Boundary algebras and scaling limits for logarithmic minimal models

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Outline

Logarithmic conformal field theory

Conformal field theory describes the continuum scaling limit of many critical statistical lattice models.

CFTs are divided into several broad representation-theoretic classes:

- If the space of states is built from a finite number of irreducible representations, then the CFT is rational.
- If the space of states involves an infinite number of irreducible representations, then the CFT is irrational.
- If the space of states involves reducible but indecomposable representations, then the CFT is logarithmic.

Typically, one identifies the scaling limit of a critical statistical model as a rational CFT, *eg.* Ising model $\longrightarrow$ Virasoro minimal model $\mathsf{M}(3,4)$.

However, there is evidence that scaling limits are typically logarithmic.

Reducible but indecomposable

A representation is reducible if it has a non-zero proper subrepresentation.

A representation is decomposable if it is the direct sum of two non-zero proper subrepresentations.

For many types of algebra representations, reducible $\iff$ decomposable:

- Finite-dim. $\mathbb{C}$-reps of finite group algebras, *eg.* $\mathbb{C}\mathsf{S}_n$;
- Finite-dim. $\mathbb{C}$-reps of semisimple Lie algebras, *eg.* $\mathfrak{sl}(2)$;
- Finite-dim. $\mathbb{C}$-reps of compact semisimple Lie groups, *eg.* $\mathsf{SU}(2)$.

However, a representation may be reducible but indecomposable.

Example: Any matrix A defines a rep of $\mathbb{C}[x]$ by $x \mapsto A$.

- If A is diagonalisable, then the rep is a direct sum of irreducibles.
- If A is not diagonalisable, then there are reducible but indecomposable subreps corresponding to the non-trivial Jordan blocks of A.

Reducible but indecomposable representations are common, *eg.*:

- $\mathbb{k}$-reps of finite group algebras $\mathbb{k}\mathsf{G}$ when $\operatorname{char}\mathbb{k} \mid \operatorname{card}\mathsf{G}$;
- $\mathbb{C}$-reps of diagram algebras at roots of unity, *eg.* Temperley-Lieb;
- $\mathbb{C}$-reps of quantum groups at roots of unity, *eg.* $\mathscr{U}_q\mathfrak{sl}(2)$;
- $\mathbb{C}$-reps of simple Lie superalgebras, *eg.* $\mathfrak{gl}(1|1)$;
- Infinite-dim. $\mathbb{C}$-reps of semisimple Lie algebras and groups;
- $\mathbb{C}$-reps of affine / Virasoro / W- algebras and superalgebras.

Often, one constructs irreducible representations as quotients of indecomposable ones, *cf.* null vectors in CFT.

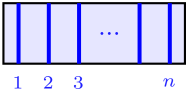
In integrable lattice models, the indecomposability of quantum group reps is relevant for functional relations and T-systems.

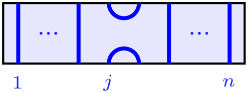
In CFT, the indecomposability of Virasoro representations has measurable consequences for correlation functions.

Temperley-Lieb indecomposables

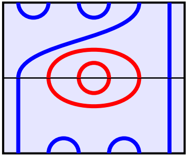
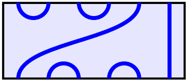
The Temperley-Lieb algebra $\mathsf{TL}_n(\beta)$, $\beta = q + q^{-1}$, has reducible but indecomposable representations when q is a root of unity.[1]

Recall that $\mathsf{TL}_n(\beta)$ is spanned by planar diagrams and generated by

$I =$ 

, $e_j =$ 

and multiplication is stacking diagrams and replacing each closed loop by a factor of β:

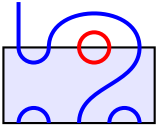
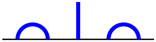
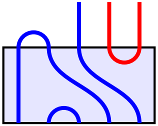

$= \beta^2$

.

[1] This is true unless $q = \pm 1$ or n is odd and $q = \pm \mathrm{i}$.

$\mathsf{TL}_n(\beta)$ has a family of standard representations S_n^d spanned by link states (half-diagrams) and parametrised by the number of defects d.

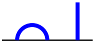
For $n = 5$ and $d = 1$, S_n^d is spanned by the following link states:

The action is again by stacking and replacing each closed loop by β, but cupped defects are assigned a factor of 0:

 $= \beta$  ,  $= 0.$

When q is not a root of unity, every standard representation is irreducible and every irreducible representation is standard.

When q is a root of unity, most standard representations are reducible but indecomposable. *eg.*, S_3^1 with $\beta = 1$ ($q = \mathrm{e}^{\mathrm{i}\pi/3}$):

$$\mathsf{S}_3^1 = \mathbb{C}\Big\{$$  ,  $$\Big\},$$

$$e_1 \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e_2 \longmapsto \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

$\mathsf{L} = \mathbb{C}\{$  $-$  $\}$ is the only non-zero proper subrepresentation. The quotient $\mathsf{N} = \mathsf{S}_3^1/\mathsf{L}$ is also irreducible.

S_3^1 is thus reducible but indecomposable, when $\beta = 1$, with structure

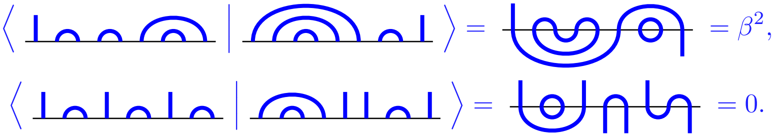
$$\mathsf{S}_3^1: \qquad \begin{array}{c} \mathsf{N} \\ \downarrow \\ \mathsf{L} \end{array}.$$

This structure (two irreducibles glued into an indecomposable) is typical for standard $\mathsf{TL}_n(\beta)$-representations at roots of unity.

Gram determinants

A standard representation is reducible iff its Gram determinant $\det_n^d$ vanishes. (It is always indecomposable.)

This is the determinant of the bilinear form defined on link states by reflecting one horizontally, stacking, and then assigning factors of β to closed loops, 0 to capped or cupped defects, and 1 to everything else:



This determinant has a closed form expression:

$$\det_n^d = \prod_{j=1}^{(n-d)/2} \left(\frac{[d+1+j]_q}{[j]_q} \right)^{\dim \mathsf{S}_n^{d+2j}} .$$

Virasoro indecomposables

The Virasoro representations one needs for the minimal models are irreducible and highest-weight.

But, the Virasoro algebra has lots of reducible but indecomposable representations, *eg.* the Verma module V_0 with $h = c = 0$:

$$
\mathsf{V}_0: \quad
\begin{tikzcd}
 & \mathsf{L}_0 \arrow[dl] \arrow[dr] & \\
\mathsf{L}_1 \arrow[d] \arrow[drr] & & \mathsf{L}_2 \arrow[d] \arrow[dll] \\
\mathsf{L}_5 \arrow[d] \arrow[drr] & & \mathsf{L}_7 \arrow[d] \arrow[dll] \\
\mathsf{L}_{12} \arrow[d] \arrow[drr] & & \mathsf{L}_{15} \arrow[d] \arrow[dll] \\
\vdots & & \vdots
\end{tikzcd}
$$

Here, L_h is the irreducible $c = 0$ highest-weight Virasoro-module of conformal dimension h.

A Virasoro Verma module is reducible iff its Kac determinant $\det_{h,c}^{(g)}$ vanishes at some conformal grade g. (It is always indecomposable.)

This is the determinant of the Shapovalov form defined by

$$\langle v_h | v_h \rangle = 1 \quad \text{and} \quad \langle L_n v | w \rangle = \langle v | L_{-n} w \rangle.$$

eg.,

$$\begin{aligned} \langle L_{-2} v_h | L_{-2} v_h \rangle &= \langle v_h | L_2 L_{-2} v_h \rangle = \langle v_h | [L_2, L_{-2}] v_h \rangle \\ &= \langle v_h | (4L_0 + \tfrac{1}{2} C) v_h \rangle = 4h + \tfrac{c}{2}. \end{aligned}$$

The Kac determinant also has a closed form expression:

$$\det_{h,c}^{g} = \prod_{\substack{r,s \geq 1 \\ rs \leq g}} \big(h - h_{r,s}(t)\big)^{p(g-rs)},$$

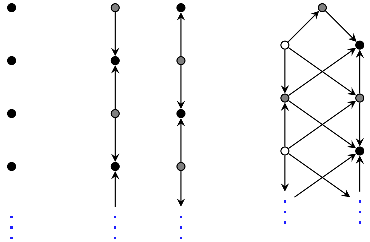
$$c = 13 - 6\big(t + t^{-1}\big), \quad h_{r,s}(t) = \frac{r^2-1}{4} t - \frac{rs-1}{2} + \frac{s^2-1}{4} t^{-1}.$$

Feigin-Fuchs modules

The Coulomb gas approach to Virasoro minimal models produces Feigin-Fuchs modules $\mathsf{F}_{r,s}$ instead of Verma modules.

These have the same characters as Verma modules, but different indecomposable structures in general.

$\mathsf{F}_{r,s}$:



$h = h_{r,s}$

The structure of $\mathsf{F}_{r,s}$, for given $r, s \in \mathbb{Z}_+$, is that for which the node of conformal dimension $h_{r,s} + rs$ has all arrows pointing outwards.

Logarithmic minimal models

Many statistical lattice models, *eg.* Ising, 3-state Potts, have scaling limits described by the (rational) Virasoro minimal model CFTs, *ie.* only irreducible representations appear.

Some, *eg.* polymers, percolation, are better described by logarithmic versions of these CFTs, *ie.* indecomposables appear.

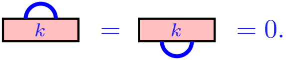
These logarithmic minimal models have conjectural lattice realisations [PRZ '06] that are Yang-Baxter integrable, so we can try to study them on the lattice and in the continuum.

The diagram algebra relevant to these realisations appears to be Temperley-Lieb, but is actually a quotient $\mathsf{B}_{n,k}(\beta)$ of the one-boundary Temperley-Lieb algebra [Morin-Duchesne-Rasmussen-DR].

Does the indecomposable structure of $\mathsf{B}_{n,k}(\beta)$-representations predict the indecomposable structure of the appropriate Virasoro representations?

The lattice realisation involves the Temperley-Lieb algebra and its Wenzl-Jones projectors, indicated diagrammatically by $\boxed{k}$.

These are elements of TL_k that project onto the trivial irreducible representation, so annihilate links:



Examples:

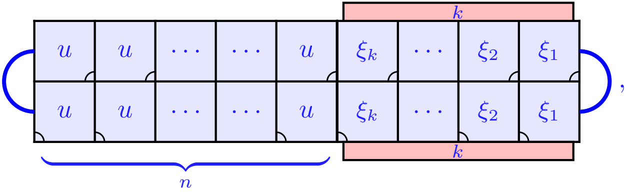
- $\boxed{1} = I;$
- $\boxed{2} = I - \dfrac{1}{\beta} e_1;$
- $\boxed{3} = I - \dfrac{\beta}{\beta^2 - 1}(e_1 + e_2) + \dfrac{1}{\beta^2 - 1}(e_1 e_2 + e_2 e_1).$

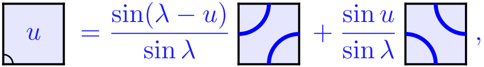
These expressions are singular when q is a root of unity (*eg.*, $\beta = 0,\ \pm 1,\ \pm\sqrt{2}, \ldots$).

The transfer tangle

The integrable structure of the lattice realisations is captured by the double row transfer tangle

$\mathbf{D}^{(k)}(u,\xi) \propto$



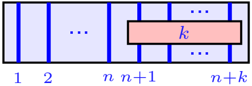


$$\xi_j = u + \xi + j\lambda, \qquad \beta = 2\cos\lambda.$$

This is a very complicated linear combination of Temperley-Lieb diagrams. The important feature for us is the boundary seam of width k on the right, flanked by the Wenzl-Jones projectors.

The hamiltonian $H^{(k)}$ is obtained by expanding $\mathbf{D}^{(k)}(u,\xi)$ in u:

$$\mathbf{D}^{(k)}(u,\xi) = I^{(k)} + \frac{2u}{\sin\lambda}\Big((\beta^{-1} - n\cos\lambda)I^{(k)} - H^{(k)}\Big) + O(u^2),$$

$I^{(k)} =$  ,

$$H^{(k)} = -I^{(k)}\sum_{j=1}^{n-1} e_j + \frac{\sin\lambda\sin(k\lambda)}{\sin\xi\sin(\xi+(k+1)\lambda)} I^{(k)} e_n I^{(k)}.$$

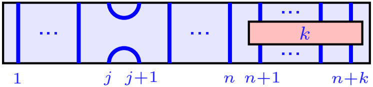
This hamiltonian, suitably shifted and rescaled, is expected to become the CFT hamiltonian $L_0 - \frac{c}{24}$ in the scaling limit.

We also expect that there exist other linear combinations of Temperley-Lieb diagrams that become the other L_n in the scaling limit [*cf.* Koo-Saleur].

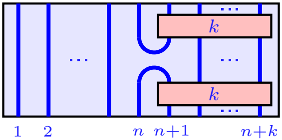
The boundary seam algebra

The transfer tangle and hamiltonian are constructed as elements of $\mathsf{TL}_{n+k}(\beta)$, but this is not the correct algebra for analysing logarithmic minimal models.

To account for the WJ-projectors, define the boundary seam algebra $\mathsf{B}_{n,k}(\beta)$ as the subalgebra of $\mathsf{TL}_{n+k}(\beta)$ with unit $I^{(k)}$ and generators

$E_j^{(k)} =$ 

$(j = 1, \ldots, n-1),$

$E_n^{(k)} = [k]_q$ 

.

Both $D^{(k)}(u, \xi)$ and $H^{(k)}$ belong to $\mathsf{B}_{n,k}(\beta)$.

The $E_j^{(k)}$ satisfy the defining relations of the one-boundary Temperley-Lieb algebra $\mathsf{TL}_n^{(1)}(\beta, [k]_q, [k+1]_q)$.

However, $\mathsf{B}_{n,k}(\beta) \cong \mathsf{TL}_n^{(1)}(\beta, [k]_q, [k+1]_q)$ iff $n \le k$.

For $n > k$, $\mathsf{B}_{n,k}(\beta)$ is always a proper quotient and there is always just one additional independent relation, *eg.*

- $\mathsf{B}_{n,0}(\beta) = \mathsf{TL}_n(\beta)$, as $E_n^{(0)} = 0$ by definition.;
- $\mathsf{B}_{n,1}(\beta) = \mathsf{TL}_{n+1}(\beta)$, as $\boxed{\;1\;} = I$, so $E_n^{(1)} E_{n-1}^{(1)} E_n^{(1)} = E_n^{(1)}$;
- $\mathsf{B}_{n,2}(\beta) \subsetneq \mathsf{TL}_{n+2}(\beta)$ — the extra relation is

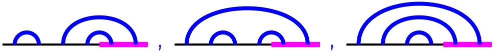
$$([2]_q E_{n-2}^{(2)} - E_n^{(2)} E_{n-1}^{(2)} E_{n-2}^{(2)} - I^{(2)})([2]_q E_n^{(2)} - E_n^{(2)} E_{n-1}^{(2)} E_n^{(2)}) = 0.$$

Since $n \to \infty$ in the scaling limit, we need the case $n > k$.

Standard $\mathsf{B}_{n,k}(\beta)$-representations

Since $\mathsf{B}_{n,k}(\beta) \subseteq \mathsf{TL}_{n+k}(\beta)$, the boundary seam algebra acts on link states.

The link states with d defects and no links between the k rightmost nodes span the standard $\mathsf{B}_{n,k}(\beta)$-representation $\mathsf{S}^d_{n,k}$, *eg.*

$\mathsf{S}^0_{4,2} = \mathbb{C}\Big\{$  $\Big\}.$

We get a natural bilinear form on $\mathsf{S}^d_{n,k}$ by inserting $I^{(k)}$ into the bilinear form for standard $\mathsf{TL}_{n+k}(\beta)$-representations.

Its Gram determinant $\det^d_{n,k}$ again controls the reducibility of the standard representations $\mathsf{S}^d_{n,k}$:

$$\det{}^d_{n,k} = \prod_{i=1}^{\lfloor k/2 \rfloor} \left(\frac{[i]_q}{[k-i+1]_q} \right)^{\dim \mathsf{S}^d_{n,k-2i}} \cdot \prod_{j=1}^{\frac{1}{2}(n+k-d)} \left(\frac{[d+j+1]_q}{[j]_q} \right)^{\dim \mathsf{S}^{d+2j}_{n,k}} .$$

Virasoro Kac representations

We expect that a standard $\mathsf{B}_{n,k}(\beta)$-representation is replaced by a Virasoro representation in the scaling limit $n \to \infty$.

Our main result precisely identifies this Virasoro representation.

Conjecture

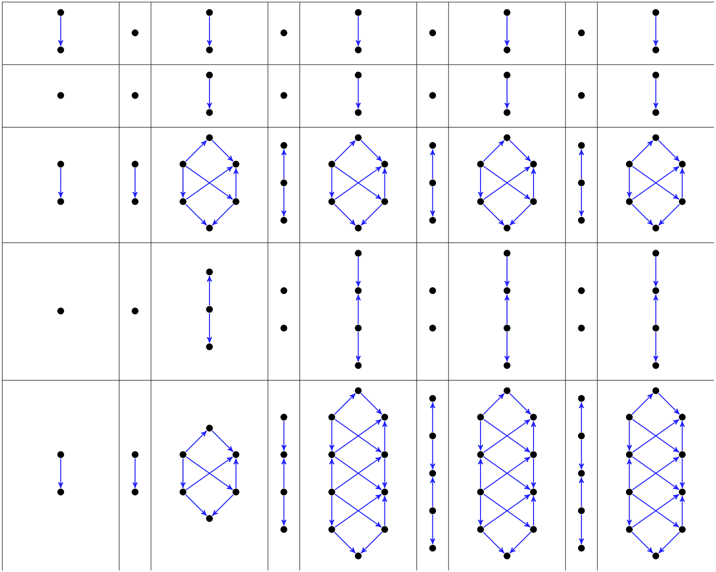
The scaling limit of the standard $\mathsf{B}_{n,k}(\beta)$-representation $\mathsf{S}_{n,k}^{d}$ is the subrepresentation $\mathsf{K}_{r,s}$ of the Feigin-Fuchs module $\mathsf{F}_{r,s}$ generated by the subsingular vectors whose grades are strictly less than rs. Here,

$$r = \left\lceil \frac{k+1}{\pi} \cos^{-1} \frac{-\beta}{2} \right\rceil, \qquad s = d+1,$$

provided that ξ is positive and sufficiently small.

The $\mathsf{K}_{r,s}$ are called Kac representations.

Kac representation structures



Character results

We can numerically diagonalise the action of the hamiltonian, for various ξ, on the standard $\mathsf{S}_{n,k}(\beta)$-representations [cf. Pearce-Rasmussen].

Ordering the (real) eigenvalues, we rigidly shift and rescale them so that the first two are 0 and 1 (or 0 and 2).

This is interpreted as approximating $q^{-(h_{r,s}-c/24)}$ times a Virasoro character. The $h_{r,s} - c/24$ factor is studied in [Pearce-Tartaglia-Couvreur].

This diagonalisation was performed for system sizes $n + k = 19$ or 20.

The results are consistent with the characters of the Kac representations from our conjecture, but they are not as convincing as one might like.

We also studied cases where ξ was not “sufficiently small”, but the pattern to the results is not as easy to discern.

$$
\begin{array}{c|ll}
(p,p')=(3,5) & n=14: & 1+q^2+q^{2.92}+q^{3.74}+q^{3.88}+q^{4.42}+q^{4.68}+q^{4.93}+q^{5.24}+q^{5.35}+\cdots \\
k=0 & n=16: & 1+q^2+q^{2.94}+q^{3.80}+q^{3.90}+q^{4.54}+q^{4.75}+q^{5.15}+q^{5.49}+q^{5.60}+\cdots \\
d=0 & n=18: & 1+q^2+q^{2.95}+q^{3.84}+q^{3.92}+q^{4.63}+q^{4.80}+q^{5.32}+q^{5.59}+q^{5.72}+\cdots \\
\xi\ \text{n/a} & n=20: & 1+q^2+q^{2.96}+q^{3.87}+q^{3.94}+q^{4.70}+q^{4.84}+q^{5.44}+q^{5.66}+q^{5.77}+\cdots \\
(r,s)=(1,1) & n\to\infty: & 1+q^2+q^3+2q^4+2q^5+4q^6+\cdots
\end{array}
$$

$$
\begin{array}{c|ll}
(p,p')=(3,4) & n=11: & 1+q+q^{1.68}+q^{1.88}+q^{2.44}+q^{2.63}+q^{2.80}+q^{3.08}+q^{3.21}+q^{3.27}+\cdots \\
k=1 & n=13: & 1+q+q^{1.71}+q^{1.90}+q^{2.51}+q^{2.70}+q^{2.83}+q^{3.21}+q^{3.35}+q^{3.37}+\cdots \\
d=2 & n=15: & 1+q+q^{1.74}+q^{1.92}+q^{2.56}+q^{2.75}+q^{2.85}+q^{3.30}+q^{3.42}+q^{3.49}+\cdots \\
\xi=\frac{\pi}{4} & n=17: & 1+q+q^{1.76}+q^{1.93}+q^{2.60}+q^{2.79}+q^{2.87}+q^{3.37}+q^{3.47}+q^{3.57}+\cdots \\
(r,s)\to(2,3) & n\to\infty: & 1+q+2q^2+3q^3+5q^4+7q^5+10q^6+\cdots
\end{array}
$$

$$
\begin{array}{c|ll}
(p,p')=(4,5) & n=12: & 1+q+q^{1.90}+q^{2.01}+q^{2.65}+q^{2.90}+q^{3.22}+q^{3.57}+q^{3.64}+q^{3.86}+\cdots \\
k=2 & n=14: & 1+q+q^{1.92}+q^{2.01}+q^{2.73}+q^{2.92}+q^{3.40}+q^{3.72}+q^{3.89}+q^{3.90}+\cdots \\
d=0 & n=16: & 1+q+q^{1.94}+q^{2.00}+q^{2.79}+q^{2.93}+q^{3.53}+q^{3.77}+q^{3.91}+q^{4.12}+\cdots \\
\xi=\frac{\pi}{5} & n=18: & 1+q+q^{1.95}+q^{2.00}+q^{2.83}+q^{2.94}+q^{3.62}+q^{3.81}+q^{3.92}+q^{4.29}+\cdots \\
(r,s)\to(3,1) & n\to\infty: & 1+q+2q^2+2q^3+4q^4+5q^5+8q^6+\cdots
\end{array}
$$

Module structure results

Knowing the character of a representation is not usually sufficient to determine its indecomposable structure.

eg., a sum $\mathrm{ch}[\mathsf{L}] + \mathrm{ch}[\mathsf{L}']$ of two irreducible characters corresponds to three inequivalent module structures:

$$\begin{matrix} \mathsf{L} \\ \oplus \\ \mathsf{L}' \end{matrix} \qquad \begin{matrix} \mathsf{L} \\ \downarrow \\ \mathsf{L}' \end{matrix} \qquad \begin{matrix} \mathsf{L} \\ \uparrow \\ \mathsf{L}' \end{matrix} \quad .$$

We cannot distinguish these structures using the hamiltonian $H^{(k)}$ (or transfer tangle), but we can if we include information about the bilinear form on the standard $\mathsf{B}_{n,k}(\beta)$-representations $\mathsf{S}^d_{n,k}$.

The kernel of this bilinear form is a subrepresentation of $\mathsf{S}^d_{n,k}$, so we can check whether each eigenvector of $H^{(k)}$ belongs to this kernel.

Example: We consider the standard $\mathsf{B}_{n,0}(1)$-representations $\mathsf{S}_{n,0}^2$, *ie.* $k=0$, $d=2$ and $\beta=1$, recalling that $\mathsf{B}_{n,0}(1)=\mathsf{TL}_n(1)$. Now,

$$\det{}_n^2=\prod_{j=1}^{\frac{1}{2}n-1}\left(\frac{[j+3]_q}{[j]_q}\right)^{\dim\mathsf{S}_n^{2(j+1)}}=\prod_{j=1}^{\frac{1}{2}n-1}(-1)^{\dim\mathsf{S}_n^{2(j+1)}}\neq 0,$$

since $q=\mathsf{e}^{\pm\mathrm{i}\pi/3}$. Thus, $\mathsf{S}_{n,0}^2=\mathsf{S}_n^2$ is irreducible for all $n\in 2\mathbb{Z}_+$.

The character analysis suggests that the scaling limit is an irreducible $c=0$ Virasoro module of conformal dimension $h_{1,3}=\frac{1}{3}$.

This is consistent with our conjecture that the scaling limit is the Kac module $\mathsf{K}_{r,s}$ with $r=\lceil\frac{0+1}{\pi}\cos^{-1}(-\frac{1}{2})\rceil=1$ and $s=2+1=3$.

It is likewise consistent with the expectation that the irreducible $\mathsf{B}_{n,0}(1)$-representations will become an irreducible Virasoro representation in the scaling limit.

Example: We consider the standard $\mathsf{B}_{n,1}(\sqrt{2})$-representations $\mathsf{S}^1_{n,1}$, *ie.* $k=1$, $d=1$ and $\beta=\sqrt{2}$, recalling that $\mathsf{B}_{n,1}(\sqrt{2})=\mathsf{TL}_{n+1}(\sqrt{2})$.

This time, $\det^1_{n+1}=0$ for $n=4,6,8,\ldots$, so $\mathsf{S}^1_{n,1}=\mathsf{S}^1_{n+1}$ is reducible.

The character analysis suggests a sum of two irreducible characters of conformal dimensions $\frac{1}{16}$ and $\frac{33}{16}$. The indecomposable structure is therefore ambiguous.

The ground state is not in the kernel of the bilinear form on $\mathsf{S}^1_{n,1}$; eigenstates in this kernel appear from grade 2.

This selects the following indecomposable structure, consistent with the Kac module $\mathsf{K}_{2,2}$ given by our conjecture:

$$\begin{array}{c} \mathsf{L}_{1/16} \\ \downarrow \\ \mathsf{L}_{33/16} \end{array} .$$

Observations:

- The structure of the standard $\mathsf{B}_{n,k}(\beta)$-representation need not match that of the Virasoro representation: the (rescaled) eigenvalues of certain eigenstates may diverge in the scaling limit.
- For $k \geq 2$, we do not know the complete indecomposable structure of the standard $\mathsf{B}_{n,k}(\beta)$-representations. Nevertheless, the Gram determinant analysis can still be applied to get partial information.
- The standard $\mathsf{B}_{n,k}(\beta)$-representations need not be indecomposable, correlating with decomposable Kac representations for $k \geq 2$.
- The Gram determinant of a standard $\mathsf{B}_{n,k}(\beta)$-representation may vanish identically or it may diverge. In both cases, it may be renormalised and the analysis is then performed as before.
- We have analysed all cases with $k \leq 3$, $d \leq 4$ and q an m-th root of unity, $m = 2, 3, 4, 5$, and the Gram determinant analysis is always consistent with our conjecture for the scaling limit.

Continuum results

One can also generalise the lattice setup to admit boundary seams on both sides of the double row transfer tangle [Pearce-Rasmussen].

This corresponds to fusing the single boundary cases, leading to a definition of lattice fusion [Gainutdinov-Vasseur] that is expected to give CFT fusion in the scaling limit.

If this is so, then the lattice setup implies that the Kac representations should have the following fusion rules:

$$\mathsf{K}_{r,1} \times \mathsf{K}_{1,s} = \mathsf{K}_{r,s}.$$

We have verified this continuum prediction in many cases using the Nahm-Gaberdiel-Kausch fusion algorithm for Virasoro representations.

This allowed us to explore the consistency of our conjecture in cases where the indecomposable structure was far more complicated than those accessible with a lattice treatment.

Summary

We have resolved two outstanding issues concerning the logarithmic minimal models, namely the identification of the underlying representations in the lattice realisation and in the continuum.

The lattice identification required introducing an algebraic framework in terms of a new (?) diagram algebra $\mathsf{B}_{n,k}(\beta)$ and its standard representations.

The known analysis of the eigenvalues of the lattice hamiltonian was reformulated and then extended in this framework to study the indecomposable structure through Gram determinant methods.

Our results indicate that one can indeed study indecomposable structures in the scaling limit by restricting to indecomposable structures on the lattice, though much care is needed.

We have also confirmed the consistency of our results directly in the continuum by performing highly non-trivial fusion calculations.

Outlook

All of the approaches we have taken have potential to be strengthened:

- For characters, the transfer tangles considered are known to satisfy functional relations [Morin-Duchesne], though these have only been solved (and eigenvalues extracted) when $\beta = 0$ [Pearce-Rasmussen-Ruelle].
- We have only used rather coarse information about the standard representations of the boundary seam algebras $\mathsf{B}_{n,k}(\beta)$ with $k \geq 2$. A thorough study of their indecomposable structures [*cf.* Martin-Woodcock] would facilitate the identification of scaling limits.
- Lattice fusion suggests that one should study certain quotients of the two-boundary Temperley-Lieb algebras. However, this may also be encoded in terms of a quotient of the one-boundary Temperley-Lieb algebra.
- The fact that Kac representations are so closely related to Feigin-Fuchs modules suggests that the fusion rules of the logarithmic minimal models may be accessible using Coulomb gas methods.

Also: What happens when ξ isn't "sufficiently small"? What about other conformal boundary conditions? What about other loop models?