Modular Transformations, Representation Theory and Physics

David Ridout

Department of Theoretical Physics
& Mathematical Sciences Institute,
Australian National University

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Some Very Old Number Theory

Question: In how many ways may a given integer be represented as a sum of two squares?

$$
\begin{aligned}
0 &= 0^2 + 0^2, \\
1 &= 1^2 + 0^2 = 0^2 + 1^2 = 0^2 + (-1)^2 = (-1)^2 + 0^2, \\
2 &= 1^2 + 1^2 = 1^2 + (-1)^2 = (-1)^2 + 1^2 = (-1)^2 + (-1)^2, \\
&\vdots
\end{aligned}
$$

Note that 3 cannot be written as the sum of two squares! Continuing by brute force, one eventually arrives at:

0	1	2	3	4	5	6	7	8	9	10	$\cdots$
1	4	4	0	4	8	0	0	4	4	8	$\cdots$

Jacobi Theta Functions

Jacobi noticed that if one takes the generating function

$$\vartheta\left(x\right) = 1 + 2x + 2x^4 + 2x^9 + 2x^{16} + \cdots = \sum_{j\in\mathbb{Z}} x^{j^2}$$

for the number of ways to write an integer as the sum of one square, then

$$\vartheta\left(x\right)^2 = 1 + 4x + 4x^2 + 4x^4 + 8x^5 + 4x^8 + 4x^9 + 8x^{10} + \cdots$$

is the generating function for the number of ways to write an integer as the sum of two squares! Moreover,

$$\vartheta\left(x\right)^2 = 1 + 4\sum_{j,k=1}^{\infty} \left(-1\right)^{k-1} x^{j(2k-1)}.$$

Theta Functions and Kac-Moody Algebras

Theta functions admit nice product formulae, *eg*. Jacobi's triple product identity:

$$\overbrace{\sum_{n\in\mathbb{Z}} z^n q^{n^2/2}}^{\vartheta_3(z;q)} = \prod_{i=1}^{\infty} \left(1 + zq^{i-1/2}\right)\left(1 - q^i\right)\left(1 + z^{-1} q^{i-1/2}\right)$$

$$\Rightarrow \quad \sum_{n\in\mathbb{Z}} (-1)^n z^{2n} q^{n(n+1)/2} = \prod_{i=1}^{\infty} \left(1 - z^2 q^i\right)\left(1 - q^i\right)\left(1 - z^{-2} q^{i-1}\right).$$

Macdonald (see also Kac and Moody) noted that this is the denominator identity for the trivial (level 0) representation of the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}(2)$.

The Lie Algebra $\mathfrak{sl}(2)$

$\mathfrak{sl}(2)$ is the Lie algebra of traceless complex 2×2 matrices with bracket given by the matrix commutator. The standard basis is:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

A representation is a homomorphism π from $\mathfrak{sl}(2)$ into $\operatorname{End} V$. For $\dim V < \infty$, its character is the (formal) sum

$$\chi_V(z) = \operatorname{tr}_V z^{\pi(H)} \overset{\text{Weyl}}{=} \frac{\sum_{w \in W} \det w \, z^{w \cdot \lambda}}{\prod_{\alpha \in \Delta_+} (1 - z^{-\alpha})},$$

where $\lambda \in \mathbb{N}$ is the highest weight of the representation, $W \cong C_2$ is the Weyl group and $\Delta_+ = \{2\}$ is a set of positive roots.

Denominator Identities

The trivial representation π_0, acting on V_0, has $\lambda = 0$, $\dim V_0 = 1$ and $\pi_0(H) = 0$. Therefore,

$$\chi_{V_0}(z) = 1 = \frac{z^0 - z^{-2}}{1 - z^{-2}}.$$

Well, this is obviously true!

The consequence

$$\text{numerator} = \text{denominator}$$

is known as a denominator identity. As you might guess, such identities become less and less obvious as the Lie algebra becomes more and more complicated, *eg.* $\mathfrak{sl}(3)$ gives

$$\begin{aligned} 1 - z_1^{-1} - z_2^{-1} + z_1^{-2} z_2^{-1} + z_1^{-1} z_2^{-2} - z_1^{-2} z_2^{-2} \\ = \left(1 - z_1^{-1}\right)\left(1 - z_2^{-1}\right)\left(1 - z_1^{-1} z_2^{-1}\right). \end{aligned}$$

The Kac-Moody Algebra $\widehat{\mathfrak{sl}}(2)$

Kac and Moody (independently) introduced a class of Lie algebras including

$$\widehat{\mathfrak{sl}}(2) = \mathfrak{sl}(2) \otimes \mathbb{C}\left[t; t^{-1}\right] \oplus \mathbb{C}K \oplus \mathbb{C}L_0.$$

The Lie bracket is given by

$$\left[J \otimes t^m, J' \otimes t^n\right] = \left[J, J'\right] \otimes t^{m+n} + m \,\mathrm{tr}\left(JJ'\right) \delta_{m+n,0} K,$$
$$\left[J \otimes t^m, K\right] = \left[K, L_0\right] = 0, \qquad \left[L_0, J \otimes t^m\right] = -mJ \otimes t^m,$$

where $J, J' \in \mathfrak{sl}(2)$.

This is an infinite-dimensional Lie algebra with an infinite Weyl group $\widehat{W} \cong C_2 \ltimes \mathbb{Z}$ and an infinite set $\widehat{\Delta}_+$ of positive roots.

More Denominator Identities

The character of an (integrable) $\widehat{\mathfrak{sl}}(2)$-representation π is

$$\chi_V(z;q) = \mathrm{tr}_V\, z^{\pi(H\otimes 1)} q^{\pi(L_0 - K/8(K+2))}$$

and Kac wrote this in the form

$$\frac{\text{infinite alternating sum over } \widehat{W}}{\text{infinite product over } \widehat{\Delta}_+.}$$

For the trivial representation, the character becomes

$$1 = \frac{\sum_{n\in\mathbb{Z}} (-1)^n z^{2n} q^{n(n+1)/2}}{\prod_{i=1}^{\infty} (1 - z^2 q^i)(1 - q^i)(1 - z^{-2} q^{i-1})},$$

which is Jacobi's triple product identity.

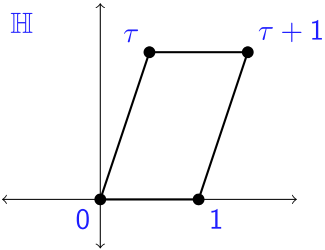
Other Kac-Moody (super)algebras give rise to new identities!

Riemann Surfaces

These are one-dimensional complex manifolds, *ie*. complex curves, meaning real-analytic surfaces admitting a complex structure.

They are designed to generalise the notion of holomorphic functions from $\mathbb{C}$ to other 2D manifolds, *eg*. S^2, T^2.

S^2 admits a unique complex structure, T^2 admits an uncountable infinity of them!



$$T^2 \cong \overline{\mathbb{H}}/\mathrm{SL}\,(2;\mathbb{Z})\,,$$

as complex manifolds, and complex structures on T^2 are parametrised by

$$[\tau] \in \mathbb{H}/\mathrm{SL}\,(2;\mathbb{Z})\,.$$

The Modular Group $\mathrm{SL}(2;\mathbb{Z})$

This is the group of 2×2 matrices with integral entries and unit determinant. It acts on $\tau \in \overline{\mathbb{H}}$ via

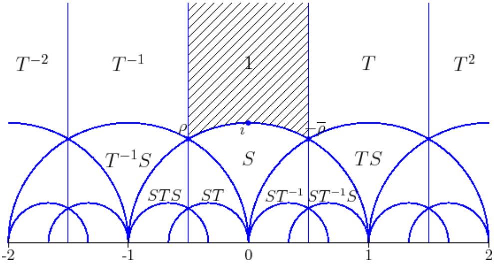
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

A convenient presentation is

$$\mathrm{SL}(2;\mathbb{Z}) = \left\langle S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ : \ S^4 = \mathbf{1},\ (ST)^3 = S^2 \right\rangle,$$

so

$$S \cdot \tau = -\frac{1}{\tau},$$
$$T \cdot \tau = \tau + 1.$$



Modular Forms and Functions

Meromorphic functions from $\overline{\mathbb{H}}$ to $\mathbb{C}$ which are invariant under $\mathrm{SL}(2;\mathbb{Z})$ are called modular functions. They define meromorphic functions on complex tori.

Theorem: The modular functions are precisely the rational functions of the Hauptmodul

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots \quad (q = \mathrm{e}^{2\pi\mathrm{i}\tau}).$$

One therefore defines a modular form to be a holomorphic function f from $\overline{\mathbb{H}}$ to $\mathbb{C}$ which satisfies

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(\tau) = f\left(\frac{a\tau+b}{c\tau+d}\right) = \mu\begin{pmatrix} a & b \\ c & d \end{pmatrix}(c\tau+d)^{\ell} f(\tau),$$

where $\ell \in \mathbb{Q}$ is the weight of f and the multiplier μ satisfies $\left|\mu\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| = 1$.

Examples

- Dedekind $\eta(\tau) = q^{1/24} \prod_{i=1}^{\infty} \left(1 - q^i\right)$ (as always, $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$):

$$\eta(-1/\tau) = \sqrt{-\mathrm{i}\tau}\ \eta(\tau), \quad \eta(\tau+1) = \mathrm{e}^{\mathrm{i}\pi/12}\ \eta(\tau) \quad (\ell = \tfrac{1}{2}).$$

- Jacobi $\vartheta_1(\zeta;\tau) = -\mathrm{i} \sum_{n\in\mathbb{Z}} (-1)^n z^{n+\frac{1}{2}} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}$ (with $z = \mathrm{e}^{2\pi\mathrm{i}\zeta}$):

$$\begin{aligned} \vartheta_1(\zeta/\tau;-1/\tau) &= -\mathrm{i}\sqrt{-\mathrm{i}\tau}\ \mathrm{e}^{\mathrm{i}\pi\zeta^2/\tau}\vartheta_1(\zeta;\tau), \\ \vartheta_1(\zeta;\tau+1) &= \mathrm{e}^{\mathrm{i}\pi/4}\ \vartheta_1(\zeta;\tau) \end{aligned} \qquad (\ell = \tfrac{1}{2}).$$

- Ratio:

$$\frac{\vartheta_1(\zeta/\tau;-1/\tau)}{\eta(-1/\tau)} = -\mathrm{i}\mathrm{e}^{\mathrm{i}\pi\zeta^2/\tau}\frac{\vartheta_1(\zeta;\tau)}{\eta(\tau)} \qquad (\ell = 0).$$

A Different Type of Example

There are three other Jacobi theta functions which combine to give a vector-valued modular form:

$$\Theta(\zeta;\tau) = \begin{bmatrix} \vartheta_2(\zeta;\tau) \\ \vartheta_3(\zeta;\tau) \\ \vartheta_4(\zeta;\tau) \end{bmatrix} = \begin{bmatrix} \sum_{n\in\mathbb{Z}} z^{n+1/2} q^{(n+1/2)^2/2} \\ \sum_{n\in\mathbb{Z}} z^n q^{n^2/2} \\ \sum_{n\in\mathbb{Z}} (-1)^n z^n q^{n^2/2} \end{bmatrix},$$

$$\begin{aligned} \Theta(\zeta/\tau;-1/\tau) &= \mathsf{S}\,\sqrt{-\mathrm{i}\tau}\,\Theta(\zeta;\tau), \\ \Theta(\zeta;\tau+1) &= \mathsf{T}\,\Theta(\zeta;\tau) \end{aligned} \qquad (\ell = \tfrac{1}{2}),$$

$$\mathsf{S} = \mathrm{e}^{\mathrm{i}\pi\zeta^2/\tau} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \mathsf{T} = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\pi/4} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

In this case, the multiplier $\mu\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a unitary matrix.

Kac-Moody Algebras Again!

Recall that theta functions appear in the denominator formula for $\widehat{\mathfrak{sl}}(2)$, itself derived from the Weyl-Kac character formula for the trivial representation.

The characters of many other $\widehat{\mathfrak{sl}}(2)$-representations involve (weight 0, vector-valued) modular forms:

$$\chi_{V_0}(z;q) = \frac{\vartheta_3(z^2;q^2)}{\eta(q)} \qquad (\pi_0(K) = \mathrm{id}),$$

$$\chi_{V_0}(z;q) = \frac{\vartheta_1(z^2;q^3)}{\vartheta_1(z^2;q)} \qquad (\pi_0(K) = -\tfrac{4}{3}\,\mathrm{id}),$$

$$\chi_{V_0}(z;q) = \frac{1}{2}\left[\frac{\eta(q)}{\vartheta_4(z^2;q)} + \frac{\eta(q)}{\vartheta_3(z^2;q)}\right] \qquad (\pi_0(K) = -\tfrac{1}{2}\,\mathrm{id}).$$

Modular forms crop up surprisingly often when considering Kac-Moody algebras (and their generalisations). Why?

Behind the Scenes: CFTs (VOAs)

Conformal field theories are (relativistic) quantum field theories in which the Lorentz symmetry is enhanced to a conformal symmetry. In two-dimensions, this manifests through the Virasoro algebra:

$$\left[L_m, L_n\right] = (m-n)\,L_{m+n} + \frac{1}{12}\left(m^3-m\right)\delta_{m+n,0}\,C, \quad \left[L_m, C\right] = 0.$$

The space of quantum states is then a representation of (two copies of) this infinite-dimensional Lie algebra.

A CFT also possesses a quantum field for every quantum state between which there is a natural, though highly non-trivial, correspondence.

A vertex operator algebra is a mathematically rigorous axiomatisation of a (chiral) CFT, capturing both states and fields.

Conformal Field Theory

In physics, there are two main *raison d'être*'s as far as CFT is concerned:

- Scaling limits of critical lattice models.
- Quantised string theories.

In both cases, one needs to be able to study the CFT on a cylinder (Riemann sphere / complex plane) and on tori (periodic boundary conditions).

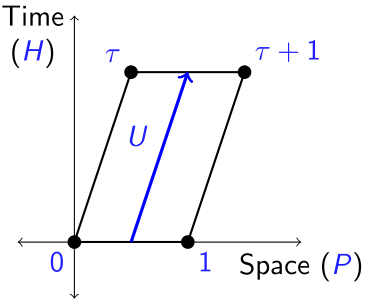
CFTs make much use of complex analysis — changing the complex torus changes the CFT (slightly).

The $\mathrm{SL}(2;\mathbb{Z})$-action preserves the complex structure of the torus, hence the CFT must be invariant under modular transformations.

Partition Functions...

One of the most important objects in any quantum theory is the partition function. This is the trace, over the space of quantum states $\mathcal{H}$, of the evolution operator U.

U is the exponential of a linear combination of the energy and momentum operators describing the infinitesimal time-evolution around the torus.



$$Z(\tau) = \mathrm{tr}_{\mathcal{H}}\, U,$$

$$\begin{aligned} U &= e^{2\pi i\left(\tau(L_0 - C/24) - \overline{\tau}\left(\overline{L}_0 - \overline{C}/24\right)\right)} \\ &= q^{L_0 - C/24}\, \overline{q}^{\overline{L}_0 - \overline{C}/24}. \end{aligned}$$

... are Modular Functions

Conclusion: A CFT defined on a (topological) torus is inconsistent unless the partition function is $\mathrm{SL}(2;\mathbb{Z})$-invariant:

$$Z(-1/\tau) = Z(\tau), \qquad Z(\tau+1) = Z(\tau).$$

Moreover, the quantum state space $\mathcal{H}$ is a representation of the Virasoro algebra (or some larger algebra containing the Virasoro algebra), so the partition function is a character:

$$Z(\tau) = \sum_{V,W} M_{VW} \overline{\chi_V(\tau)} \chi_W(\tau).$$

This explains why characters of infinite-dimensional Lie algebras are often modular forms of weight 0 — they have to sum up to a modular function!

Example: The Ising Model

The critical point of the Ising model is described by a CFT on which both C and $\overline{C}$ act as $\frac{1}{2}\,\mathrm{id}$. The Lie algebra is the Virasoro algebra, but there are only three allowed representations:

$$\chi_{V_0}(\tau) = \frac{1}{2}\left[\sqrt{\frac{\vartheta_3(0;\tau)}{\eta(\tau)}} + \sqrt{\frac{\vartheta_4(0;\tau)}{\eta(\tau)}}\right],$$

$$\chi_{V_{1/16}}(\tau) = \frac{1}{\sqrt{2}}\sqrt{\frac{\vartheta_2(0;\tau)}{\eta(\tau)}},$$

$$\chi_{V_{1/2}}(\tau) = \frac{1}{2}\left[\sqrt{\frac{\vartheta_3(0;\tau)}{\eta(\tau)}} - \sqrt{\frac{\vartheta_4(0;\tau)}{\eta(\tau)}}\right].$$

The only (normalised) $\mathrm{SL}(2;\mathbb{Z})$-invariant modular partition function is

$$\begin{aligned} Z(\tau) &= \left|\chi_{V_0}(\tau)\right|^2 + \left|\chi_{V_{1/16}}(\tau)\right|^2 + \left|\chi_{V_{1/2}}(\tau)\right|^2 \\ &= \left|\frac{\vartheta_2(0;\tau)}{2\eta(\tau)}\right| + \left|\frac{\vartheta_3(0;\tau)}{2\eta(\tau)}\right| + \left|\frac{\vartheta_4(0;\tau)}{2\eta(\tau)}\right|, \end{aligned}$$

corresponding to the state space decomposition

$$\mathcal{H} \cong (V_0 \otimes V_0) \oplus (V_{1/16} \otimes V_{1/16}) \oplus (V_{1/2} \otimes V_{1/2}).$$

Example: The WZW model on $\mathrm{SU}(2)$

The Wess-Zumino-Witten model on $\mathrm{SU}(2)$ is a CFT with Lie algebra $\widehat{\mathfrak{sl}}(2)$ for which K acts as $k\,\mathrm{id}$, $k \in \mathbb{N}$.

The Virasoro algebra is constructed as a quadratic expression in $\widehat{\mathfrak{sl}}(2)$-generators and C acts as $3k/(k+2)$.

When $k = 1$, only two representations are allowed:

$$\chi_{V_0}(\zeta;\tau) = \frac{\vartheta_3(2\zeta;2\tau)}{\eta(\tau)}, \qquad \chi_{V_1}(\zeta;\tau) = \frac{\vartheta_2(2\zeta;2\tau)}{\eta(\tau)}.$$

The partition function is

$$Z(\zeta;\tau) = \left|\frac{\vartheta_2(2\zeta;2\tau)}{\eta(\tau)}\right|^2 + \left|\frac{\vartheta_3(2\zeta;2\tau)}{\eta(\tau)}\right|^2,$$

corresponding to $\mathcal{H} \cong (V_0 \otimes V_0) \oplus (V_1 \otimes V_1)$.

Example: The Free Bosonic String

This CFT admits the Kac-Moody algebra $\widehat{\mathfrak{gl}}(1)$,

$$\left[a_m, a_n\right] = m\delta_{m+n,0},$$

from which one may construct a Virasoro algebra with $C = \mathrm{id}$:

$$L_n = \begin{cases} \frac{1}{2}\sum_{r\in\mathbb{Z}} a_r a_{n-r} & \text{if } n \neq 0, \\ \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_{-n}a_n & \text{if } n = 0. \end{cases}$$

There are no constraints on the representations and

$$\chi_{V_p}(\tau) = \frac{\mathrm{e}^{\mathrm{i}\pi\tau p^2}}{\eta(\tau)} \qquad \text{(not a modular form!)}.$$

One is led to try a state space of the form

$$\mathcal{H} = \int_{\mathbb{R}} V_p \otimes V_p \, \mathrm{d}p,$$

which leads to the partition function

$$\begin{aligned} Z(\tau) &= \int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}\pi\tau p^2}}{\eta(\tau)} \frac{\mathrm{e}^{-\mathrm{i}\pi\overline{\tau} p^2}}{\overline{\eta(\tau)}} \, \mathrm{d}p = \frac{1}{|\eta(\tau)|^2} \int_{\mathbb{R}} \mathrm{e}^{-2\pi \operatorname{Im}\tau p^2} \, \mathrm{d}p \\ &= \frac{1}{\sqrt{2 \operatorname{Im}\tau} \, |\eta(\tau)|^2} \qquad (\text{since } \operatorname{Im}\tau > 0). \end{aligned}$$

This is an $\mathrm{SL}(2;\mathbb{Z})$-invariant (modular function):

$$\begin{aligned} \sqrt{\operatorname{Im}(-1/\tau)} \, |\eta(-1/\tau)|^2 &= \sqrt{\operatorname{Im}\left(-\overline{\tau}/|\tau|^2\right)} \left|\sqrt{-\mathrm{i}\tau}\eta(\tau)\right|^2 \\ &= \sqrt{\operatorname{Im}\tau} \, |\eta(\tau)|^2 . \end{aligned}$$

The Verlinde Formula

The characters of a "nice" CFT (VOA) form a vector-valued modular form of weight 0:

$$\chi(-1/\tau) = \mathsf{S}\chi(\tau).$$

The Verlinde formula states that the numbers obtained from the entries of the matrix S by

$$\mathbf{N}_{ij}{}^{k} = \sum_{\ell} \frac{\mathsf{S}_{i\ell}\mathsf{S}_{j\ell}\mathsf{S}^{\dagger}_{k\ell}}{\mathsf{S}_{0\ell}},$$

where ℓ runs over the set indexing S and 0 refers to the "vacuum character" (VOA), are non-negative integers.

This is now a theorem of Huang for "rational" VOAs.

Fusion Coefficients

These fusion coefficients $\mathbf{N}_{ij}{}^{k}$ count:

- Dimensions of spaces of conformal blocks.
- Dimensions of spaces of generalised theta functions.
- Dimensions of spaces of sections of powers of determinant line bundles over the moduli space of G-bundles over Riemann surfaces with marked points.

They form the structure constants of a commutative ring which, for WZW models, agrees with the twisted equivariant K-theory of G.

The matrices $\mathbf{N}_i$ with entries $\mathbf{N}_{ij}{}^{k}$ are simultaneously diagonalised by S. This follows from a detailed study of the consistency of correlation functions on general Riemann surfaces, *ie*. the Verlinde formula is expected to hold (in some form) in any consistent CFT.

Outlook

There has been a lot of recent progress in studying non-rational CFTs. Motivations for abandoning rationality include:

- Non-local observables of lattice models require non-rational CFT.
- Strings on non-compact spacetimes are usually not rational.
- Neither are strings on supersymmetric spacetimes.
- Schramm-Loewner Evolution corresponds to non-rational CFTs.

Generic features of non-rational CFTs seem to include:

- Continuous spectrum of representations (*cf.* bosonic string).
- Appearance of reducible indecomposable representations.
- Logarithmic singularities in correlation functions.

From the perspective of modularity:

- Continuous spectrum logarithmic CFTs appear to behave — characters are not quite modular forms, but integration of anomalies takes care of extra τ-factors (*cf.* bosonic string). Verlinde works!
- Examples include $\widehat{\mathfrak{gl}}(1|1)$, $\widehat{\mathfrak{sl}}(2)$ with fractional level, *etc*...
- Discrete spectrum logarithmic CFTs do not behave — characters are weight $\ell \neq 0$ modular forms, but no integration to deal with τ factors. Verlinde fails (fusion coefficients are not even diagonalisable), but see many proposed modifications.
- Nevertheless, modular invariant partition functions exist!
- Examples include triplet models $W(p, q)$ (advertised as logarithmic minimal models).

Presumably, discrete spectrum logarithmic CFTs are consistent. How does this change the Verlinde formula? Nobody knows...

And while I'm here...

There will be an MSI special year workshop:

The Mathematics of CFT

Australian National University, July 13–17, 2015.

Speakers include: Gaberdiel, Gannon, Mason, Pearce, Runkel, Saleur, Semikhatov!