The standard module formalism: modularity and logarithmic CFT

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Outline

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Modularity and rational CFT

One of the most intriguing properties of a conformal field theory (CFT)[1] is that its bulk partition function

$$Z(\tau, \dots) = \operatorname{tr}\Big(q^{L_0 - c/24} \overline{q}^{\overline{L}_0 - \overline{c}/24} \cdots\Big), \quad q = \mathsf{e}^{2\pi \mathfrak{i} \tau},$$

must be invariant under the action of the modular group $\mathsf{SL}_2(\mathbb{Z})$:

$$Z\left(\frac{a\tau + b}{c\tau + d}\right) = Z(\tau) \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}).$$

If the CFT is rational, then the space spanned by the irreducible characters of the chiral algebra (vertex operator algebra) is a finite-dimensional representation of $\mathsf{SL}_2(\mathbb{Z})$. In this basis:

- $\mathcal{T} \colon \tau \mapsto \tau + 1$ is diagonal and unitary.
- $\mathcal{S} \colon \tau \mapsto -1/\tau$ is symmetric and unitary.
- $\mathcal{C} = \mathcal{S}^2 = (\mathcal{S}\mathcal{T})^3$ is a permutation of order at most 2.

[1] [Here, we assume that the chiral algebra is bosonic and $\mathbb{Z}$-graded.]

Example: The Ising model CFT has 3 irreducible highest-weight representations whose characters are

$$\mathrm{ch}_0 = \frac{1}{2}\left(\sqrt{\frac{\vartheta_3(0,\tau)}{\eta(\tau)}} + \sqrt{\frac{\vartheta_4(0,\tau)}{\eta(\tau)}}\right), \qquad \mathrm{ch}_{1/16} = \sqrt{\frac{\vartheta_2(0,\tau)}{2\eta(\tau)}},$$
$$\mathrm{ch}_{1/2} = \frac{1}{2}\left(\sqrt{\frac{\vartheta_3(0,\tau)}{\eta(\tau)}} - \sqrt{\frac{\vartheta_4(0,\tau)}{\eta(\tau)}}\right),$$

The subscripts give the conformal weights of the highest-weight states. With respect to the ordered basis $[\mathrm{ch}_0, \mathrm{ch}_{1/16}, \mathrm{ch}_{1/2}]$, we have

$$\mathcal{T} = \begin{pmatrix} \mathsf{e}^{-\mathsf{i}\pi/24} & 0 & 0 \\ 0 & \mathsf{e}^{\mathsf{i}\pi/12} & 0 \\ 0 & 0 & -\mathsf{e}^{-\mathsf{i}\pi/24} \end{pmatrix},$$
$$\mathcal{S} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Modularity is useful

Given a rational CFT, assume that we know the action of $\mathcal{S}$ and $\mathcal{T}$ on its irreducible characters. Then, we can:

- Strongly constrain the structure of the bulk state space. If its character, the partition function, has the form

$$Z(\tau) = \sum_{i,j} M_{ij} \operatorname{ch}_i(q) \operatorname{ch}_j(\overline{q}), \quad M_{ij} \in \mathbb{Z}_{\geqslant 0},$$

  then it will be $\mathsf{SL}_2(\mathbb{Z})$-invariant if and only if $\mathcal{S}^\dagger M \mathcal{S} = \mathcal{T}^\dagger M \mathcal{T} = M$.
- Compute the fusion rules. If we write these rules in the form

$$\operatorname{ch}_i \times \operatorname{ch}_j = \sum_k \mathcal{N}_{ij}{}^k \operatorname{ch}_k, \quad \mathcal{N}_{ij}{}^k \in \mathbb{Z}_{\geqslant 0},$$

  then the Verlinde formula gives

$$\mathcal{N}_{ij}{}^k = \sum_\ell \frac{\mathcal{S}_{i\ell} \mathcal{S}_{j\ell} \overline{\mathcal{S}_{k\ell}}}{\mathcal{S}_{0\ell}},$$

  where ch_0 denotes the vacuum character.

Modularity and logarithmic CFT

This all works beautifully for rational theories, so we expect it to work in some fashion more generally.

However, life isn't meant to be easy.

We saw previously that the log-rational symplectic fermions CFT has vacuum supercharacter whose S-transform has a τ-dependent coefficient:

$$\mathrm{sch}\big[\mathsf{L}_0\big] = \eta(\tau)^2, \quad \Rightarrow \quad \mathrm{sch}\big[\mathsf{L}_0\big](-1/\tau) = -\mathfrak{i}\tau\,\mathrm{sch}\big[\mathsf{L}_0\big](\tau).$$

The space spanned by the characters and supercharacters does not give a representation of $\mathsf{SL}_2(\mathbb{Z})$, so we cannot even try to constrain the bulk state space or compute the fusion rules.

This doesn't bode well.

The triplet model

Modularity is easier to study for bosonic $\mathbb{Z}$-graded chiral algebras, so we pass from symplectic fermions to its bosonic orbifold: the triplet model.

It is generated by three dimension 3 Virasoro primaries $W^{\pm}$ and W^{0}. The OPEs are messy but preserve the superscript grading (T is grade 0).

Each symplectic fermion irreducible splits into two triplet irreducibles, giving four in total:

	Neveu-Schwarz	Ramond
singlet	W_0	$\mathsf{W}_{-1/8}$
doublet	W_1	$\mathsf{W}_{3/8}$

The symplectic fermions staggered module likewise splits into two triplet staggered modules P_0 and P_1.

$$\begin{tikzcd} & \mathsf{W}_0 \arrow[dl] \arrow[dr] & \\ \mathsf{W}_1 \arrow[dr] & \mathsf{P}_0 & \mathsf{W}_1 \arrow[dl] \\ & \mathsf{W}_0 & \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{W}_1 \arrow[dl] \arrow[dr] & \\ \mathsf{W}_0 \arrow[dr] & \mathsf{P}_1 & \mathsf{W}_0 \arrow[dl] \\ & \mathsf{W}_1 & \end{tikzcd}$$

The irreducible triplet characters do not span an $\mathsf{SL}_2(\mathbb{Z})$-representation:

$$\mathrm{ch}\big[\mathsf{W}_0\big] = \frac{1}{2}\left(\frac{\vartheta_2(0,\tau)}{2\eta(\tau)} + \eta(\tau)^2\right), \quad \mathrm{ch}\big[\mathsf{W}_{-1/8}\big] = \frac{\vartheta_3(0,\tau)+\vartheta_4(0,\tau)}{2\eta(\tau)},$$
$$\mathrm{ch}\big[\mathsf{W}_1\big] = \frac{1}{2}\left(\frac{\vartheta_2(0,\tau)}{2\eta(\tau)} - \eta(\tau)^2\right), \quad \mathrm{ch}\big[\mathsf{W}_{3/8}\big] = \frac{\vartheta_3(0,\tau)-\vartheta_4(0,\tau)}{2\eta(\tau)}.$$

The staggered triplet characters do not have an η^2,

$$\mathrm{ch}\big[\mathsf{P}_0\big] = \mathrm{ch}\big[\mathsf{P}_1\big] = 2\big(\mathrm{ch}\big[\mathsf{W}_0\big] + \mathrm{ch}\big[\mathsf{W}_1\big]\big) = \frac{\vartheta_2(0,\tau)}{\eta(\tau)},$$

and $\mathrm{ch}\big[\mathsf{P}_0\big]$, $\mathrm{ch}\big[\mathsf{W}_{-1/8}\big]$ and $\mathrm{ch}\big[\mathsf{W}_{3/8}\big]$ span an $\mathsf{SL}_2(\mathbb{Z})$-representation.

This gives an $\mathsf{SL}_2(\mathbb{Z})$-invariant (candidate) partition function

$$\frac{1}{2}\left|\mathrm{ch}\big[\mathsf{P}_0\big]\right|^2 + \left|\mathrm{ch}\big[\mathsf{W}_{-1/8}\big]\right|^2 + \left|\mathrm{ch}\big[\mathsf{W}_{3/8}\big]\right|^2,$$

but doesn't help with fusion and the Verlinde formula.

Torus Amplitudes

Characters are examples of torus amplitudes. In general, one only expects an action of $\mathsf{SL}_2(\mathbb{Z})$ on the space of *all* torus amplitudes.

For rational CFTs, this space coincides with the span of the characters. For logarithmic CFTs, it doesn't!

For the triplet model, with four irreducible characters, the space of torus amplitudes has dimension 5. We may choose the missing generator to be

$$-\mathfrak{i}\tau\big(\mathrm{ch}\big[\mathsf{W}_0\big]-\mathrm{ch}\big[\mathsf{W}_1\big]\big).$$

With characters being formal power series in $q = \mathsf{e}^{2\pi\mathfrak{i}\tau}$, the prefactor $\tau \sim \log q$ above arises from solving an ODE whose indicial equation has repeated roots, *cf.* logarithms in sphere amplitudes (correlators).

This is bad, very bad...

Generalising from characters to torus amplitudes seems natural and necessary for log-rational CFTs, there are still reasons to be dissatisfied:

- Partition functions are characters, so finding modular invariants means forcing the coefficients of non-character torus amplitudes to vanish.
- The $\mathsf{SL}_2(\mathbb{Z})$-representation is not unitary, so it isn't easy to find canonical modular invariants, *eg.* diagonal, charge conjugation.
- Must the tensor product of the torus amplitude representation of $\mathsf{SL}_2(\mathbb{Z})$ and its conjugate even contain a trivial subrepresentation (a modular invariant)?
- There is no canonical basis of general torus amplitudes, which is bad news for a Verlinde formula.

For the triplet, the torus amplitude representation has two independent modular invariants. One involves the "non-character".

Moreover, this representation has $\mathcal{S}_{00} = 0$, so we cannot divide by the vacuum S-matrix elements in the Verlinde formula.

The singlet model

The triplet model is the bosonic orbifold of symplectic fermions,

$$J^+(z)J^-(w) \sim \frac{\mathbf{1}}{(z-w)^2},$$

but it is also the orbifold with respect to the automorphism that negates the symplectic fermion fields: $J^{\pm}(z) \mapsto -J^{\pm}(z)$.

This automorphism generalises to $J^{\pm}(z) \mapsto \omega^{\pm 1} J^{\pm}(z)$, where $\omega \neq 0$.

If ω is a root of unity, then the resulting orbifold is log-rational. Otherwise, the orbifold is not log-rational: it is called the singlet model.

The singlet model is generated by a single dimension 3 Virasoro primary W^0. (Symplectic fermions becomes the doublet model in this language.)

The four triplet irreducibles break up into a countably infinite number of singlet irreducibles:

$$W_0 = \bigoplus_{\lambda \in 2\mathbb{Z}} A_\lambda, \qquad W_{-1/8} = \bigoplus_{\lambda \in 2\mathbb{Z}+1/2} T_\lambda,$$

$$W_1 = \bigoplus_{\lambda \in 2\mathbb{Z}+1} A_\lambda, \qquad W_{3/8} = \bigoplus_{\lambda \in 2\mathbb{Z}-1/2} T_\lambda.$$

However, the number of singlet irreducibles is actually uncountably infinite. Aside from the A_λ, for which $\lambda \in \mathbb{Z}$, they are denoted by T_λ, where $\lambda \in \mathbb{R} \setminus \mathbb{Z}$. (Note: conformal dimensions are quadratic in λ.)

The reason for the notational distinction is that there is a version of the T_λ with $\lambda \in \mathbb{Z}$.

These are reducible but indecomposable.

$$\begin{tikzcd} & A_\lambda \arrow[d] \\ T_\lambda & \\ & A_{\lambda-1} \end{tikzcd}$$

We construct these indecomposables by decomposing the triplet staggered modules into singlet staggered modules:

$$\mathsf{P}_0 = \bigoplus_{\lambda \in 2\mathbb{Z}} \mathsf{Q}_\lambda, \quad \mathsf{P}_1 = \bigoplus_{\lambda \in 2\mathbb{Z}+1} \mathsf{Q}_\lambda.$$

$$\begin{tikzcd}
 & \mathsf{A}_\lambda \arrow[dl] \arrow[dr] & \\
\mathsf{A}_{\lambda+1} \arrow[dr] & \mathsf{Q}_\lambda & \mathsf{A}_{\lambda-1} \arrow[dl] \\
 & \mathsf{A}_\lambda &
\end{tikzcd}$$

The T_λ with $\lambda \in \mathbb{Z}$ are then submodules and quotients:

$$\mathsf{T}_{\lambda+1} \subset \mathsf{Q}_\lambda, \quad \frac{\mathsf{Q}_\lambda}{\mathsf{T}_{\lambda+1}} = \mathsf{T}_\lambda.$$

The characters explain why the notation for these subquotients is natural:

$$\mathrm{ch}\big[\mathsf{T}_\lambda\big] = \frac{q^{(\lambda-1/2)^2/2}}{\eta(q)}, \quad \lambda \in \mathbb{R}.$$

The characters for the A_λ are more complicated and will not be needed.

Modularity

Surprisingly, the T_λ-characters carry an $\mathsf{SL}_2(\mathbb{Z})$-action:

$$\mathrm{ch}\big[\mathsf{T}_\lambda\big](\mathcal{A}\cdot\tau) = \int_{\mathbb{R}} \mathcal{A}_{\lambda\mu}\mathrm{ch}\big[\mathsf{T}_\mu\big](\tau)\,\mathrm{d}\mu, \quad \mathcal{A}\in\mathsf{SL}_2(\mathbb{Z}).$$

Moreover,

- $\mathcal{T}_{\lambda\mu} = \mathsf{e}^{\mathrm{i}\pi(\lambda(\lambda-1)+1/6)}\delta(\lambda-\mu)$ is diagonal and unitary,
- $\mathcal{S}_{\lambda\mu} = \mathsf{e}^{-2\pi\mathrm{i}(\lambda-1/2)(\mu-1/2)}$ is symmetric and unitary,
- $\mathcal{C}_{\lambda\mu} = \delta(\lambda+\mu-1)$ is a permutation of order 2.

This extends to the A_λ-characters by noting that

$$\begin{aligned}\mathrm{ch}\big[\mathsf{A}_\lambda\big] &= \mathrm{ch}\big[\mathsf{T}_\lambda\big]-\mathrm{ch}\big[\mathsf{A}_{\lambda-1}\big] = \mathrm{ch}\big[\mathsf{T}_\lambda\big]-\mathrm{ch}\big[\mathsf{T}_{\lambda-1}\big]+\mathrm{ch}\big[\mathsf{A}_{\lambda-2}\big] \\ &= \sum_{n=0}^{\infty}(-1)^n\mathrm{ch}\big[\mathsf{T}_{\lambda-n}\big] \quad \Rightarrow \quad \mathcal{S}_{\underline{\lambda}\mu} = \frac{\mathsf{e}^{-2\pi\mathrm{i}\lambda(\mu-1/2)}}{2\cos[\pi(\mu-1/2)]}.\end{aligned}$$

(The underline $\underline{\lambda}$ indicates an A-type label.)

The vacuum is A_0 and its S-matrix entries,

$$\mathcal{S}_{\underline{0}\mu} = \frac{1}{2\cos[\pi(\mu - 1/2)]},$$

diverge for $\mu \in \mathbb{Z}$. However, this is a set of (Lebesgue) measure zero.

Let's try the Verlinde formula:

$$\mathcal{N}_{\lambda\mu}{}^{\nu} = \int_{\mathbb{R}} \frac{\mathcal{S}_{\lambda\rho}\mathcal{S}_{\mu\rho}\overline{\mathcal{S}_{\nu\rho}}}{\mathcal{S}_{\underline{0}\rho}} \, \mathrm{d}\rho = \delta(\nu = \lambda + \mu) + \delta(\nu = \lambda + \mu - 1).$$

It gives non-negative (Grothendieck) fusion coefficients:

$$\mathrm{ch}\big[\mathsf{T}_\lambda\big] \times \mathrm{ch}\big[\mathsf{T}_\mu\big] = \int_{\mathbb{R}} \mathcal{N}_{\lambda\mu}{}^{\nu} \mathrm{ch}\big[\mathsf{T}_\nu\big] \, \mathrm{d}\nu = \mathrm{ch}\big[\mathsf{T}_{\lambda+\mu}\big] + \mathrm{ch}\big[\mathsf{T}_{\lambda+\mu-1}\big].$$

Similarly, we get

$$\mathrm{ch}\big[\mathsf{A}_\lambda\big] \times \mathrm{ch}\big[\mathsf{T}_\mu\big] = \mathrm{ch}\big[\mathsf{T}_{\lambda+\mu}\big], \quad \mathrm{ch}\big[\mathsf{A}_\lambda\big] \times \mathrm{ch}\big[\mathsf{A}_\mu\big] = \mathrm{ch}\big[\mathsf{A}_{\lambda+\mu}\big].$$

This is good, very good!!!

- The T_λ, with $\lambda \in \mathbb{R}$, provide a canonical basis of characters for which $\mathcal{T}$, $\mathcal{S}$ and $\mathcal{C}$ have the expected properties.
- The $\mathsf{SL}_2(\mathbb{Z})$-representation is unitary, so the diagonal partition function is modular invariant. There is no need to look for additional torus amplitudes.
- The Verlinde formula gives sensible results and with a little work, one can deduce the fusion rules:

$$
\begin{aligned}
&\mathsf{A}_\lambda \times \mathsf{A}_\mu = \mathsf{A}_{\lambda+\mu} && (\lambda, \mu \in \mathbb{Z}), \\
&\mathsf{A}_\lambda \times \mathsf{T}_\mu = \mathsf{T}_{\lambda+\mu} && (\lambda \in \mathbb{Z},\ \mu \notin \mathbb{Z}), \\
&\mathsf{T}_\lambda \times \mathsf{T}_\mu = \begin{cases} \mathsf{T}_{\lambda+\mu} \oplus \mathsf{T}_{\lambda+\mu-1} & \text{if } \lambda+\mu \notin \mathbb{Z}, \\ \mathsf{Q}_{\lambda+\mu-1} & \text{if } \lambda+\mu \in \mathbb{Z} \end{cases} && (\lambda, \mu \notin \mathbb{Z}), \\
&\mathsf{A}_\lambda \times \mathsf{Q}_\mu = \mathsf{Q}_{\lambda+\mu} && (\lambda, \mu \in \mathbb{Z}), \\
&\mathsf{T}_\lambda \times \mathsf{Q}_\mu = \mathsf{T}_{\lambda+\mu+1} \oplus 2\mathsf{T}_{\lambda+\mu} \oplus \mathsf{T}_{\lambda+\mu-1} && (\lambda \notin \mathbb{Z},\ \mu \in \mathbb{Z}), \\
&\mathsf{Q}_\lambda \times \mathsf{Q}_\mu = \mathsf{Q}_{\lambda+\mu+1} \oplus 2\mathsf{Q}_{\lambda+\mu} \oplus \mathsf{Q}_{\lambda+\mu-1} && (\lambda \in \mathbb{Z},\ \mu \in \mathbb{Z}).
\end{aligned}
$$

Bulk state space

Finally, we can consider the bulk quantum state space. The diagonal partition function is just

$$Z = \int_{\mathbb{R}} \left| \operatorname{ch}\left[\mathsf{T}_\lambda \right] \right|^2 \mathrm{d}\lambda,$$

so it is natural to propose that the bulk state space has the form

$$\mathsf{H} = \mathsf{H}_{\mathbb{Z}} \oplus \oint_{\mathbb{R} \setminus \mathbb{Z}} \left(\mathsf{T}_\lambda \otimes \mathsf{T}_\lambda \right) \mathrm{d}\lambda,$$

where $\mathsf{H}_{\mathbb{Z}}$ is the part in which reducible but indecomposable representations appear.

The obvious guess

$$\mathsf{H}_{\mathbb{Z}} = \bigoplus_{\lambda \in \mathbb{Z}} \left(\mathsf{T}_\lambda \otimes \mathsf{T}_\lambda \right)$$

is wrong (the 2-point functions are degenerate).

$$\begin{tikzcd}
& \left|\phi_\lambda\right\rangle \otimes \left|\phi_\lambda\right\rangle \arrow[dl] \arrow[dr] & \\
\left|\chi_\lambda\right\rangle \otimes \left|\phi_\lambda\right\rangle \arrow[dr] & & \left|\phi_\lambda\right\rangle \otimes \left|\chi_\lambda\right\rangle \arrow[dl] \\
& \left|\chi_\lambda\right\rangle \otimes \left|\chi_\lambda\right\rangle &
\end{tikzcd}$$

The problem is that the singular vectors of the T_λ, with $\lambda \in \mathbb{Z}$, don't have partners to render them non-null.

We fix this by using the staggered modules Q_λ instead, noting that

$$\mathrm{ch}\Big[\bigoplus_{\lambda\in\mathbb{Z}}(\mathsf{T}_\lambda \otimes \mathsf{T}_\lambda)\Big] = \mathrm{ch}\Big[\bigoplus_{\lambda\in\mathbb{Z}}(\mathsf{Q}_\lambda \otimes \mathsf{A}_\lambda)\Big] = \mathrm{ch}\Big[\bigoplus_{\lambda\in\mathbb{Z}}(\mathsf{A}_\lambda \otimes \mathsf{Q}_\lambda)\Big].$$

The result is manifestly local with non-degenerate 2-point functions:

$$\begin{tikzcd}
 & \cdots & \mathsf{A}_{\lambda+1}\otimes\mathsf{A}_{\lambda+1} \arrow[dl] \arrow[dr] \arrow[drr, dashed] & & & \mathsf{A}_{\lambda}\otimes\mathsf{A}_{\lambda} \arrow[dll, dashed] \arrow[dl] \arrow[dr] \arrow[drr, dashed] & & & \mathsf{A}_{\lambda-1}\otimes\mathsf{A}_{\lambda-1} \arrow[dll, dashed] \arrow[dl] \arrow[dr] & \cdots & \\
\cdots & \mathsf{A}_{\lambda+2}\otimes\mathsf{A}_{\lambda+1} \arrow[dr] & & \mathsf{A}_{\lambda}\otimes\mathsf{A}_{\lambda+1} \arrow[dl] \arrow[drr, dashed] & \mathsf{A}_{\lambda+1}\otimes\mathsf{A}_{\lambda} \arrow[dll, dashed] \arrow[dr] & & \mathsf{A}_{\lambda-1}\otimes\mathsf{A}_{\lambda} \arrow[dl] \arrow[drr, dashed] & \mathsf{A}_{\lambda}\otimes\mathsf{A}_{\lambda-1} \arrow[dll, dashed] \arrow[dr] & & \mathsf{A}_{\lambda-2}\otimes\mathsf{A}_{\lambda-1} \arrow[dl] & \cdots \\
 & \cdots & \mathsf{A}_{\lambda+1}\otimes\mathsf{A}_{\lambda+1} & & & \mathsf{A}_{\lambda}\otimes\mathsf{A}_{\lambda} & & & \mathsf{A}_{\lambda-1}\otimes\mathsf{A}_{\lambda-1} & \cdots & \\
 & & & & & \mathsf{H}_{\mathbb{Z}} & & & & &
\end{tikzcd}$$

The standard module formalism

To recap for the singlet model, we have identified a collection of (indecomposable) standard modules T_λ, where $\lambda \in \mathbb{R}$.

These are partitioned into irreducibles with $\lambda \notin \mathbb{Z}$ (the typical modules) and indecomposables with $\lambda \in \mathbb{Z}$ (the atypical standard modules).

Representations with $\lambda \in \mathbb{Z}$ like A_λ and Q_λ are also called atypical.

This is one instance of the standard module formalism:

- The standard characters form a basis for the space of all characters and carry a representation of $\mathsf{SL}_2(\mathbb{Z})$.
- The (standard) Verlinde formula returns non-negative integer (Grothendieck) fusion coefficients.
- The logarithmic behaviour (reducibility but indecomposability) is confined to the atypical sector. The typical sector behaves like rational CFTs and the free boson.

The key feature of the standard module formalism is that the atypical sector constitutes a measure zero subset of the parameter space.

When integrating over the standard modules / characters, as in the Verlinde formula, the logarithmic difficulties are thus irrelevant.

Rational CFTs and the free boson may be regarded as instances of the standard module formalism in which there are no atypical representations.

Among logarithmic CFTs, the standard module formalism holds for

- Singlet models (parametrised by $p, p' \in \mathbb{Z}_{>0}$).
- Logarithmic minimal models (Virasoro and $N = 1$).
- Fractional level WZW models (*eg.*, $\widehat{\mathfrak{sl}}_2$).
- Super-WZW models (*eg.*, $\widehat{\mathfrak{gl}}(1|1)$).
- Bosonic ghosts.

The triplet model again

The standard module formalism does not apply to the triplet models.

Of the irreducibles, $\mathcal{W}_0$ and $\mathcal{W}_1$ should be regarded as atypical while $\mathcal{W}_{-1/8}$ and $\mathcal{W}_{3/8}$ should be regarded as typical.

The atypical standards have the form $\begin{array}{c} \mathcal{W}_0 \\ \downarrow \\ \mathcal{W}_1 \end{array}$ and $\begin{array}{c} \mathcal{W}_1 \\ \downarrow \\ \mathcal{W}_0 \end{array}$.

The parameter space is discrete and the atypicals have non-zero measure.

The same is true for the other log-rational orbifolds of symplectic fermions: there are finitely many standard modules and a non-zero number of them are reducible but indecomposable.

In fact, all the logarithmic CFTs that are known to admit the standard module formalism have log-rational infinite order simple current extensions that do not.

Log-rationality and modularity

We can exploit this relationship between the "good" logarithmic CFTs and their "evil" log-rational cousins to compute fusion in the latter.

$$
\begin{tikzcd}
\text{triplet} \arrow[d, shift left, "\text{orbifold}"] \\
\text{singlet} \arrow[u, shift left, "\text{simple current extension}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times B \arrow[d, "\text{restriction}"'] & & C \\
a \times b \arrow[rr, "\text{standard}", "\text{Verlinde}"'] & & c \arrow[u, "\text{induction}"']
\end{tikzcd}
$$

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\text{triplet} \arrow[d, shift left, "\text{orbifold}"] \\
\text{singlet} \arrow[u, shift left, "\text{simple current extension}"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
{\color{blue}\mathrm{A} \times \mathrm{B}} \arrow[r, red, "\text{log-rational}", "\text{Verlinde}"'] \arrow[d, "\text{restriction}"'] & {\color{blue}\mathrm{C}} \\
{\color{blue}\mathrm{a} \times \mathrm{b}} \arrow[r, "\text{standard}", "\text{Verlinde}"'] & {\color{blue}\mathrm{c}} \arrow[u, "\text{induction}"']
\end{tikzcd}$$

In fact, the standard Verlinde formula can be lifted to a log-rational Verlinde formula that computes (Grothendieck) fusion coefficients.

The input is the log-rational S-matrix of the characters (not the torus amplitudes), τ-dependent coefficients included.

Bulk state space

Finally, one can lift the local, non-degenerate proposal for the bulk state space of the singlet model to a similar proposal for the triplet:

$$\mathsf{H} = \mathsf{H}_{\mathbb{Z}} \oplus \mathsf{W}_{-1/8} \otimes \mathsf{W}_{-1/8} \oplus \mathsf{W}_{3/8} \otimes \mathsf{W}_{3/8}.$$

$$\begin{tikzcd}
 & \mathsf{W}_0 \otimes \mathsf{W}_0 \arrow[dl] \arrow[dr] \arrow[drr, dashed] \arrow[drrrr, dashed] & & & \mathsf{W}_1 \otimes \mathsf{W}_1 \arrow[dllll, dashed] \arrow[dll, dashed] \arrow[dl] \arrow[dr] & \\
\mathsf{W}_1 \otimes \mathsf{W}_0 \arrow[dr] \arrow[drrrr, dashed] & & \mathsf{W}_1 \otimes \mathsf{W}_0 \arrow[dl] \arrow[drr, dashed] & \mathsf{W}_0 \otimes \mathsf{W}_1 \arrow[dll, dashed] \arrow[dr] & & \mathsf{W}_0 \otimes \mathsf{W}_1 \arrow[dllll, dashed] \arrow[dl] \\
 & \mathsf{W}_0 \otimes \mathsf{W}_0 & & & \mathsf{W}_1 \otimes \mathsf{W}_1 &
\end{tikzcd}$$

$\mathsf{H}_{\mathbb{Z}}$

It is likewise local and non-degenerate.

Conclusions

- Modularity is subtle for log-rational CFTs, *eg.* the triplet.
- We do not have an $\mathsf{SL}_2(\mathbb{Z})$-action on the characters and the Verlinde formula does not work.
- However, it appears that standard modules provide a good organising principle in non-log-rational examples, *eg.* the singlet.
- We have an $\mathsf{SL}_2(\mathbb{Z})$-action on the standard characters and the (standard) Verlinde formula works.
- In all known examples, log-rational and non-log-rational CFTs are related by infinite order orbifolds and simple current extensions.
- We can exploit this to compute (Grothendieck) fusion rules of log-rational theories using induction and restriction.
- We can also deduce log-rational versions of the standard Verlinde formula that apply directly to CFTs like the triplet model.
- We expect that these versions will lead to a better understanding of modularity in general.

ToDo...

- Develop a rigorous mathematical setting for the standard module formalism.
- Understand how invariant measures naturally arise on the space that parametrises the standard characters.
- Figure out what modularity (and projectivity and rigidity and ...) means for general logarithmic CFTs.
- Find log-rational CFTs with no infinite order automorphisms or explain why such models do not exist.
- Construct moar examples, moar!!!

"*Only those who attempt the absurd will achieve the impossible.*"

- M C Escher