

# Non-rational CFTs and the Verlinde formula

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CFTs, VOAs and the Verlinde formula

Dropping log-rationality

The standard module formalism

A log-rational Verlinde formula?

## Rational CFT and the Verlinde formula

Two of the ingredients of CFT are:

- A **vertex operator algebra** (VOA)  $V$ .
- A **physical category**  $\mathcal{C}$  of  $V$ -modules that is
  - closed under conjugation  $\mathcal{C}$ ,
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For rational CFTs, the  $S$ -transform of the irreducible characters satisfies:

- $S^T = S$ ,  $S^\dagger = S^{-1}$ ,  $S^2 = C$ .
- $S$  diagonalises the fusion rules through the **Verlinde formula** [Huang]:

$$\mathcal{L}_i \times \mathcal{L}_j = \bigoplus_k \begin{bmatrix} k \\ i \ j \end{bmatrix} \mathcal{L}_k, \quad \begin{bmatrix} k \\ i \ j \end{bmatrix} = \sum_\ell \frac{S_{i\ell} S_{j\ell} S_{k\ell}^*}{S_{0\ell}}.$$

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How does the formalism of rational CFT, especially Verlinde, generalise to non-rational and logarithmic CFT?



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If the goal is to decompose fusion products, the Verlinde formula helps!

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But, there is [Fuchs-Hwang-Semikhatov-Tipunin] a  $W(1, p)$  Verlinde-like formula for simple characters (automorphy factor cancels  $\tau$ -dependence).

## A non-logarithmic non-rational CFT

The free boson:  $\mathcal{V}$  = Heisenberg VOA:  $[a_m, a_n] = m\delta_{m+n=0}\mathbf{1}$ .

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- $\text{ch}_{\mathcal{F}_p} = \text{tr}_{\mathcal{F}_p} y^{\mathbf{1}} z^{a_0} q^{L_0 - \mathbf{1}/24} = \frac{yz^p q^{p^2/2}}{\eta(q)}$ .



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- $\begin{bmatrix} r \\ p & q \end{bmatrix} = \int_{-\infty}^{\infty} \frac{S_{ps} S_{qs} S_{rs}^*}{S_{0s}} ds = \delta(r = p + q)$ ,

$$\Rightarrow \mathcal{F}_p \times \mathcal{F}_q = \int_{-\infty}^{\infty} \begin{bmatrix} r \\ p & q \end{bmatrix} \mathcal{F}_r dr = \mathcal{F}_{p+q}. \quad \checkmark$$

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- $S\{\text{ch}_{\mathcal{M}}\} = \int_{-\infty}^{\infty} S_{\mathcal{M}\mathcal{F}_q} \text{ch}_{\mathcal{F}_q} dq$ :

$$S_{\mathcal{F}_p\mathcal{F}_q} = e^{-2\pi i(p-\frac{1}{2})(q-\frac{1}{2})}, \quad S_{\mathcal{L}_p\mathcal{F}_q} = \frac{e^{-2\pi i p(q-\frac{1}{2})}}{2 \cos[\pi(q-\frac{1}{2})]}.$$

↑  
**note: pole!**

- The Verlinde formula  $\left[ \begin{array}{c} \mathcal{F}_r \\ \mathcal{M} \quad \mathcal{N} \end{array} \right] = \int_{-\infty}^{\infty} \frac{S_{\mathcal{M}\mathcal{F}_s} S_{\mathcal{N}\mathcal{F}_s} S_{\mathcal{F}_r\mathcal{F}_s}^*}{S_{\mathcal{L}_0\mathcal{F}_s}} ds$  gives

$$\left[ \begin{array}{c} \mathcal{F}_r \\ \mathcal{L}_p \quad \mathcal{L}_q \end{array} \right] = \sum_{n=1}^{\infty} (-1)^{n-1} \delta(r = p + q + n),$$

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$\Rightarrow$  **(Grothendieck)**  $\mathcal{L}_p \times \mathcal{L}_q = \mathcal{L}_{p+q}$ ,  $\mathcal{L}_p \times \mathcal{F}_q = \mathcal{F}_{p+q}$ ,  $\checkmark$   
 fusion rules  $[\mathcal{F}_p \times \mathcal{F}_q] = [\mathcal{F}_{p+q}] + [\mathcal{F}_{p+q-1}]$ .



## The standard module formalism

In all known examples of non-log-rational CFTs, we have identified (indecomposable) **standard modules** with excellent modular properties.

We partition them into irreducible (**typical**) and reducible (**atypical**).

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5. If  $\text{ch}_{\mathcal{M}} = \sum_m a_m \text{ch}_m$ , define  $S_{\mathcal{M}n} = \sum_m a_m S_{mn}$ . This sum converges for all typical  $n$  ( $n \notin A$ ).

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6. The vacuum module  $\Omega$  satisfies  $S_{\Omega n} \neq 0$ , for all  $n \notin A$ .



Now, define a product  $\boxtimes$  using the **standard Verlinde formula**:

$$\begin{aligned}\mathrm{ch}_{\mathcal{M}} \boxtimes \mathrm{ch}_{\mathcal{N}} &= \int_M \left[ \begin{array}{c} p \\ \mathcal{M} \quad \mathcal{N} \end{array} \right] \mathrm{ch}_p \, d\mu(p), \\ \left[ \begin{array}{c} p \\ \mathcal{M} \quad \mathcal{N} \end{array} \right] &= \int_M \frac{S_{\mathcal{M}q} S_{\mathcal{N}q} S_{pq}^*}{S_{\Omega q}} \, d\mu(q).\end{aligned}$$

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Rational CFTs form the “trivial” examples of this formalism:

- Standard = irreducible, so no atypicals ( $A = \emptyset$ ).
- The measurable space  $M$  is finite and  $\mu$  is counting measure.
- Grothendieck fusion = fusion.

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Logarithmic conformal field theory	Fusion known?
Virasoro logarithmic minimal models $LM(p, p')$	Many examples
$N = 1$ logarithmic minimal models $LSM(p, p')$	Some examples
Singlet models $I(p, p') = W_{2, (2p-1)(2p'-1)}$	?
Admissible level $\widehat{\mathfrak{sl}}(2)_k$	$k = -\frac{1}{2}, -\frac{4}{3}$
Bosonic $\beta\gamma$ ghosts	✓
$GL(1 1)$ Wess-Zumino-Witten model	✓

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The singlet model results imply Grothendieck fusion rules for the log-rational triplet models  $W(p, p')$ . These are consistent with the known triplet fusion rules (and conjectures).

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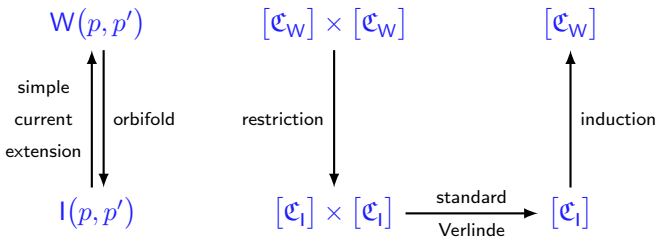


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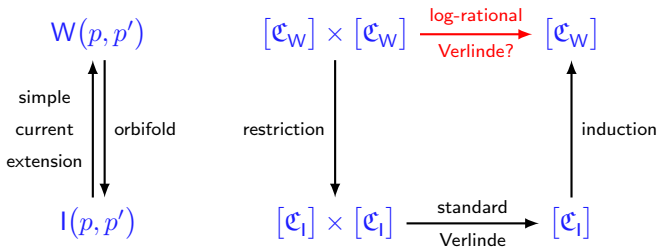


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This log-rational Verlinde formula is currently being worked out for the triplet models [Melville-DR].

# Thank you!

“Only those who attempt the absurd will achieve the impossible.”

- M C Escher