Two-dimensional superconformal algebras (still crazy after all these years)

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- Conformal field theories
- Vertex operator algebras
- Examples

II. Superconformal algebras

III. Minimal models

- Sectors
- Examples

IV. A new hope

V. Outlook

2D Conformal Field Theory

A conformal field theory (CFT) is a quantum field theory whose symmetries not only include the (infinitesimal) length-preserving transformations, the Lorentz algebra, but also the (infinitesimal) angle-preserving transformations, the conformal algebra.

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Physical applications include string theory and critical points of statistical mechanics models.

Mathematical applications include monstrous moonshine, infinite-dimensional Lie algebras, Schramm-Loewner evolution, modular forms, knot theory, subfactors, combinatorics, enumerative geometry, quantum groups, algebraic geometry, *etc*...

Superconformal field theory

Physics utilises both bosonic and fermionic fields:

$$A(z)B(w) = (-1)^{\bar{A}\bar{B}} B(w)A(z), \qquad \bar{A} = \begin{cases} 0 & \text{if } A \text{ is bosonic,} \\ 1 & \text{if } A \text{ is fermionic.} \end{cases}$$

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Examples include the CFTs underlying superstring theories as well as certain statistical models (tricritical Ising, Ashkin-Teller, *etc*...).

Most of the standard CFTs are built from infinite-dimensional Lie algebras. SCFTs then correspond to infinite-dimensional Lie superalgebras (which are less well understood!).

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2. A vacuum state $|0\rangle \in V^0$.
3. A conformal state $|T\rangle \in V^0$.
4. A parity-preserving state-field correspondence from the states of V to the fields in $\mathrm{End}(V)[[z, z^{-1}]]$:

$$|A\rangle \longmapsto A(z), \qquad A(z)|0\rangle\big|_{z=0} = |A\rangle.$$

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$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{12}(m^3-m)\delta_{m+n=0}c\,\mathbf{1}, \quad c \in \mathbb{C}.$$

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The L_0-eigenvalue h of a state is called its conformal weight.

Example: the free boson

We start with the Heisenberg algebra $\widehat{\mathfrak{gl}}(1)$:

$$[a_m, a_n] = m\delta_{m+n=0}\,\mathbf{1} \quad (m, n \in \mathbb{Z}).$$

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- The field product (operator product expansion or OPE) is

$$a(z)a(w) = \frac{1}{(z-w)^2} + :a(z)a(w):.$$

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This time, we start with an infinite-dimensional Lie superalgebra:

$$\{\psi_m, \psi_n\} = \delta_{m+n=0}\, \mathbf{1} \quad (m, n \in \mathbb{Z} - \tfrac{1}{2}).$$

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$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w},$$

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The central charge c is free.

Example: an affine VOA

The affine Kac-Moody algebra $\widehat{\mathfrak{sl}}(2)$ is a central extension of the loop algebra of $\mathfrak{sl}(2)$ ($k \in \mathbb{C}$ is the level):

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Let $J \otimes t^n \equiv J_n$, for $J \in \mathfrak{sl}(2)$. Then, the Lie bracket of $\widehat{\mathfrak{sl}}(2)$ is

$$\left[J_m^a, J_n^b\right] = f^{ab}{}_c J_{m+n}^c + m\kappa^{ab}\delta_{m+n=0} k\,\mathbf{1},$$

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The $\widehat{\mathfrak{sl}}(2)$ Verma module of highest weight 0 does not admit the structure of a vertex operator algebra. Instead, we must quotient by the Verma submodule of highest weight -2.

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- The Sugawara construction gives

$$T(z) = \frac{1}{2(k+2)} \kappa_{ab} : J^a(z) J^b(z) : , \quad c = \frac{3k}{k+2}.$$

There is no conformal structure if $k=-2$.

Superconformal algebras

A superconformal algebra is not just the conformal algebra (Virasoro) extended by fermions.

One takes N Grassmann variables θ_i and works with superfields in

$$\mathrm{End}(V)[[z, z^{-1}]] \otimes \bigwedge(\theta_1, \ldots, \theta_N).$$

There is then an energy-momentum superfield $\mathbb{T}(z; \theta_1, \ldots, \theta_N)$ whose OPE with itself indicates superconformal invariance.

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To study the representation theory, it is convenient to expand in the θ_i and work directly with their component fields:

$$\mathbb{T}(z; \theta_1, \ldots, \theta_N) = \theta_1 \cdots \theta_N \, T(z) + \sum_{i=1}^{N} \theta_1 \cdots \widehat{\theta_i} \cdots \theta_N \, G_i(z) + \cdots .$$

The $N = 1$ superconformal algebra

$N = 0$ is just Virasoro. For $N = 1$, we have

$$\mathbb{T}(z;\theta) = \theta T(z) + \frac{1}{2}G(z).$$

$T(z)$ is bosonic, of conformal weight 2, whereas $G(z)$ is fermionic, of conformal weight $\frac{3}{2}$. Each θ_i has effective weight $-\frac{1}{2}$.

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The component OPEs are

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w},$$

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$$G(z)G(w) \sim \frac{2c/3}{(z-w)^3} + \frac{2\,T(w)}{z-w}.$$

$G(z)$ is a Virasoro primary.

The $N = 2$ superconformal algebra

With two Grassmann variables, we have

$$\mathbb{T}(z;\theta_+,\theta_-) = \theta_+\theta_- \overbrace{T(z)}^{\text{B, wt } 2} + \frac{1}{2}\theta_+ \overbrace{G^-(z)}^{\text{F, wt } \frac{3}{2}} + \frac{1}{2}\theta_- \overbrace{G^+(z)}^{\text{F, wt } \frac{3}{2}} + \frac{1}{2} \overbrace{H(z)}^{\text{B, wt } 1}.$$

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$$T(z)G^{\pm}(w) \sim \frac{\frac{3}{2}G^{\pm}(w)}{(z-w)^2} + \frac{\partial G^{\pm}(w)}{z-w}, \quad T(z)H(w) \sim \frac{H(w)}{(z-w)^2} + \frac{\partial H(w)}{z-w},$$

$$H(z)G^{\pm}(w) \sim \frac{\pm G^{\pm}(w)}{z-w}, \quad H(z)H(w) \sim \frac{c/3}{(z-w)^2}, \quad G^{\pm}(z)G^{\pm}(w) \sim 0,$$

$$G^+(z)G^-(w) \sim \frac{2c/3}{(z-w)^3} + \frac{2\,H(w)}{(z-w)^2} + \frac{2\,T(w) + \partial H(w)}{z-w}.$$

$H(z)$ is a free boson; the $G^{\pm}(z)$ are Virasoro/free boson primaries.

The $N = 3$ superconformal algebra(s)

With three Grassmann variables, the field content is

$$\overbrace{T(z)}^{\text{B, wt } 2}, \quad \overbrace{G^a(z)}^{\text{F, wt } \frac{3}{2}}, \quad \overbrace{J^a(z)}^{\text{B, wt } 1}, \quad \overbrace{\psi(z)}^{\text{F, wt } \frac{1}{2}} \qquad (a \in \{+, 0, -\}).$$

The $J^a(z)$ give an $\widehat{\mathfrak{sl}}(2)$ subalgebra and the $G^a(z)$ define an adjoint representation of $\mathfrak{sl}(2)$.

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$\psi(z)$ is a free fermion... but it decouples! The $N = 3$ VOA is the tensor product of the free fermion VOA and a reduced VOA:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \qquad (c = \tfrac{1}{2}(3k-1)),$$

$$T(z)G^b(w) \sim \frac{\frac{3}{2}\,G^b(w)}{(z-w)^2} + \frac{\partial G^b(w)}{z-w}, \quad J^a(z)G^b(w) \sim \frac{f^{ab}{}_c G^c(w)}{z-w},$$

$$T(z)J^b(w) \sim \frac{J^b(w)}{(z-w)^2} + \frac{\partial J^b(w)}{z-w}, \quad J^a(z)J^b(w) \sim \frac{\kappa^{ab}\,k}{(z-w)^2} + \frac{f^{ab}{}_c J^c(w)}{z-w},$$

$$G^a(z)G^b(w) \sim \frac{2\,\kappa^{ab}(k-1)}{(z-w)^3} + \frac{2(k-1)\,f^{ab}{}_c J^c(w)/k}{(z-w)^2} + \frac{4\,\kappa^{ab}T(w) + f^{ab}{}_c \partial J^c(w) - 2 :J^a(w)J^b(w): /k}{z-w}.$$

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are altogether too ugly to write down!

One has T, four G^a, six J^i, four ψ^a and S. The J^i generate a copy of $\widehat{\mathfrak{so}}(4) = \widehat{\mathfrak{sl}}(2) \oplus \widehat{\mathfrak{sl}}(2)$. The G^a then decompose into two $\mathfrak{sl}(2)$ doublets, as do the ψ^a. S has conformal weight 0.

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The ψ^a and S may be consistently decoupled; this also removes three of the J^i, leaving one copy of $\widehat{\mathfrak{sl}}(2)$.

Sometimes these fields are not decoupled and then one adds an extra field of conformal weight 1.

There also seems to be at least one other $N = 4$ superalgebra intermediate between these possibilities...

Twisted sectors

Fermions admit periodic and antiperiodic boundary conditions:

$G^i(z) = \sum_{n\in\mathbb{Z}+1/2} G^i_n z^{-n-3/2}$ (Neveu-Schwarz),

$G^i(z) = \sum_{n\in\mathbb{Z}} G^i_n z^{-n-3/2}$ (Ramond).

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One can also consider sectors that mix the fermions. *eg.*, one can have $N = 2$ sectors in which $H(z)$ is antiperiodic, but $T(z)$ is periodic. Physicality not clear — required for consistency?

Minimal models

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The free boson and free fermion VOAs are simple and universal; the corresponding (S)CFTs are thus not minimal models.

Virasoro minimal models

The universal Virasoro VOA is simple unless the central charge is

$$c = 1 - \frac{6(p'-p)^2}{pp'}, \qquad p, p' \in \mathbb{Z}_{\geqslant 2}, \quad \gcd\{p, p'\} = 1.$$

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The only Virasoro modules that are modules of the minimal model VOA are the simple highest weight modules of highest weight

$$h_{r,s} = \frac{(p'r-ps)^2 - (p'-p)^2}{4pp'}, \qquad \begin{array}{l} r = 1, 2, \dots, p-1, \\ s = 1, 2, \dots, p'-1 \end{array}$$

and direct sums thereof.

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and direct sums thereof.

Because the representation theory is semisimple and there are a finite number of simple modules, the Virasoro minimal models are rational CFTs. When $|p - p'| = 1$, these CFTs are also unitary.

$\widehat{\mathfrak{sl}}(2)$ minimal models

The universal $\widehat{\mathfrak{sl}}(2)$ VOA is defined for all levels $k \neq -2$.

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When $v > 1$, the minimal model VOA is non-unitary and non-rational: there are finitely many simple highest weight modules, but an uncountable infinity of simple parabolic highest weight modules.

Moreover, the representation theory is then non-semisimple: the minimal model VOA underlies a logarithmic CFT.

$N = 1$ minimal models

The universal $N = 1$ VOA is simple unless the central charge is

$$c = \frac{3}{2} - \frac{3(p'-p)^2}{pp'}, \qquad p, p' \in \mathbb{Z}_{\geqslant 2}, \qquad \begin{gathered} p = p' \bmod 2, \\ \gcd\left\{p, \tfrac{1}{2}(p'-p)\right\} = 1. \end{gathered}$$

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The minimal model VOA is rational and the simple modules are highest weight with

$$h_{r,s} = \frac{(p'r-ps)^2 - (p'-p)^2}{8pp'} + \frac{1}{16}\delta_{r \neq s \bmod 2}, \qquad \begin{aligned} r &= 1, 2, \dots, p-1, \\ s &= 1, 2, \dots, p'-1. \end{aligned}$$

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The proof is somewhat indirect, utilising the minimal models of $\widehat{\mathfrak{sl}}(2)$ and a coset construction.

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Why is $N = 2$ so much harder? Because basic questions about its representations have still not been completely settled.

$N > 2$ minimal models

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Otherwise, we know very little about the $N = 3$ and $N = 4$ minimal models, largely because the representation theories of the superconformal algebras are very poorly understood.

Obstacles for $N > 1$ include:

- submodules of Verma modules that are generated by subsingular vectors,
- submodules of Verma modules not being Verma,
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$N = 1$ Verma modules also exhibit these features, but only in the Ramond sector and only in the Verma module of conformal highest weight $h = c/24$ (and its submodules) — toy model?

Quantum hamiltonian reduction

We've seen that the $N \leqslant 2$ superconformal VOAs are related to $\widehat{\mathfrak{sl}}(2)$. However, the $\widehat{\mathfrak{sl}}(2)$ minimal models were, until recently, infamous for confusing both physicists and mathematicians.

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A more general relationship is provided by quantum hamiltonian reduction, a construction that applies to affine VOAs:

Affine VOA	$\widehat{\mathfrak{sl}}(2)$	$\widehat{\mathfrak{osp}}(1\|2)$	$\widehat{\mathfrak{sl}}(2\|1)$	$\widehat{\mathfrak{osp}}(3\|2)$	$\widehat{\mathfrak{psl}}(2\|2)$
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While the representation theories still present obstacles, affine symmetry is expected to be easier to analyse.

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Programme testing nearly complete for $\widehat{\mathfrak{sl}}(2)$ and Vir. We are currently extending to $\widehat{\mathfrak{osp}}(1|2)$ and $N = 1$.

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Complete results will be relevant to mock/quantum modular forms, mirror symmetry, CFT dualities and generalised moonshine.

And while I'm here...

There will be an MSI special year workshop:

The Mathematics of CFT

Australian National University, July 13–17, 2015.

Speakers include: Gaberdiel, Gannon, Mason, Pearce, Runkel, Saleur, Semikhatov!