Symmetric Jack polynomials and fractional level WZW models

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Logarithmic CFTs

Logarithmic CFTs are those for which there are correlation functions with logarithmic singularities.

They are built from representations that need not be semisimple.

In particular, the Virasoro zero mode L_0 is not diagonalisable.

Applications include:

- Non-local observables in statistical models.
- Supersymmetric/non-compact string theories.
- Ghosts.
- Holographic duals of chiral gravity models.
- Fractional/Integer quantum Hall effect.

Archetypal examples include:

- $\mathsf{GL}(1|1)$ WZW model.
- bc ghost systems.
- Symplectic fermions ($\mathsf{PSL}(1|1)$ WZW model).
- Singlet/Triplet models $W(p,p')$ ($p,p' \in \mathbb{Z}_+$, $(p,p')=1$).
- Logarithmic minimal models $LM(p,p')$ ($p,p' \in \mathbb{Z}_+$, $(p,p')=1$).
- Fractional level $\widehat{\mathfrak{sl}}(2)_k$ ($k = \dfrac{u}{v} - 2$, $u \in \mathbb{Z}_{\geqslant 2}$, $v \in \mathbb{Z}_+$, $(u,v)=1$).
- $\beta\gamma$ ghost systems.

But, aside from the $LM(p,p')$ and $W(p,p')$ with $p,p' \geqslant 2$, all these examples are very similar structurally.

Need to lift our game and study some different examples!

How to analyse a CFT

Abstractly, one can start with a vertex operator algebra and constrain the associated CFTs through consistency conditions.

1. Determine the representation theory of the VOA.
2. Determine the modular transformations of the characters.
3. Classify the modular invariant partition functions.
4. Check the crossing symmetry of the 4-point functions.

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Each of these is pretty hard in general... Aim a bit lower:

1. Determine the standard modules of the VOA.
2. Determine the modular transformations of the characters.
3. Check that the Verlinde formula gives non-negative integers.
4. Find some modular invariant partition functions.

2., 3. and 4. are mostly under control. What about 1.?

Affine VOA-modules

Unique affine VOA $\mathsf{U}_k(\widehat{\mathfrak{g}})$ (generic level k, $k \neq -\mathsf{h}^\vee$).

$\mathsf{U}_k(\widehat{\mathfrak{g}})$-mod $\simeq$ $\widehat{\mathfrak{g}}$-mod (roughly).

Simple quotient $\mathsf{L}_k(\widehat{\mathfrak{g}})$ for admissible k — set singular vector to zero.

$\mathsf{L}_k(\widehat{\mathfrak{g}})$-mod $\subsetneq$ $\widehat{\mathfrak{g}}$-mod.

How to get singular vector? Virasoro minimal model analogy $\Rightarrow$ Wakimoto free field realisation, Jack symmetric polynomials.

Which $\widehat{\mathfrak{g}}$-mod category? Highest weight / category $\mathscr{O}$ not good enough!

Need (maximal) parabolic categories $\mathscr{O}_{\mathfrak{p}}$ of $\widehat{\mathfrak{g}}$.

Introduce the relaxed category $\mathscr{R}$. Relaxed highest weight modules can have an infinite-dimensional space of ground states.

Wakimoto for $\widehat{\mathfrak{sl}}(2)$

Universal VOA $\mathsf{U}_k\big(\widehat{\mathfrak{sl}}(2)\big)$: Conformal structure is

$$T(z) = \frac{1}{2(k+2)}\left(\frac{1}{2} : h(z)h(z) : - : e(z)f(z) : - : f(z)e(z) : \right),$$

($k \neq -\mathsf{h}^\vee = -2$) and operator product expansions are

$$h(z)e(w) \sim \frac{+2e(w)}{z-w}, \quad h(z)h(w) \sim \frac{2k}{(z-w)^2},$$
$$h(z)f(w) \sim \frac{-2f(w)}{z-w}, \quad e(z)f(w) \sim \frac{-k}{(z-w)^2} - \frac{h(w)}{z-w}.$$

For $k = -2 + \frac{u}{v}$ admissible ($u \in \mathbb{Z}_{\geqslant 2}$, $v \in \mathbb{Z}_+$, $(u,v) = 1$), get simple VOA $\mathsf{L}_k\big(\widehat{\mathfrak{sl}}(2)\big)$ by setting singular vector of $\mathfrak{sl}(2)$-weight $2(u-1)$ and conformal weight $(u-1)v$ to zero.

Wakimoto free field realisation: One deformed free boson $a(z)$, one pair of bosonic ghosts $\beta(z), \gamma(z)$:

$$a(z)a(w) \sim \frac{1}{(z-w)^2}, \qquad \gamma(z)\beta(w) \sim \frac{1}{z-w},$$

$$T^{\text{bos.}}(z) = \frac{1}{2} : a(z)a(z) : - \frac{1}{\alpha}\partial a(z), \quad T^{\text{gh.}}(z) = - : \beta(z)\partial\gamma(z) : .$$

$\mathsf{U}_k\big(\widehat{\mathfrak{sl}}(2)\big)$ is a sub-VOA of the Wakimoto VOA when the deformation parameter is $\alpha = \sqrt{2(k+2)}$:

$$e(z) = \beta(z), \qquad h(z) = 2 : \beta(z)\gamma(z) : + \alpha\, a(z),$$

$$f(z) = \; : \beta(z)\gamma(z)\gamma(z) : + \alpha\, a(z)\gamma(z) + \Big(\frac{\alpha^2}{2} - 2\Big)\partial\gamma(z),$$

$$T(z) = T^{\text{bos.}}(z) + T^{\text{gh.}}(z).$$

Wakimoto modules

Wakimoto modules restrict to $\mathsf{U}_k\big(\widehat{\mathfrak{sl}}(2)\big)$-modules. Let:

- $\mathcal{F}_p$ = free bosonic Fock space of momentum p.
- $\mathcal{G}$ = ghost vacuum module.
- $\mathcal{G}_q$ = relaxed highest weight module of ghost number $q \in \mathbb{R}/\mathbb{Z}$ (recently classified in DR–SW).

Define Wakimoto modules

$$\mathcal{W}_p = \mathcal{F}_p \otimes \mathcal{G}, \qquad \mathcal{W}_{p;q} = \mathcal{F}_p \otimes \mathcal{G}_q.$$

Vacuum module is $\mathcal{W}_0$.

Singular vectors explicitly constructed using screening operators.

Screening operators

Wakimoto screening field: $\mathscr{Q}(z) = \mathscr{V}_{-2/\alpha}(z)\beta(z)$.

($\mathscr{V}_p(z)$ = free boson vertex operator of momentum p.)

$\Rightarrow \mathscr{Q}^{[1]} = \oint_0 \mathscr{Q}(z) \frac{\mathrm{d}z}{2\pi\mathfrak{i}}$ is a screening operator.

Commutes with $\mathsf{U}_k\big(\widehat{\mathfrak{sl}}(2)\big)$ when contour about 0 closes.

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More general screening operators:

$$\mathscr{Q}^{[r]}(z) = \mathscr{Q}^{[r]}(z_1, \dots, z_k) = \mathscr{Q}(z_1) \cdots \mathscr{Q}(z_r),$$
$$\mathscr{Q}^{[r]} = \int_{[\Delta_r]} \mathscr{Q}^{[r]}(z_1, \dots, z_r)\,\mathrm{d}z_1 \cdots \mathrm{d}z_r.$$

$[\Delta_r]$ is a certain (normalised) r-dimensional cycle.

$r \in \mathbb{Z}_+$, $s \in \mathbb{Z} \Rightarrow$ screening well-defined:

$$\mathscr{Q}^{[r]} \colon \mathcal{W}_{p_{r,s}} \to \mathcal{W}_{p_{-r,s}}, \qquad \mathscr{Q}^{[r]} \colon \mathcal{W}_{p_{r,s};q} \to \mathcal{W}_{p_{-r,s};q},$$

Parametrisation: $p_{r,s} = \frac{1}{\alpha}\big(r - 1 - (k+2)s\big)$.

As in minimal models, symmetric polynomials naturally arise:

$$\mathscr{Q}^{[r]}(z)\Big|_{\mathcal{W}_{p_{r,s}}} = \overbrace{\prod_{1 \leqslant i \neq j \leqslant r} \left(1 - \frac{z_i}{z_j}\right)^{1/(k+2)}}^{\text{integration kernel}} \cdot \overbrace{\prod_{i=1}^{r} z_i^s}^{\text{Jack poly.}} \cdot \prod_{i=1}^{r} \beta(z_i)$$
$$\cdot \prod_{m \geqslant 1} \exp\Big(-\frac{2}{\alpha} \underbrace{\sum_{i=1}^{r} z_i^m}_{\text{power sum in } z} \frac{a_{-m}}{m}\Big) \exp\Big(\frac{2}{\alpha} \underbrace{\sum_{i=1}^{r} z_i^{-m}}_{\text{power sum in } z^{-1}} \frac{a_m}{m}\Big)\Big|_{\mathcal{W}_{p_{-r,s}}}.$$

Let $\underbrace{|p\rangle}_{\widehat{\mathfrak{sl}}(2)\text{ hwv}} = \underbrace{u_p}_{\text{Fock vac}} \otimes \underbrace{v}_{\text{ghost vac}}.$

Theorem 1

For $r \in \mathbb{Z}_+$ and $s \in \mathbb{Z}$, $\mathscr{Q}^{[r]} \colon \mathcal{W}_{p_{r,s}} \to \mathcal{W}_{p_{-r,s}}$ is non-zero and

$$\mathscr{Q}^{[r]} \left|p_{r,s}\right\rangle = (\rho_{-\alpha} \circ \sigma_r) \left(\mathsf{Q}^{k+2}_{[(-s-1)^r]}(x,y)\right) \left|p_{-r,s}\right\rangle \quad (s \leqslant -1)$$

is singular in $\mathcal{W}_{p_{-r,s}}$. Here:
$\mathsf{Q}^{\alpha^2/2}_{[(-s-1)^r]}(x,y)$ = *dual Jack polynomial in* $x_1, \dots, x_r, y_1, \dots, y_r$.

$$\rho_{-\alpha}\left(\sum_{i=1}^r y^m\right) = -\alpha a_{-m}, \quad \sigma_r\left(\mathrm{g}^t_\nu(x)\right) = \beta_{-\nu_1-1} \cdots \beta_{-\nu_r-1}$$

(g^t_ν dual to monomial symmetric polynomials).

Similar formula for relaxed singular vectors in $\mathcal{W}_{p_{-r,s};q}$, $s \leqslant 0$.

Vacuum singular vector

Determine (simple) highest weight and relaxed highest weight modules for quotient VOA $\mathsf{L}_k\big(\widehat{\mathfrak{sl}}(2)\big)$ at admissible level.

Vacuum singular vector is $\mathscr{Q}^{[u-1]}\big|p_{u-1,-v}\big\rangle$, $\mathfrak{sl}(2)$-weight is $2(u-1)$. Use weight 0 descendant:

$$\big|\chi\big\rangle = f_0^{u-1}\mathscr{Q}^{[u-1]}\big|p_{u-1,-v}\big\rangle \propto \mathscr{Q}^{[u-1]}\gamma_0^{u-1}\big|p_{u-1,-v}\big\rangle$$

$$\Rightarrow \quad \chi(z) = \int_{[\Delta_r]} \mathscr{V}_{-2/\alpha}(z_1+w)\cdots\mathscr{V}_{-2/\alpha}(z_{u-1}+w)\mathscr{V}_{p_{u-1,-v}}(w) \cdot \beta(z_1+w)\cdots\beta(z_{u-1}+w)\gamma(w)^{u-1}\,\mathrm{d}z_1\cdots\mathrm{d}z_{u-1}.$$

$\mathsf{L}_k\big(\widehat{\mathfrak{sl}}(2)\big)$-modules $=$ $\widehat{\mathfrak{sl}}(2)$-modules on which $\chi(z)$ acts as zero.

Highest weight classification

Recall $\underbrace{|p\rangle}_{\widehat{\mathfrak{sl}}(2)\text{ hwv}} = \underbrace{u_p}_{\text{Fock vac}} \otimes \underbrace{v}_{\text{ghost vac}}.$

$$\langle p|\chi(w)|p\rangle = \int_{[\Delta_r]} \langle u_p, \mathscr{V}_{-2/\alpha}(z_1+w)\cdots\mathscr{V}_{-2/\alpha}(z_{u-1}+w)\mathscr{V}_{p_{u-1,-v}}(w)\, u_p\rangle \cdot\langle v, :\beta(z_1+w)\cdots\beta(z_{u-1}+w): :\gamma(w)^{u-1}: v\rangle\, \mathrm{d}z_1\cdots\mathrm{d}z_{u-1}.$$

Free boson contribution computed using Jack magic (as in Virasoro minimal models). Ghost contribution uses Wick's theorem.

Proposition 2

Let $\lambda_{r,s} = \lambda_{p_{r,s}} = r-1-ts$. *Then, any highest weight module over the simple VOA* $\mathsf{L}_k\big(\widehat{\mathfrak{sl}}(2)\big)$ *is a simple* $\widehat{\mathfrak{sl}}(2)$*-module of highest weight* $\lambda_{r,s}$*, where* $r = 1,2,\dots,u-1$ *and* $s = 0,1,2,\dots,v-1$*.*

Relaxed highest weight classification

Relaxed highest weight $\widehat{\mathfrak{sl}}(2)$-vectors: $|p;q\rangle = u_p \otimes v_q$, $v_q =$ relaxed highest weight vector with ghost number q.

Free boson contribution to $\langle p;q|\chi(w)|p;q\rangle$ same as before. Ghost contribution messier. Final constraint on p, q is

$$\prod_{(r,s)\in K(u,v)} (\Delta_p - \Delta_{r,s}) \cdot \sum_{\ell=0}^{u-1} \binom{q-1}{\ell}\binom{u-1+\ell}{u-1}\binom{\alpha p + u}{u-1-\ell} = 0,$$

where $\Delta_p = \frac{1}{2}p\left(p+\frac{2}{\alpha}\right)$, $\Delta_{r,s} = \Delta_{p_{r,s}} = \dfrac{(vr-us)^2 - v^2}{4uv}$, and

$K(u,v)$ is the set of $(r,s) \in \{1,\ldots,u-1\} \times \{1,\ldots,v-1\}$ with (r,s) and $(u-r, v-s)$ identified ("Kac table").

Proposition 3

The sum over binomials in the constraint equation is a polynomial of the $\mathfrak{sl}(2)$-weight $\lambda_{p;q} = \alpha p + 2q$ and conformal weight Δ_p:

$$\sum_{\ell=0}^{u-1} \binom{q-1}{\ell} \binom{u-1+\ell}{u-1} \binom{\alpha p + u}{u-1-\ell} = g_u(\lambda_{p;q}, \Delta_p),$$

$$g_{u+2}(\lambda, \Delta) = \frac{(2u+1)\lambda}{(u+1)^2} g_{u+1}(\lambda, \Delta) - \frac{4t\Delta - (u-1)(u+1)}{(u+1)^2} g_u(\lambda, \Delta).$$

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Corollary 4

The Zhu algebra of $\mathsf{L}_k\big(\widehat{\mathfrak{sl}}(2)\big)$ is isomorphic to the quotient of the universal enveloping algebra of $\mathfrak{sl}(2)$ by the ideal generated by the polynomial

$$\prod_{(r,s)\in K(u,v)} (L_0 - \Delta_{r,s}\, \mathbf{1}) \cdot g_u(h_0, L_0).$$

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- *The simple $\widehat{\mathfrak{sl}}(2)$-modules generated by a relaxed highest weight vector of $\mathfrak{sl}(2)$-weight λ and conformal weight $\Delta_{r,s}$, where $(r, s) \in K(u, v)$ and $\lambda \neq \lambda_{r,s} \bmod 2$.*

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There are, for each $(r,s) \in K(u,v)$, two non-semisimple relaxed highest weight $\mathsf{L}_k\big(\widehat{\mathfrak{sl}}(2)\big)$-modules corresponding to the latter case, but with $\lambda = \lambda_{r,s} \bmod 2$. They each have two composition factors.

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Since $K(u,v)$ is non-empty whenever $v > 1$, the relaxed category $\mathscr{R}$ of $\mathsf{L}_k\big(\widehat{\mathfrak{sl}}(2)\big)$-modules is then not semisimple, explaining why fractional level $\widehat{\mathfrak{sl}}(2)$ CFTs are logarithmic.

To the future...

These results should generalise straightforwardly to the VOAs $\mathsf{L}_k(\widehat{\mathfrak{sl}}(n))$ and (perhaps less easily) to other admissible level $\mathsf{L}_k(\widehat{\mathfrak{g}})$.

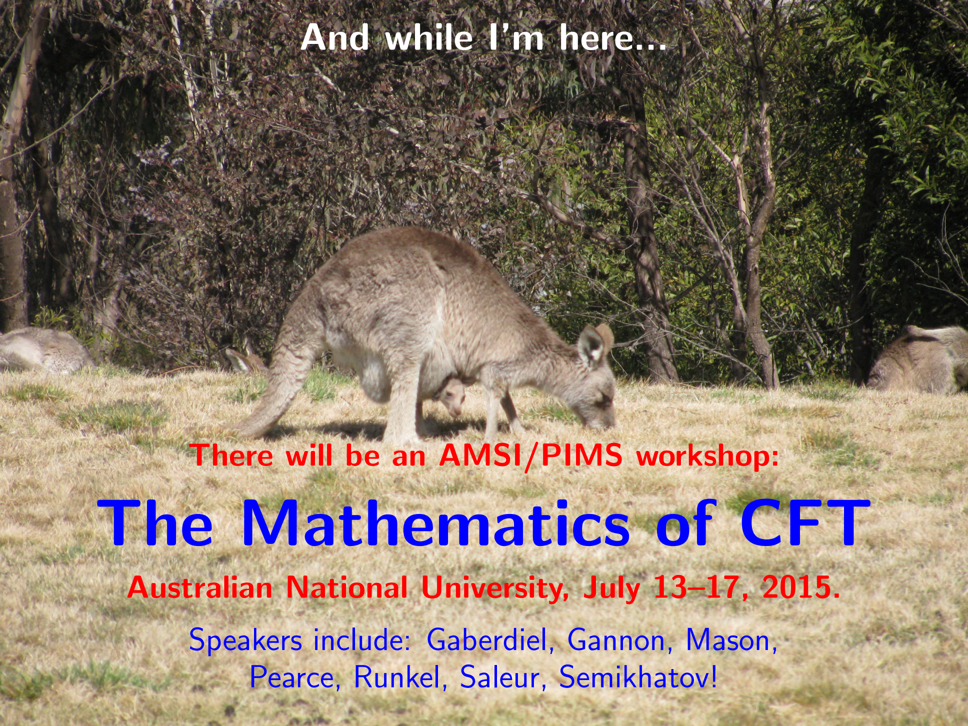
Currently pursuing supersymmetric generalisations, *eg*., $\mathsf{L}_k(\widehat{\mathfrak{sl}}(2|1))$ and superconformal algebras.

Symmetric polynomials and Wakimoto should lead to structural results for relaxed (*ie*. parabolic) highest weight modules, *eg*., Shapovalov determinants, embedding diagrams, ...

Aim to deduce character formulae for standard modules — main input for our formalism for computing modular transformations and Verlinde fusion.

Explicit knowledge of Zhu ideal generators can give information about projectives in non-semisimple VOA categories.

Goal (ambitious of course): Understand logarithmic CFTs for all admissible level Kac-Moody superalgebras.



And while I'm here...

There will be an AMSI/PIMS workshop:

The Mathematics of CFT

Australian National University, July 13–17, 2015.

Speakers include: Gaberdiel, Gannon, Mason,
Pearce, Runkel, Saleur, Semikhatov!