Non-rational CFTs and the Verlinde formula

David Ridout

Department of Theoretical Physics
& Mathematical Sciences Institute,
Australian National University

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CFTs, VOAs and the Verlinde formula

Dropping log-rationality

The standard module formalism

A log-rational Verlinde formula?

Rational CFT and the Verlinde formula

Two of the ingredients of CFT are:

- A vertex operator algebra (VOA) V.
- A physical category $\mathfrak{C}$ of V-modules that is
  - closed under conjugation C,
  - closed under fusion $\times$, and
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For rational CFTs, the S-transform of the irreducible characters satisfies:

- $\mathsf{S}^\top = \mathsf{S}$, $\mathsf{S}^\dagger = \mathsf{S}^{-1}$, $\mathsf{S}^2 = \mathsf{C}$.
- S diagonalises the fusion rules through the Verlinde formula [Huang]:

$$\mathscr{L}_i \times \mathscr{L}_j = \bigoplus_k \begin{bmatrix} & k & \\ i & & j \end{bmatrix} \mathscr{L}_k, \quad \begin{bmatrix} & k & \\ i & & j \end{bmatrix} = \sum_\ell \frac{\mathsf{S}_{i\ell}\mathsf{S}_{j\ell}\mathsf{S}^*_{k\ell}}{\mathsf{S}_{0\ell}}.$$

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> How does the formalism of rational CFT, especially Verlinde, generalise to non-rational and logarithmic CFT?

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If the goal is to decompose fusion products, the Verlinde formula helps!

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Worse, there is no canonical basis of torus amplitudes in which to try to express a Verlinde formula.

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But, there is [Fuchs-Hwang-Semikhatov-Tipunin] a $\mathsf{W}(1,p)$ Verlinde-like formula for simple characters (automorphy factor cancels τ-dependence).

A non-logarithmic non-rational CFT

The free boson: $\mathsf{V} =$ Heisenberg VOA: $[a_m, a_n] = m\delta_{m+n=0}\mathbf{1}$.

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- $\mathrm{ch}_{\mathscr{F}_p} = \mathrm{tr}_{\mathscr{F}_p}\, y^{\mathbf{1}} z^{a_0} q^{L_0 - \mathbf{1}/24} = \dfrac{\mathsf{y}\mathsf{z}^p \mathsf{q}^{p^2/2}}{\eta(\mathsf{q})}$.

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- $\begin{bmatrix} & r & \\ p & & q \end{bmatrix} = \int_{-\infty}^{\infty} \dfrac{\mathsf{S}_{ps}\mathsf{S}_{qs}\mathsf{S}_{rs}^*}{\mathsf{S}_{0s}}\,\mathrm{d}s = \delta(r = p+q)$,

$$\Rightarrow \quad \mathscr{F}_p \times \mathscr{F}_q = \int_{-\infty}^{\infty} \begin{bmatrix} & r & \\ p & & q \end{bmatrix} \mathscr{F}_r\,\mathrm{d}r = \mathscr{F}_{p+q}. \qquad \checkmark$$

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- $\displaystyle \mathsf{S}\left\{\mathrm{ch}_{\mathscr{M}}\right\} = \int_{-\infty}^{\infty} \mathsf{S}_{\mathscr{M}\mathscr{F}_q} \mathrm{ch}_{\mathscr{F}_q} \, \mathrm{d}q$:

$$\mathsf{S}_{\mathscr{F}_p\mathscr{F}_q} = \mathsf{e}^{-2\pi\mathrm{i}(p-\frac{1}{2})(q-\frac{1}{2})}, \quad \mathsf{S}_{\mathscr{L}_p\mathscr{F}_q} = \frac{\mathsf{e}^{-2\pi\mathrm{i}p(q-\frac{1}{2})}}{2\cos[\pi(q-\frac{1}{2})]}.$$

$\uparrow$

note: pole!

- The Verlinde formula $\begin{bmatrix} & \mathscr{F}_r & \\ \mathscr{M} & & \mathscr{N} \end{bmatrix} = \int_{-\infty}^{\infty} \frac{\mathsf{S}_{\mathscr{M}\mathscr{F}_s}\mathsf{S}_{\mathscr{N}\mathscr{F}_s}\mathsf{S}^*_{\mathscr{F}_r\mathscr{F}_s}}{\mathsf{S}_{\mathscr{L}_0\mathscr{F}_s}}\, \mathrm{d}s$ gives

$$\begin{bmatrix} & \mathscr{F}_r & \\ \mathscr{L}_p & & \mathscr{L}_q \end{bmatrix} = \sum_{n=1}^{\infty} (-1)^{n-1} \delta(r = p + q + n),$$

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$\Rightarrow$ (Grothendieck) fusion rules $\quad \mathscr{L}_p \times \mathscr{L}_q = \mathscr{L}_{p+q}, \quad \mathscr{L}_p \times \mathscr{F}_q = \mathscr{F}_{p+q},$
$[\mathscr{F}_p \times \mathscr{F}_q] = [\mathscr{F}_{p+q}] + [\mathscr{F}_{p+q-1}].$ ✓

The standard module formalism

In all known examples of non-log-rational CFTs, we have identified (indecomposable) standard modules with excellent modular properties.

We partition them into irreducible (typical) and reducible (atypical).

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5. If $\mathrm{ch}_{\mathscr{M}} = \sum_m a_m \mathrm{ch}_m$, define $\mathsf{S}_{\mathscr{M}n} = \sum_m a_m \mathsf{S}_{mn}$. This sum converges for all typical n ($n \notin A$).

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6. The vacuum module Ω satisfies $\mathsf{S}_{\Omega n} \neq 0$, for all $n \notin A$.

Now, define a product $\boxtimes$ using the standard Verlinde formula:

$$\mathrm{ch}_{\mathscr{M}} \boxtimes \mathrm{ch}_{\mathscr{N}} = \int_M \begin{bmatrix} & p & \\ \mathscr{M} & & \mathscr{N} \end{bmatrix} \mathrm{ch}_p \, \mathrm{d}\mu(p),$$

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Rational CFTs form the "trivial" examples of this formalism:

- Standard $=$ irreducible, so no atypicals ($A = \varnothing$).
- The measurable space M is finite and μ is counting measure.
- Grothendieck fusion $=$ fusion.

Ah, but does it work?

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Logarithmic conformal field theory	Fusion known?
Virasoro logarithmic minimal models $\mathsf{LM}(p, p')$	Many examples
$N = 1$ logarithmic minimal models $\mathsf{LSM}(p, p')$	Some examples
Singlet models $\mathsf{I}(p, p') = \mathsf{W}_{2,(2p-1)(2p'-1)}$	?
Admissible level $\widehat{\mathfrak{sl}}(2)_k$	$k = -\frac{1}{2},\ -\frac{4}{3}$
Bosonic $\beta\gamma$ ghosts	✓
$\mathrm{GL}(1\|1)$ Wess-Zumino-Witten model	✓

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The singlet model results imply Grothendieck fusion rules for the log-rational triplet models $\mathsf{W}(p,p')$. These are consistent with the known triplet fusion rules (and conjectures).

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But, we can exploit the relation between the singlet and triplet models!

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\mathsf{W}(p,p') \arrow[d, shift left, "\text{orbifold}"] \\
\mathsf{I}(p,p') \arrow[u, shift left, "\substack{\text{simple}\\ \text{current}\\ \text{extension}}"]
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\qquad
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{[\mathfrak{C}_{\mathsf{I}}] \times [\mathfrak{C}_{\mathsf{I}}]} \arrow[rr, "\text{standard}", "\text{Verlinde}"'] & & {[\mathfrak{C}_{\mathsf{I}}]} \arrow[u, "\text{induction}"']
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{[\mathfrak{C}_{\mathsf{W}}] \times [\mathfrak{C}_{\mathsf{W}}]} \arrow[r, "\text{log-rational}", "\text{Verlinde?}"'] \arrow[d, "\text{restriction}"'] & {[\mathfrak{C}_{\mathsf{W}}]} \\
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\end{tikzcd}$$

This log-rational Verlinde formula is currently being worked out for the triplet models [Melville-DR].

Thank you!

"Only those who attempt the absurd will achieve the impossible."

- M C Escher