# Parabolic Verma modules, bosonic ghost systems and logarithmic CFT

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# Background and Motivation

Logarithmic CFT refers to correlation functions having logarithmic singularities. [Gurarie, Rozansky–Saleur]

Equivalently, the spectrum includes modules upon which the Virasoro zero mode $L_0$ (or $L_0 + \overline{L}_0$) acts non-semisimply.

Such modules may be indecomposable, but are always reducible: The span of the $L_0$-eigenvectors is a proper submodule.

Examples of logarithmic CFTs include:

- Symplectic fermions. [Kausch]
- The triplet models. [Gaberdiel–Kausch, Feigin *et al*]
- Super-WZW models. [Rozansky–Saleur–Schomerus–Quella]
- Fractional level WZW models. [Gaberdiel, Lesage *et al*, DR]

One wants to extend the formalism of rational CFT to the logarithmic case. But, some things don't work as well...

*eg*, the characters need not span an $\mathsf{SL}(2;\mathbb{Z})$-module: S-transforms have $\tau$-dependent coefficients. [Flohr]

Recently, a formalism has been proposed for logarithmic CFTs within which one can derive constant S-matrix coefficients and apply the Verlinde formula. [Creutzig–DR–Wood]

This formalism has been tested successfully for:

- Singlet/triplet models. [DR–Wood]
- The $\widehat{\mathfrak{gl}}(1|1)$ WZW model. [Creutzig–DR, Babichenko–DR]
- Fractional level $\widehat{\mathfrak{sl}}(2)$. [Creutzig–DR]
- Logarithmic minimal models. [Morin-Duchesne–Rasmussen–DR]
- $N = 1$ log. minimal models. [Canagasabey–Rasmussen–DR]

The formalism rests upon a set of standard modules parametrised by a measurable space.

The typical modules are the simple standards. They correspond to a subset of parameters of full measure. Typicals are (conjecturally) projective in the physically relevant module category.

All other (indecomposable) modules are atypical. The logarithmic structure of the CFT is concentrated in this measure zero subset of the parameter space.

The characters of the standards form a (topological) basis. They have nice modular properties. The atypical characters have more complicated modular properties.

The standard modules might be familar, *eg*. Verma or Feigin-Fuchs modules, or unfamiliar: For fractional level WZW models, the standards are quotients of maximally parabolic Verma modules.

# Bosonic Ghosts at $c = 2$

We outline the application of this formalism in the easiest case, the bosonic $\beta\gamma$ ghost system of central charge $c = 2$. The current is $J = \,:\beta\gamma:$ and the Virasoro field is $T = -:\beta\partial\gamma:$.

OPEs:
$$\beta(z)\gamma(w) \sim -\frac{1}{z-w},$$

$$J(z)\beta(w) \sim +\frac{\beta(w)}{z-w}, \qquad T(z)\beta(w) \sim \frac{\beta(w)}{(z-w)^2} + \frac{\partial\beta(w)}{z-w},$$
$$J(z)\gamma(w) \sim -\frac{\gamma(w)}{z-w}, \qquad T(z)\gamma(w) \sim \frac{\partial\gamma(w)}{z-w}.$$

*ie*, $\beta$ has charge $+1$ and dimension $1$; $\gamma$ has charge $-1$ and dimension $0$.

This ghost theory appears, in particular, in the Wakimoto free field realisation for WZW models.

Auts: Conjugation Spectral flow

$$\begin{aligned} \mathsf{w}(\beta_n) &= \gamma_n, & \sigma^\ell(\beta_n) &= \beta_{n-\ell}, \\ \mathsf{w}(\gamma_n) &= -\beta_n, & \sigma^\ell(\gamma_n) &= \gamma_{n+\ell}, \\ \mathsf{w}(J_n) &= \delta_{n=0} - J_n, & \sigma^\ell(J_n) &= J_n + \ell\delta_{n=0}, \\ \mathsf{w}(L_n) &= L_n - nJ_n, & \sigma^\ell(L_n) &= L_n - \ell J_n - \tfrac{\ell(\ell-1)}{2}\delta_{n=0}. \end{aligned}$$

We may twist the action on any ghost module $M$ by these automorphisms:

- $\mathsf{w}(M)$ is the conjugate of $M$.
- The $\sigma^\ell(M)$ are the spectral flow images of $M$.

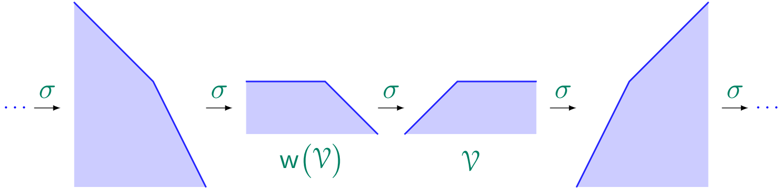
We take $\ell \in \mathbb{Z}$ so that the ghost fields act on $\sigma^\ell(M)$ with integer moding: $\beta(z) = \sum_{n\in\mathbb{Z}} \beta_n z^{-n-1}$, $\gamma(z) = \sum_{n\in\mathbb{Z}} \gamma_n z^{-n}$.

# Ghost Modules

Fix a Cartan subalgebra spanned by the unit $\mathbf{1}$. Spectral flow relates the possible Borel subalgebras.

The vacuum $\Omega$ satisfies $\beta_n \Omega = \gamma_{n+1} \Omega = 0$, for $n \geqslant 0$, and generates the only highest weight module $\mathcal{V}$.

Spectral flow gives infinitely many non-isomorphic modules $\sigma^{\ell}(\mathcal{V})$.




The vacuum module is not self-conjugate: $\mathsf{w}(\mathcal{V}) = \sigma^{-1}(\mathcal{V})$.

There is also a continuous family of parabolic Verma modules $\mathcal{W}_\lambda$ parametrised by $\lambda \in \mathbb{R}/\mathbb{Z}$. They are simple for $\lambda \neq [0]$.

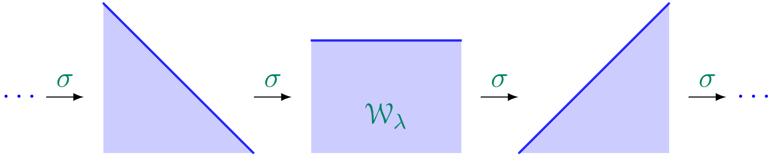
For $\lambda = [0]$, there are three inequivalent parabolic modules

$$0 \to \mathcal{V} \to \mathcal{W}_{[0]}^{+} \to \mathsf{w}(\mathcal{V}) \to 0, \quad 0 \to \mathsf{w}(\mathcal{V}) \to \mathcal{W}_{[0]}^{-} \to \mathcal{V} \to 0,$$

and $\mathcal{V} \oplus \mathsf{w}(\mathcal{V})$.

Their ground states are spanned by vectors $v_j$, $j \in \lambda$, of charge $j$ and dimension $0$.

Spectral flow gives non-isomorphic modules $\sigma^\ell(\mathcal{W}_\lambda)$, $\sigma^\ell(\mathcal{W}_{[0]}^{\pm})$.



# Characters

Standard modules: $\sigma^{\ell}\big(\mathcal{W}_{\lambda}\big)$, $\sigma^{\ell}\big(\mathcal{W}^{\pm}_{[0]}\big)$ $(\ell \in \mathbb{Z})$.

Typical modules: $\sigma^{\ell}\big(\mathcal{W}_{\lambda}\big)$ $(\lambda \neq [0],\ \ell \in \mathbb{Z})$.

Atypical modules: $\sigma^{\ell}\big(\mathcal{W}^{\pm}_{[0]}\big)$, $\sigma^{\ell}\big(\mathcal{V}\big)$ $(\ell \in \mathbb{Z})$.

$$\operatorname{ch}\big[M\big]\big(z;q\big) = \operatorname{tr}_{M} z^{J_0} q^{L_0 - c/24}.$$

$$\Rightarrow \qquad \operatorname{ch}\big[\mathcal{V}\big] = -\mathfrak{i} z^{1/2} \frac{\eta\big(q\big)}{\vartheta_1\big(z;q\big)},$$

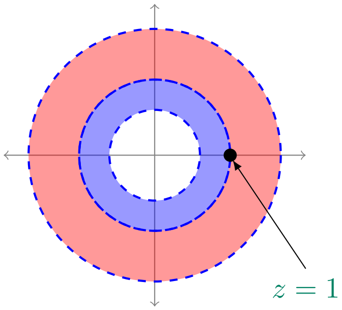
$$\operatorname{ch}\big[\mathsf{w}\big(\mathcal{V}\big)\big] = -\mathfrak{i} z^{1/2} \frac{\eta\big(q\big)}{\vartheta_1\big(z^{-1};q\big)} = +\mathfrak{i} z^{1/2} \frac{\eta\big(q\big)}{\vartheta_1\big(z;q\big)}$$

$$\Rightarrow \qquad \operatorname{ch}\big[\mathcal{W}^{\pm}_{[0]}\big] = \operatorname{ch}\big[\mathcal{V}\big] + \operatorname{ch}\big[\mathsf{w}\big(\mathcal{V}\big)\big] = 0. \qquad ???$$

However, $\eta(q)/\vartheta_1(z;q)$ has a pole at $z=1$. We find that

$$\mathrm{ch}\big[\mathcal{V}\big] = -\mathrm{i}z^{1/2}\frac{\eta(q)}{\vartheta_1(z;q)} \qquad \text{for } |q|<1,\ 1<|z|<|q|^{-1},$$

$$\mathrm{ch}\big[\mathsf{w}(\mathcal{V})\big] = +\mathrm{i}z^{1/2}\frac{\eta(q)}{\vartheta_1(z;q)} \qquad \text{for } |q|<1,\ |q|<|z|<1.$$




These regions are disjoint, so summing the characters is incorrect.

$$\mathrm{ch}\big[\mathcal{W}^{\pm}_{[0]}\big] = \frac{\sum_{n\in\mathbb{Z}} z^n}{\eta(q)^2} = \frac{\sum_{m\in\mathbb{Z}} \delta(\zeta-m)}{\eta(q)^2}$$

is a singular distribution supported at $\zeta\in\mathbb{Z}$, *ie*. $z=\mathrm{e}^{2\pi\mathrm{i}\zeta}=1$.

$\boxed{\text{Characters are not meromorphic functions, but distributions.}}$

# Modular Transformations

The standard modules have nice S-transforms:

$$\mathrm{ch}\left[\sigma^{\ell}\left(\mathcal{W}_{\lambda}\right)\right]\left(\zeta\middle|\tau\right) = \frac{\mathsf{e}^{\mathsf{i}\pi\ell(\ell-1)\tau}}{\eta\left(\tau\right)^2}\sum_{n\in\mathbb{Z}}\mathsf{e}^{2\pi\mathsf{i}n\lambda}\delta\left(\zeta-n\tau-\ell\right)$$

$$\Rightarrow \quad \mathrm{ch}\left[\sigma^{\ell}\left(\mathcal{W}_{\lambda}\right)\right]\left(\zeta/\tau\middle|-1/\tau\right)$$

$$= A\left(\zeta\middle|\tau\right)\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}/\mathbb{Z}}\mathsf{S}\left[\sigma^{\ell}\left(\mathcal{W}_{\lambda}\right)\to\sigma^{m}\left(\mathcal{W}_{\mu}\right)\right]\mathrm{ch}\left[\sigma^{m}\left(\mathcal{W}_{\mu}\right)\right]\left(\zeta\middle|\tau\right)\,\mathrm{d}\mu,$$

where the automorphy factor and S-matrix elements are

$$A\left(\zeta\middle|\tau\right) = \frac{\left|\tau\right|}{-\mathsf{i}\tau}\mathsf{e}^{\mathsf{i}\pi\zeta^2/\tau}\mathsf{e}^{-\mathsf{i}\pi\zeta/\tau}\mathsf{e}^{\mathsf{i}\pi\zeta},$$

$$\mathsf{S}\left[\sigma^{\ell}\left(\mathcal{W}_{\lambda}\right)\to\sigma^{m}\left(\mathcal{W}_{\mu}\right)\right] = \left(-1\right)^{\ell+m}\mathsf{e}^{-2\pi\mathsf{i}(\ell\mu+m\lambda)}.$$

This S-matrix is symmetric, unitary, and squares to conjugation.

The atypical modules must first be resolved into standards:

$$\cdots \to \sigma^{\ell+3}\big(\mathcal{W}_0^+\big) \to \sigma^{\ell+2}\big(\mathcal{W}_0^+\big) \to \sigma^{\ell+1}\big(\mathcal{W}_0^+\big) \to \sigma^{\ell}\big(\mathcal{V}\big) \to 0$$

$$\Rightarrow \quad \mathsf{S}\big[\sigma^{\ell}(\mathcal{V}) \to \sigma^{m}(\mathcal{W}_{\mu})\big] = (-1)^{\ell+m+1} \frac{\mathsf{e}^{-2\pi\mathsf{i}(\ell+1/2)\mu}}{\mathsf{e}^{\mathsf{i}\pi\mu} - \mathsf{e}^{-\mathsf{i}\pi\mu}}.$$

Note the pole when $\mu = [0]$.

The Verlinde formula gives the (Grothendieck) fusion coefficients:

$$\mathrm{ch}\big[M\big] \boxtimes \mathrm{ch}\big[N\big] = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \begin{bmatrix} \sigma^{n}(\mathcal{W}_{\nu}) \\ M \quad N \end{bmatrix} \mathrm{ch}\big[\sigma^{n}(\mathcal{W}_{\nu})\big] \, \mathrm{d}\nu,$$

$$\begin{bmatrix} \sigma^{n}(\mathcal{W}_{\nu}) \\ M \quad N \end{bmatrix} = \sum_{r\in\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{\mathsf{S}\big[M \to \sigma^{r}(\mathcal{W}_{\rho})\big] \mathsf{S}\big[N \to \sigma^{r}(\mathcal{W}_{\rho})\big] \mathsf{S}\big[\sigma^{n}(\mathcal{W}_{\nu}) \to \sigma^{r}(\mathcal{W}_{\rho})\big]^{*}}{\mathsf{S}\big[\mathcal{V} \to \sigma^{r}(\mathcal{W}_{\rho})\big]} \, \mathrm{d}\rho.$$

These coefficients give non-negative integer multiplicities.

# Fusion

The Verlinde formula gives the Grothendieck fusion rules:

$$\begin{aligned}
\left[\sigma^{\ell}(\mathcal{V})\right] \boxtimes \left[\sigma^{m}(\mathcal{V})\right] &= \left[\sigma^{\ell+m}(\mathcal{V})\right], \\
\left[\sigma^{\ell}(\mathcal{V})\right] \boxtimes \left[\sigma^{m}(\mathcal{W}_{\mu})\right] &= \left[\sigma^{\ell+m}(\mathcal{W}_{\mu})\right], \\
\left[\sigma^{\ell}(\mathcal{W}_{\lambda})\right] \boxtimes \left[\sigma^{m}(\mathcal{W}_{\mu})\right] &= \left[\sigma^{\ell+m}(\mathcal{W}_{\lambda+\mu})\right] + \left[\sigma^{\ell+m-1}(\mathcal{W}_{\lambda+\mu})\right].
\end{aligned}$$

These imply the following (genuine) fusion rules:

$$\begin{aligned}
\sigma^{\ell}(\mathcal{V}) \times \sigma^{m}(\mathcal{V}) &= \sigma^{\ell+m}(\mathcal{V}), \\
\sigma^{\ell}(\mathcal{V}) \times \sigma^{m}(\mathcal{W}_{\mu}) &= \sigma^{\ell+m}(\mathcal{W}_{\mu}) && (\mu \neq [0]), \\
\sigma^{\ell}(\mathcal{W}_{\lambda}) \times \sigma^{m}(\mathcal{W}_{\mu}) &= \sigma^{\ell+m}(\mathcal{W}_{\lambda+\mu}) \oplus \sigma^{\ell+m-1}(\mathcal{W}_{\lambda+\mu}) && (\lambda \neq -\mu).
\end{aligned}$$

It remains to compute the typical fusion rules with $\lambda + \mu = [0]$.

This is where the logarithmic nature of the theory manifests:

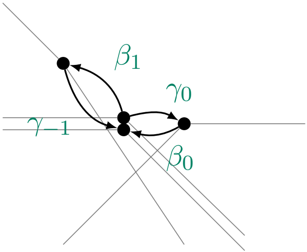
$$[\sigma^\ell(\mathcal{W}_\lambda)] \boxtimes [\sigma^m(\mathcal{W}_{-\lambda})] = [\sigma^{\ell+m-2}(\mathcal{V})] + 2\,[\sigma^{\ell+m-1}(\mathcal{V})] + [\sigma^{\ell+m}(\mathcal{V})],$$

but $$\sigma^\ell(\mathcal{W}_\lambda) \times \sigma^m(\mathcal{W}_{-\lambda}) = \sigma^{\ell+m}(\mathcal{P}).$$

Associativity fixes the fusion rules involving $\mathcal{P}$.

$L_0$ acts non-semisimply on $\mathcal{P}$ and its spectral flow images.

$$\begin{tikzcd}
 & \sigma^{-1}(\mathcal{V}) \arrow[dl] \arrow[dr] & \\
\sigma^{-2}(\mathcal{V}) \arrow[dr] & \mathcal{P} & \mathcal{V} \arrow[dl] \\
 & \sigma^{-1}(\mathcal{V}) &
\end{tikzcd}$$




$\mathcal{P}$ is (conjecturally) the projective cover of $\mathsf{w}(\mathcal{V}) = \sigma^{-1}(\mathcal{V})$.

# Conclusions and Future Directions

The $c = 2$ bosonic ghost system is a logarithmic CFT. This is also true at all other central charges.

Almost all of the spectrum consists of (typical) spectral flow images of parabolic Verma modules $\mathcal{W}_\lambda$. The vacuum module $\mathcal{V}$ is atypical and its projective cover $\mathcal{P}$ is logarithmic.

The characters span an uncountably-infinite-dimensional rep. of the modular group with $\mathsf{S}$ symmetric and unitary in the standard basis.

The diagonal partition function $\sum_{\ell \in \mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \left| \mathrm{ch}\left[\sigma^\ell(\mathcal{W}_\lambda)\right] \right|^2 \, \mathrm{d}\lambda$ is modular invariant. There is also an infinite series of modular invariants corresponding to the simple currents $\sigma^\ell(\mathcal{V})$.

The role of this theory in Wakimoto free field realisations suggests that this formalism is suitable for attacking fractional level (super) WZW models in general.

## And while I'm here...

There will be an AMSI/PIMS workshop:

# The Mathematics of CFT

Australian National University, July 13–17, 2015.