;\Delta\rangle = 0$, the conformal weight Δ is no longer fixed by the $\mathfrak{sl}(2)$ weight λ.

If $|\lambda;\Delta\rangle$ is a relaxed highest weight vector, so are $E_0|\lambda;\Delta\rangle$ and $F_0|\lambda;\Delta\rangle$ (when non-zero).

Relaxed Verma modules are then constructed by inducing from an $\mathfrak{sl}(2)$ module of relaxed highest weight vectors.

The $\mathfrak{sl}(2)$ module will generically not be highest or lowest weight. If it is, then Δ is fixed in terms of λ.

Admissible relaxed highest weight modules

Adamović-Milas likewise classified the admissible relaxed highest weight modules. Aside from the $\mathcal{L}_{r,s}$ and $\mathsf{w}\big(\mathcal{L}_{r,s}\big)$, one also finds the (non-simple) $\mathcal{E}^{\pm}_{r,s}$ and new simple modules $\mathcal{E}_{\lambda;\Delta_{r,s}}$ with

- $\lambda \neq \pm\lambda_{r,s} \bmod 2$,
- $r = 1, 2, \ldots, u-1$,
- $s = 1, 2, \ldots, v-1$.

The $\mathcal{E}_{\lambda;\Delta_{r,s}}$ are induced from simple $\mathfrak{sl}(2)$ modules that are neither highest nor lowest weight.

The common $\mathfrak{sl}(2)$ weight mod 2 of the states of $\mathcal{E}_{\lambda;\Delta_{r,s}}$ is λ (which is therefore only defined mod 2). The conformal weight of the ground states is $\Delta_{r,s}$.

Note that $\mathsf{w}\big(\mathcal{E}^{\pm}_{r,s}\big) = \mathcal{E}^{\mp}_{u-r,v-s}$ and $\mathsf{w}\big(\mathcal{E}_{\lambda;\Delta_{r,s}}\big) = \mathcal{E}_{-\lambda;\Delta_{r,s}}$.

Spectral flow automorphisms

But wait, there's more: To "maximise" the spectrum, we should impose closure under twists by all automorphisms, not just conjugation.

The $\widehat{\mathfrak{sl}}(2)$ automorphisms preserving any given Cartan subalgebra form an infinite dihedral group generated by conjugation and the spectral flow automorphisms σ^ℓ, $\ell \in \mathbb{Z}$:

$$\sigma^\ell(E_n) = E_{n-\ell}, \quad \sigma^\ell(H_n) = H_n - \ell\delta_{n=0}k, \quad \sigma^\ell(F_n) = F_{n+\ell}.$$

When ℓ is even, σ^ℓ is an affine Weyl translation.

Spectral flow does not preserve the conformal grading:

$$\sigma^\ell(L_0) = L_0 - \frac{1}{2}\ell H_0 + \frac{1}{4}\ell^2 k.$$

Admissible twisted relaxed highest weight modules

We now twist the $\mathcal{U}_k$ action, by σ^ℓ, on the relaxed highest weight modules to obtain new modules.

However, the conformal weights of $\sigma^\ell(\mathcal{L}_{r,s})$, $\sigma^\ell(\mathcal{E}^{\pm}_{r,s})$ and $\sigma^\ell(\mathcal{E}_{\lambda;\Delta_{r,s}})$ are no longer bounded below in general.

Because of the identifications

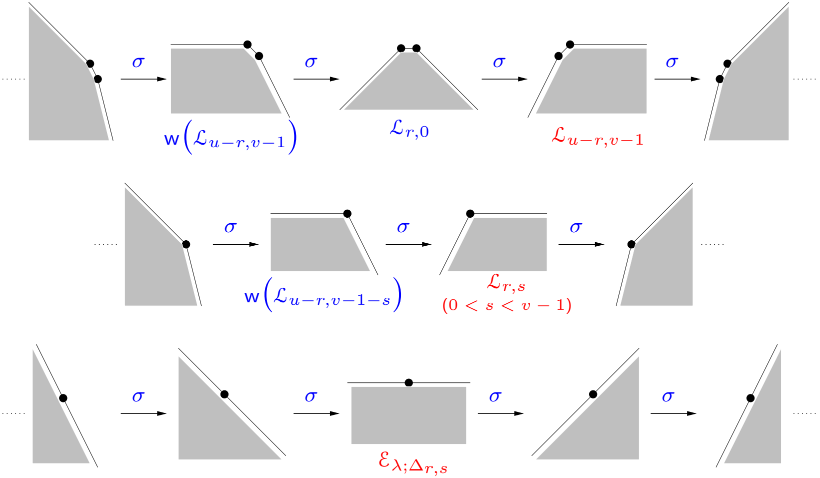
$$\sigma(\mathcal{L}_{r,0}) = \mathcal{L}_{u-r,v-1}, \qquad \sigma^{-1}(\mathcal{L}_{r,s}) = \mathsf{w}(\mathcal{L}_{u-r,v-1-s}),$$

spectral flow organises the simple admissible modules into families:

$$\sigma^\ell(\mathcal{L}_{r,s}), \qquad \sigma^\ell(\mathcal{E}_{\lambda;\Delta_{r,s}});$$
$$\ell \in \mathbb{Z}, \quad r = 1, 2, \ldots, u-1, \quad s = 1, 2, \ldots, v-1, \quad \lambda \neq \pm\lambda_{r,s}.$$

When $v = 1$, twisting by spectral flow amounts to acting with the non-trivial Dynkin symmetry of $\widehat{\mathfrak{sl}}(2)$.

Families of simple admissible modules ($v > 1$)



Are we there yet?

These families exhaust the simple admissible VOA modules.

More precisely, these are the simple VOA modules in the category of $\mathcal{U}_k$ modules which are finitely generated, H_0-semisimple (redundant?) and locally $\mathfrak{Vir}^+$-finite.

The subcategory of VOA modules is not semisimple, but it is closed under conjugation (since $\sigma \mathsf{w} = \mathsf{w} \sigma^{-1}$).

This subcategory contains extensions that are not L_0-semisimple. *eg.* for $k = -\frac{1}{2}$ ($u = 3$, $v = 2$),

$$0 \longrightarrow \sigma^{-1}\left(\mathcal{E}^+_{2,1}\right) \longrightarrow \mathcal{S}_{1,0} \longrightarrow \sigma\left(\mathcal{E}^+_{1,1}\right) \longrightarrow 0$$

is non-split with a rank 2 Jordan block for L_0 at $(\lambda, \Delta) = (0, 0)$.

The associated CFTs are therefore logarithmic.

Morals and lessons

Highest weight categories are no good for admissible level CFT.

Representations whose conformal weights are not bounded below are the rule, not the exception.

However, the conformal weights are bounded below in each $\mathfrak{sl}(2)$ weight space. This $\mathfrak{Vir}^+$-finiteness means that the powers appearing in any given OPE are bounded below.

Restricting to any subcategory of VOA modules is suspicious if not completely unphysical. The category must close under conjugation, spectral flow (?) and fusion. The families

$$\sigma^\ell(\mathcal{L}_{r,s}), \qquad \sigma^\ell(\mathcal{E}_{\lambda;\Delta_{r,s}}), \qquad \sigma^\ell(\mathcal{S}_{r,s});$$
$$\ell \in \mathbb{Z}, \quad r = 1, 2, \ldots, u-1, \quad s = 1, 2, \ldots, v-1, \quad \lambda \neq \pm\lambda_{r,s}$$

close under all three.

A logarithmic CFT framework

The $\sigma^\ell(\mathcal{E}_{\lambda;\Delta_{r,s}})$ with $\lambda \neq \pm\lambda_{r,s}$ are the typical modules.

Together with the $\sigma^\ell(\mathcal{E}^{\pm}_{r,s})$, which "fill in the gaps" $\lambda = \pm\lambda_{r,s}$, they form a continuum of standard modules. There is a common character formula for all the standard modules.

The modules $\sigma^\ell(\mathcal{L}_{r,s})$, $\sigma^\ell(\mathcal{E}^{\pm}_{r,s})$, $\sigma^\ell(\mathcal{S}_{r,s})$, *etc*... that correspond to $\lambda = \lambda_{r,s}$ constitute the atypical modules.

Typical modules are simple and projective. The simple atypicals are the $\sigma^\ell(\mathcal{L}_{r,s})$ and their projective covers are the $\sigma^\ell(\mathcal{S}_{r,s})$.

In terms of the parametrisation space

$$(\ell, \lambda, r, s) \in \mathbb{Z} \times \frac{\mathbb{R}}{2\mathbb{Z}} \times \frac{\{1, \dots, u-1\} \times \{1, \dots, v-1\}}{\mathbb{Z}_2},$$

the atypical modules correspond to a set of measure zero. In other words, the spectrum is "rational almost everywhere".

Getting the spectrum right matters

These continuous families and spectral flows might seem like overkill. But...

- The modular S-transformation is implemented on the space of characters as an integral operator.
- This operator is symmetric, unitary, and squares to give the conjugation permutation.
- The diagonal partition function is modular invariant.
- Inserting the kernel of the S-operator into the Verlinde formula gives non-negative integer fusion coefficients.
- The Grothendieck fusion rules obtained from the Verlinde formula agree with the known fusion rules.
- There is an infinite family of simple current extensions, each of which has a finite number of simple modules. C_2-cofinite?

Summary

Non-negative integer level WZW models enjoy a finite spectrum of simple modules, all of which are highest weight.

By contrast, admissible levels require a continuum of simple modules, almost all of which are not highest weight.

Almost all of the admissible modules have conformal weights that are unbounded below. Most are not even C_1-cofinite.

The non-unitary admissible level WZW models are all logarithmic.

Nevertheless, their modular properties are understood and a continuous version of the Verlinde formula computes the Grothendieck fusion coefficients.

None of this works if you artificially restrict the module category, *eg*. by considering only highest weight, or category $\mathcal{O}$, modules.