Classifying representations for conformal field theory

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Conformal field theory
Vertex operator algebras

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Conformal field theory

A CFT is a QFT with conformal invariance:

Poincaré symmetry $\leftrightarrow$ Conformal symmetry
Conformal field theory

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In 2D, the conformal symmetry gives (two commuting copies of) the Virasoro algebra $\mathfrak{vir}_c$:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n=0}.$$ 

Here, $c \in \mathbb{R}$ is the central charge of the CFT.
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CFTs are fundamental to statistical mechanics and string theory and (increasingly) to pure mathematics.
Vertex operator algebras

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The mathematical structure underlying a conformal field theory is called a \textit{vertex operator algebra (VOA)} \cite{Borcherds:1986,Borcherds:1987}.

Fields may be differentiated and admit a normally ordered product:

\[
: \phi(w) \psi(w) : = \oint_w \frac{\phi(z) \psi(w)}{z - w} \frac{dz}{2\pi i}.
\]
Examples

- The Heisenberg algebra $\widehat{h}$:

$$[a_m, a_n] = m\delta_{m+n=0}1.$$

Its weight 0 Verma module is a VOA with OPE

$$a(z)a(w) \sim \frac{1}{(z-w)^2}.$$
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- The weight 0 Verma module of the Virasoro algebra \( \text{vir}_c \) has a weight 1 Verma submodule. The quotient is a VOA with OPE

\[
L(z)L(w) \sim \frac{c}{2} \frac{1}{(z - w)^4} + \frac{2L(w)}{(z - w)^2} + \frac{\partial L(w)}{z - w}.
\]
The affine Kac-Moody algebra $\hat{\mathfrak{sl}}(2)_k$:

$$[J^a_m, J^b_n] = [J^a, J^b]_{m+n} + m\kappa(J^a, J^b)\delta_{m+n=0} k \mathbf{1}. $$

Its weight 0 Verma module has a weight $-2$ Verma submodule. For $k \neq -2$, the quotient is a VOA with OPE:

$$J^a(z) J^b(w) \sim \frac{\kappa(J^a, J^b) k \mathbf{1}}{(z-w)^2} + \frac{[J^a, J^b](w)}{z-w}. $$
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• The $\mathfrak{vir}_c$ and $\hat{\mathfrak{sl}}(2)_k$ VOAs are not simple iff

$$c = c(p, p') = 1 - \frac{6(p' - p)^2}{pp'}, \quad p, p' \in \mathbb{Z}_{\geq 2}, \gcd\{p, p'\} = 1,$$

$$k = k(u, v) = -2 + \frac{u}{v}, \quad u \in \mathbb{Z}_{\geq 2}, v \in \mathbb{Z}_{\geq 1}, \gcd\{u, v\} = 1,$$

respectively. The simple quotients are the minimal models.
VOA modules

Importantly, VOAs have representation theories.
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- This is not true for Virasoro minimal models, e.g. Virasoro!

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c = c(2, 5) = -\frac{22}{5} \quad \Rightarrow \quad |\chi\rangle := (L_{-2}^2 - \frac{3}{5} L_{-4}) |0\rangle = 0
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⇒ the only highest-weight modules of the $(2, 5)$ minimal model VOA are the irreducibles of weights $0$ and $-\frac{1}{5}$. 
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How to do this calculation for general minimal models?
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There are many families of $n$-variable symmetric polynomials.
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Partitions $\lambda = [\lambda_1, \ldots, \lambda_m]$ give more:

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_m}, \quad e_\lambda = e_{\lambda_1} \cdots e_{\lambda_m}, \quad h_\lambda = h_{\lambda_1} \cdots h_{\lambda_m}.$$
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Partitions $\lambda = [\lambda_1, \ldots, \lambda_m]$ give more:

$$p_{\lambda} = p_{\lambda_1} \cdots p_{\lambda_m}, \quad e_{\lambda} = e_{\lambda_1} \cdots e_{\lambda_m}, \quad h_{\lambda} = h_{\lambda_1} \cdots h_{\lambda_m}.$$  

And monomial, Schur, Jack, ... also indexed by partitions.
Schur polynomials

The $n$-variable Schur polynomials are defined by

$$s_\lambda(z) = \frac{\det (z_i^{\lambda_j+n-j})}{\det (z_i^{n-j})}.$$

They arise in representation theory ($S_n$, $GL(n)$, $U(n)$, ...).
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Schur polynomials have a remarkable \textbf{orthonormality} property:

$$\langle f, g \rangle_{n} = \int_{[\Delta_{n}]} \prod_{1 \leq i \neq j \leq n} \left(1 - \frac{z_{i}}{z_{j}}\right)f(z_{1}^{-1}, \ldots, z_{n}^{-1})g(z_{1}, \ldots, z_{n}) \frac{dz_{1} \cdots dz_{n}}{z_{1} \cdots z_{n}}$$

$$\Rightarrow \quad \langle s_{\lambda}, s_{\mu} \rangle_{n} = \delta_{\lambda=\mu}.$$
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$$\int_{[\Delta_n]}$$

is normalised so that $\langle s_0, s_0 \rangle_n = \langle 1, 1 \rangle_n = 1.$
Jack polynomials

Jack polynomials are deformations of Schur polynomials:

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J_{\lambda}^\kappa (z) \xrightarrow{\kappa=1} s_{\lambda} (z).
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\[ J_\lambda^\kappa (z) \overset{\kappa=1}{\rightarrow} s_\lambda (z). \]

They enjoy a similar orthogonality property:

\[ \langle f, g \rangle_\kappa = \int_{[\Delta_n]} \prod_{1 \leq i \neq j \leq n} \left( 1 - \frac{z_i}{z_j} \right)^{1/\kappa} f(z_1^{-1}, \ldots, z_n^{-1})g(z_1, \ldots, z_n) \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n} \]

\[ \Rightarrow \quad \langle J_\lambda^\kappa, J_\mu^\kappa \rangle_\kappa = \left\{ \begin{array}{ll} \text{explicit combinatorial factor} & \delta_\lambda=\mu. \end{array} \right\} \]

Special case (but crucial later):

\[ J_\kappa^\kappa (z_1, \ldots, z_n) = n \prod_{i=1}^{\kappa} z_i^{m_i} \quad \text{for all } \kappa. \]
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Special case (but crucial later):

$$J_{[m^n]}^\kappa (z_1, \ldots, z_n) = \prod_{i=1}^n z_i^m \quad (\text{for all } \kappa).$$
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The zero modes $\phi_0$ of the fields $\phi(z)$ of a VOA act on the space of ground states of a module. This action defines Zhu’s algebra.
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- For the Heisenberg VOA $\hat{\mathfrak{h}}$, Zhu’s algebra is $\mathbb{C}[a_0]$.
- For the Virasoro VOA $\mathfrak{vir}_c$, Zhu’s algebra is $\mathbb{C}[L_0]$.
- For the VOA $\hat{\mathfrak{sl}}(2)_k$, Zhu’s algebra is $\mathcal{U}\mathfrak{sl}(2)$. 
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Irreducible modules are under control for these VOAs.

But, what about their minimal models? That’s much harder!
Ancient history

Zhu’s construction preserves structure:

\[ \text{Zhu} \left( \frac{V}{I} \right) = \frac{\text{Zhu}(V)}{\text{Zhu}(I)}. \]
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- Projection of Virasoro singular vector [Feigin-Fuchs ’88]
  \( \Rightarrow \) generator of \( \text{Zhu}(I) \) (\( \text{Zhu}(V) = \mathbb{C}[L_0] \)).
  \( \Rightarrow \) classify simple Virasoro minimal model modules [Wang ’93].
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- \( \hat{\mathfrak{sl}}(2)_k \) singular vector \([\text{Malikov-Feigin-Fuchs '86}] \) projection \([\text{Fuchs '89}]\)
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- Few other examples! Projection technology doesn’t generalise?
More ancient history

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- Deformed free field realisation/construction works for $\text{vir}_c$ and Jack polynomials [Mimachi-Yamada '95].
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Recent resurgence of interest due to AGT conjecture [Alday-Gaiotto-Tachikawa '09] and “generalised” Jack polynomials [Morozov-Smirnov '13].
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Free field constructions of singular vectors use screening operators.
Why Jack polynomials?

Free field constructions of singular vectors use screening operators.

Screening operators intertwine the action of the Virasoro VOA on the free field modules.
Q is the zero mode of a product of $\hat{\mathfrak{h}}$ vertex operators:

$$V_{\alpha}(z) = e^{\alpha q} z^{\alpha a_0} \prod_{m=1}^{\infty} e^{\alpha a_m z^m / m} e^{-\alpha a_m z^{-m} / m}.$$
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To get the vacuum singular vector $|\chi\rangle$ of $\text{vir}_{c(p,p')}$, we take

$$Q = \int_{[\Delta_{p-1}]} V_\alpha(z_1) \cdots V_\alpha(z_{p-1}) \, dz_1 \cdots dz_{p-1}, \quad \alpha = \sqrt{\frac{2p'}{p}},$$

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Commute the exponentials in the vertex operators:

$$|\chi\rangle = \int_{[\Delta_{p-1}]} \prod_{1 \leq i \neq j \leq p-1} (z_i - z_j)^{\alpha^2/2} \cdot \prod_{i=1}^{p-1} z_i^{-(p-1)\alpha^2}$$

$$\cdot \prod_{m=1}^{\infty} e^{\alpha a_m p_m(z_1,\ldots,z_{p-1})/m} |0\rangle \, dz_1 \cdots dz_{p-1}. $$
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Commute the exponentials in the vertex operators:

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Commute the exponentials in the vertex operators:

$$|\chi\rangle = \int_{[\Delta_{p-1}]} \prod_{1 \leq i \neq j \leq p-1} \left(1 - \frac{z_i}{z_j}\right)^{p'/p} J_{p/p'}^{p/(p'-1)} \left(z_1^{-1}, \ldots, z_{p-1}^{-1}\right)$$

$$\cdot \prod_{m=1}^{\infty} e^{\alpha a_m p_m (z_1, \ldots, z_{p-1})/m} |0\rangle \frac{dz_1 \cdots dz_{p-1}}{z_1 \cdots z_{p-1}}.$$
If we can write $\prod_{m=1}^{\infty} e^{\alpha a_m p_m(z)/m}$ in terms of $\kappa = \frac{p}{p'}$ Jacks, their orthogonality will translate into a singular vector formula!
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The vacuum singular vector is thus

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Other singular vectors obtained similarly!
Classifying VOA modules

Idea: Given $|\chi\rangle$, solve $\langle \mu | \chi(w) | \mu \rangle = 0$ for highest weights $\mu$. 
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The product may be decomposed into Jacks using specialisation:

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Homogeneity and orthogonality finish the job!
**Theorem** [Wang, DR-Wood]: The Virasoro minimal model VOA with $c = c(p, p')$ is *rational* and the irreducibles are precisely those of highest weight

$$h_{r,s} = \frac{(p'r - ps)^2 - (p' - p)^2}{4pp'}, \quad 1 \leq r \leq p - 1, \ 1 \leq s \leq p' - 1.$$
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We have generalised this methodology to $\hat{\mathfrak{sl}}(2)_k$ minimal models (Coulomb gas $\rightarrow$ Wakimoto, ie. dress with $\beta\gamma$ ghosts).
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- Proved new singular vector formulae for parabolic highest-weight modules.
- Exhibited reducible, but indecomposable, modules for the $\hat{sl}(2)_k$ minimal models. (ie. VOA is logarithmic.)
Conformal field theory

Vertex operator algebras

A brief detour

Symmetric polynomials

History lesson

Zhu’s algebra

Classifying VOA modules

Singular vectors

The classification

Where am I going?

Short-term goals

Grander plans
(Some) short-term goals

A toolkit for studying CFTs begins by classifying VOA-modules.
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A grander scheme

Inject results into the (log)CFT toolkit [Creutzig-DR-Wood, v1.1e]:

• Classify \(N\)-graded VOA-modules.
• Construct parabolics and irreducibles through resolutions.
• Modularity and Verlinde via standard modules [Creutzig-DR].
• Construct projectives and decompose fusion products.
• Classify staggered modules, cf. [Kytölä-DR].
• Extend symmetric polynomial game to correlators.
• Profit!
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The idea is to exploit rigorous free field methods as “brute force” calculations become infeasible at higher rank.
Thank you!