Beyond rational conformal field theory

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Rational conformal field theory

Non-rational conformal field theory

Logarithmic conformal field theory

Where have I been?

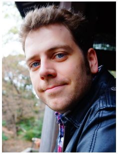
A logarithmic Verlinde formula

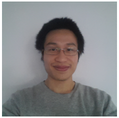
Fractional level WZW models

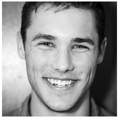
Where am I going?

Short-term goals

Long-term goals

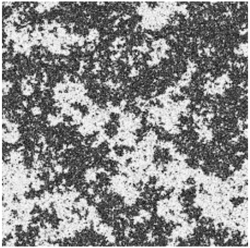





Motivating example: the Ising model

The Ising model describes a simple magnet. The 2D model undergoes a continuous phase transition at a critical temperature.



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$$\langle \sigma \rangle \sim (T_c - T)^{1/8} \quad (T < T_c)$$

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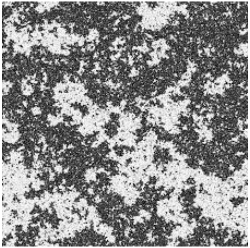

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At the critical point, there are clusters of all shapes and sizes!

Fractal structure ⇒ scale invariance ⇒ conformal invariance.

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In 2D, the conformal symmetry gives (two commuting copies of) the Virasoro algebra $\mathfrak{vir}$:

$$\left[L_m, L_n\right] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n=0}\mathbf{1}.$$

Here, $c \in \mathbb{R}$ is the central charge of the CFT.

The quantum state space

The quantum state space $\mathcal{H}$ carries a representation of $\mathfrak{vir} \oplus \mathfrak{vir}$, *ie.* $\mathcal{H}$ is a $\mathfrak{vir} \oplus \mathfrak{vir}$-module. (Larger symmetry algebras are possible!)

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eg. the continuum scaling limit of the Ising model is the minimal model CFT $\mathsf{M}(3,4)$ ($c = \frac{1}{2}$):

$$\mathcal{H} = (\mathcal{L}_0 \otimes \mathcal{L}_0) \oplus (\mathcal{L}_{1/16} \otimes \mathcal{L}_{1/16}) \oplus (\mathcal{L}_{1/2} \otimes \mathcal{L}_{1/2}).$$

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The highest weight $\frac{1}{16}$ dictates $\langle \sigma \rangle \sim (T_c - T)^{1/8}$ (partially) and $\langle \sigma_i \sigma_j \rangle \sim |i - j|^{-1/4}$ (totally).

Vertex operator algebras

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states $|\phi\rangle \in \mathcal{V}$ ⟷ (state-field correspondence) ⟷ fields $\phi(z)$ ⟷ ($\phi(z)=\sum_n \phi_n z^{-n-h}$) ⟷ modes $\phi_n \in \mathrm{End}(\mathcal{V})$

mode action $\phi_n|\psi\rangle$ (*eg.* $\mathfrak{vir}$-module) ⟷ operator product $\phi(z)\psi(w)$ expansion ⟷ commutator $[\phi_m, \psi_n]$ (*eg.* $\mathfrak{vir}$)

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Classification very difficult (impossible?).

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- The characters $\mathrm{ch}_{\mathcal{M}} = \mathsf{tr}_{\mathcal{M}}\, q^{L_0 - c/24}$ span an $\mathrm{SL}(2;\mathbb{Z})$-module [Zhu]. The partition function $\mathrm{ch}_{\mathcal{H}}$ is modular invariant.

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- The S-transform $\tau \mapsto -\frac{1}{\tau}$ is represented by a matrix S whose entries recover the fusion product via the Verlinde formula:

$$\mathcal{A} \times \mathcal{B} = \bigoplus_{\mathcal{C}} \begin{bmatrix} & \mathcal{C} & \\ \mathcal{A} & & \mathcal{B} \end{bmatrix} \mathcal{C}, \qquad \begin{bmatrix} & \mathcal{C} & \\ \mathcal{A} & & \mathcal{B} \end{bmatrix} = \sum_{\mathcal{D}} \frac{\mathsf{S}_{\mathcal{A}\mathcal{D}} \mathsf{S}_{\mathcal{B}\mathcal{D}} \mathsf{S}^*_{\mathcal{C}\mathcal{D}}}{\mathsf{S}_{\mathcal{V}\mathcal{D}}}.$$

(The VOA-modules form a modular tensor category [Huang].)

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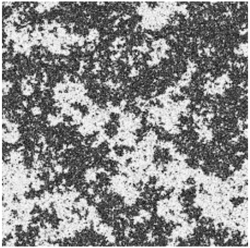

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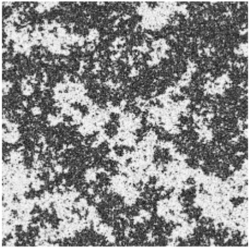

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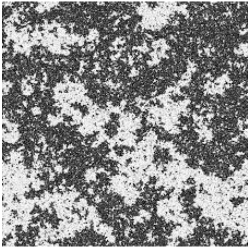

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Such questions seem to need non-rational CFT.

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Physicists tend to assume semisimplicity in non-rational CFT. This is unlikely to be a good assumption.

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All are "rank 1" - plenty of room for further investigation!

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- Projective modules are difficult to identify [Tsuchiya-Nagatomo].

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- Short-term goals
- Long-term goals

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Measure theory fails for the triplet models. But, triplet fusion obtained via simple current extensions [DR-Wood].

Triplet Verlinde formula should follow [Melville-DR].

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Admissible level $\widehat{\mathfrak{sl}}(2)_k$

The unitary minimal models $\mathsf{M}(p, p+1)$ are cosets [Goddard-Kent-Olive] of the unitary WZW models $\widehat{\mathfrak{sl}}(2)_k$ with $k \in \mathbb{Z}_{\geq 0}$.

Kent proposed non-unitary WZW models, for $k+2 = \frac{u}{v}$, $u, v \in \mathbb{Z}_{\geq 2}$, $\gcd\{u, v\} = 1$, to construct non-unitary minimal models.

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[Di Francesco-Mathieu-Sénéchal, Sec. 18.6] suggest that these non-unitary $\widehat{\mathfrak{sl}}(2)_k$ models might suffer from an intrinsic "sickness".

Diagnosis [DR]: $\mathrm{ch}_{\mathcal{M}} = \mathsf{tr}_{\mathcal{M}}\, z^{H_0} q^{L_0 - c/24}$ converges for $|q| < 1$ and z in an annulus (*cf.* unitary result: $|q| < 1$ and $z \neq 0$).

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Results consistent with the known fusion rules for $k = -\frac{1}{2}, -\frac{4}{3}$.

— Case closed: patient may be discharged —

Rational conformal field theory

Non-rational conformal field theory

Logarithmic conformal field theory

Where have I been?

A logarithmic Verlinde formula

Fractional level WZW models

Where am I going?

Short-term goals

Long-term goals

Future plans

A toolkit for studying logCFTs begins by classifying VOA-modules.

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Aim to further extend these results in (at least) three directions:

- Admissible level affine VOAs, *eg.* $\widehat{\mathfrak{osp}}(1|2)_k$, $\widehat{\mathfrak{sl}}(3)_k$, $\widehat{\mathfrak{sl}}(2|1)_k$.
- Superconformal minimal models with $N = 1, 2, \ldots$
- Extend Jack polynomial technology to correlators and fusion.

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And, of course, one day I'd like to write a book...

Thankyou!