$sl(3)$ weight modules and higher-rank logarithmic CFT

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Outline

Motivation

Rational CFT has been quite a success story for mathematical physics (and pure maths).

2D CFT describes the quantum state space ("Hilbert space") of certain massless theories as a representation of two vertex operator algebras.

Rationality means this representation is a finite direct sum of irreducibles:

$$\mathsf{H} = \bigoplus_{i=1}^{n} \mathsf{L}_i \otimes \mathsf{L}_i.$$

But what if the theory requires reducible but indecomposable representations, *eg.* polymers, percolation? We need logarithmic CFT.

Such CFTs generally have, unlike rational CFTs, logarithmic singularities in some correlators [Rozansky-Saleur '92, Gurarie '93].

But, tractable examples of logarithmic CFTs are hard to find.

Rational CFTs	Logarithmic CFTs
Compactified free bosons	Symplectic fermions
Free fermions	Bosonic ghosts
Minimal models	Triplet models?
Wess-Zumino-Witten models	Fractional-level WZW models?

WZW models form a very rich supply of well-understood rational CFTs.

Perhaps their fractional-level analogues will play a similar role for logarithmic CFTs...

They also have independent physical interest! Protected sectors of certain 4D $N = 2$ super-CFTs $\overset{\text{belief}}{\sim}$ fractional-level VOAs [Beem *et al.*'15].

Schur indices $\sim$ VOA characters, Higgs branches $\sim$ associated varieties.

Fractional-level WZW models

We know the VOA: irreducible level-k vacuum module over $\widehat{\mathfrak{g}}$, where

$$k + \mathsf{h}^\vee = \frac{u}{v}, \quad u \in \mathbb{Z}_{\geqslant 2},\ v \in \mathbb{Z}_{\geqslant 1},\ \gcd\{u, v\} = 1$$

and $k \notin \mathbb{Z}_{\geqslant 0}$ [Gorelik-Kac '06]. The representations are the problem.

For $\mathfrak{g} = \mathfrak{sl}_2$, highest-weight modules do not suffice [Koh-Sorba '88]. We need relaxed highest-weight modules (+ spectral flow + extensions) to have modular invariance [Creutzig-DR '13] and (conjecturally) closure under fusion.

Relaxed highest-weight modules are representations generated by a state that only needs to be annihilated by $\mathfrak{g}$-modes with strictly positive indices.

So they can (and often do) have infinitely many ground states.

$$\begin{matrix} \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ \cdots & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \cdots \\ \cdots & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

How do we construct irreducible relaxed highest-weight modules?

As quotients of relaxed Verma modules (of course).

Recall that Verma modules are induced from irreducible representations of the Cartan subalgebra. To get a relaxed Verma module, we induce from a weight representation of the zero-mode subalgebra ($\cong$ to $\mathfrak{g}$).

Cartan subalgebra irreps are easy: they're all 1-dimensional.

What about weight reps of $\mathfrak{g}$? That's not so easy...

We know the finite-dimensional ones and some infinite-dimensional ones, *eg.* Verma modules for $\mathfrak{g}$. Are there others?

It turns out that there are lots! Even for $\mathfrak{g} = \mathfrak{sl}_2$, we have the principal and complementary series from $\mathsf{SL}_2(\mathbb{R})$ representation theory and more.

Weight modules for $\mathfrak{sl}_2$

A weight module is one on which the Cartan subalgebra $\mathfrak{h}$ acts diagonalisably. We assume that the eigenspaces are finite-dimensional.

For $\mathfrak{sl}_2$, irreducible weight modules are easy to classify:

- Highest- and lowest-weight modules with highest weight $\mu \in \mathsf{P}_{\geqslant}$.
- Highest-weight Verma modules with highest weight $\mu \notin \mathsf{P}_{\geqslant}$.
- Lowest-weight Verma modules with lowest weight $\mu \notin -\mathsf{P}_{\geqslant}$.
- Dense modules with weight support $\lambda + \mathsf{Q}$ and Casimir eigenvalue q, where $q \neq (\mu, \mu + 2\rho)$ for any $\mu \in \lambda + \mathsf{Q}$.

($\mathsf{P}_{\geqslant} =$ dominant integral weights, $\mathsf{Q} =$ root lattice, $\rho =$ Weyl vector.)

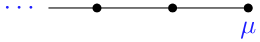
Dense modules are constructed by inducing from the centraliser of $\mathfrak{h}$ in $\mathscr{U}(\mathfrak{g})$ to $\mathscr{U}(g)$. This centraliser is just polynomials in $\mathfrak{h}$ and the Casimir.

The weight spaces ($= \mathfrak{h}$-eigenvectors) are thus one-dimensional.

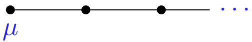
$\boxed{\text{Irreducible weight } \mathfrak{sl}_2\text{-modules}}$

Highest- and lowest-weight

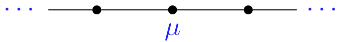
Highest-weight Verma



Lowest-weight Verma



Dense



Weight modules for simple $\mathfrak{g}$

For higher-rank $\mathfrak{g}$, the classification is not easy.

Even for $\mathfrak{sl}_3$, the centraliser of $\mathfrak{h}$ in $\mathscr{U}(\mathfrak{g})$ is non-abelian. Generators are known, but no set of relations is known to be complete [Futorny '86, '89]. Weight spaces are rarely one-dimensional.

Theorem [Fernando '90]

An irreducible weight module for $\mathfrak{g}$ is either

- Dense ($=$ torsion-free $=$ cuspidal), *ie.* the weight support is a single coset in $\mathfrak{h}^*/\mathsf{Q}$, or
- a quotient of the parabolic induction of a dense $\mathfrak{p}$-module, where $\mathfrak{p} \subset \mathfrak{g}$ is a parabolic subalgebra ($=$ contains a Borel subalgebra).

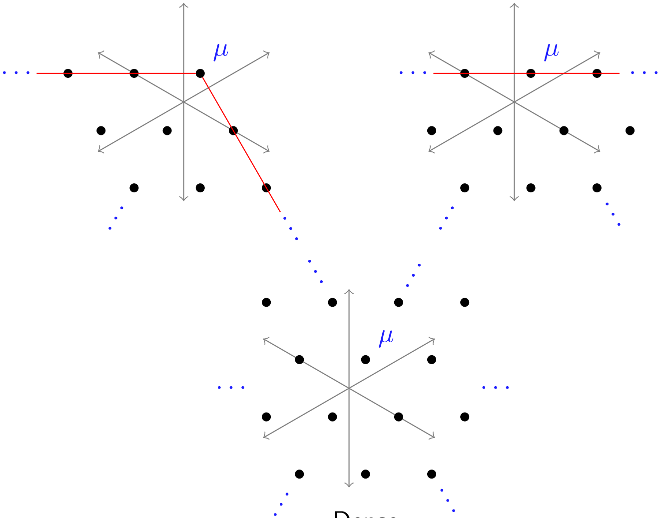
Moreover, dense modules only exist for $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$.

We can therefore classify irreducible weight modules inductively, if we can classify the irreducible dense $\mathfrak{sl}_n$- and $\mathfrak{sp}_{2n}$-modules.

Irreducible weight $\mathfrak{sl}_3$-modules

Highest-weight

Parabolically induced from $\mathfrak{sl}_2$



Dense

The classification of irreducible weight $\mathfrak{g}$-modules was completed in [Mathieu '00] using coherent families.

The key observation is that the dense modules $\mathsf{D}_{\lambda;\chi}$ are all "the same", *ie.* they fit together into families parametrised by the central character χ:

$$\mathsf{C}_\chi = \bigoplus_{\lambda \in \mathfrak{h}^*/\mathsf{Q}} \mathsf{D}_{\lambda;\chi}.$$

Facts:

1. The dimension of a weight space of C_χ is independent of the weight.
2. The action of $\mathfrak{g}$ on C_χ is polynomial.
3. Every coherent family has at least one reducible summand whose composition factors include an irreducible highest-weight $\mathfrak{g}$-module.

Because, we understand the relevant irreducible highest-weight modules, we understand coherent families and thus dense modules [Mathieu], hence we understand irreducible weight modules [Fernando].

Higher-rank logCFTs

Recall, we want to study fractional-level WZW models for higher-rank $\mathfrak{g}$, *eg.* $\mathfrak{g} = \mathfrak{sl}_3$, as archetypal examples of logarithmic CFTs.

For this, we need the irreducible relaxed highest-weight $\widehat{\mathfrak{g}}$-modules which define modules over the fractional-level VOA.

[Frenkel-Zhu '92, Zhu '96] let us classify these in terms of the irreducible weight $\mathfrak{g}$-modules which are annihilated by a certain ideal $I_{u,v}$ of $\mathscr{U}(\mathfrak{g})$.

We know the highest-weight modules that $I_{u,v}$ annihilates [Arakawa '12]. Polynomial action then tells us which coherent families are annihilated.

This leads to an inductive strategy to classify irreducible relaxed highest-weight VOA-modules given the highest-weight classification.

This strategy is currently being fleshed out for fractional level VOAs corresponding to $\mathfrak{g} = \mathfrak{sl}_3$.

Nilpotent orbit	hw. $\mathfrak{sl}_3$-mods	$\mathfrak{sl}_3$-families	VOA-mods
zero: $\begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}$	finite-dim.	finite-dim.	ordinary hw.
minimal: $\begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix}$	bounded	coherent	relaxed hw.
principal: $\begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$	unbounded	parabolic	"semi-relaxed" hw.

This not only gives an elegant proof of the relaxed classification for $\mathfrak{sl}_3$ [Arakawa-Futorny-Ramirez '16], it also gives information about indecomposable relaxed VOA-modules.

These indecomposables are essential for the standard module formalism [Creutzig-DR '13, DR-Wood '14] that describes the modular properties of the corresponding logarithmic CFTs.

Example: $\mathfrak{g} = \mathfrak{sl}_3$, $k = -\frac{3}{2}$ $(u = 3,\ v = 2)$.

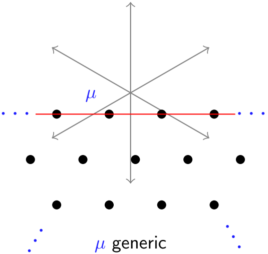
This example may be brute-force analysed [Perše '07, Adamović '14, Kawasetsu-DR-Wood '18] as $I_{3,2}$ is generated by a degree 2 element in $\mathscr{U}(\mathfrak{sl}_3)$. There is:

- 1 ordinary highest-weight VOA-module (the vacuum module),
- 3 bounded highest-weight VOA-modules (up to twists), giving
- 1 family of parabolically induced VOA-modules (up to twists), with $\mathfrak{sl}_2 \subset \mathfrak{p} \subset \mathfrak{sl}_3$), and
- 1 coherent family of relaxed highest-weight VOA-modules.

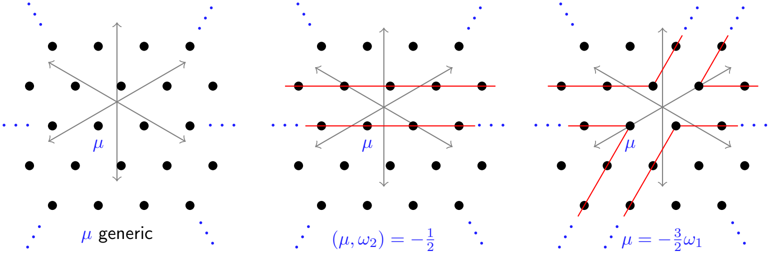
Aside from the vacuum module, these irreducibles are all classified by the minimal nilpotent orbit as $v = 2$ [Arakawa '12]. To see principal irreducibles, we would need to analyse a fractional level with $v \geqslant 3$.

Brute-forcing any other fractional level is much more challenging. However, our classification strategy gives the result relatively easily.

Ground states of the semi-relaxed highest-weight modules.



Ground states of the relaxed highest-weight modules.



Conclusions

- Fractional-level WZW models are promising candidates for sorely needed tractable examples of higher-rank logarithmic CFTs.
- The task of classifying their highest-weight modules was recently (mostly) completed by Arakawa.
- We have now shown that Mathieu's coherent families let us leverage this result to inductively deduce the classification of irreducible relaxed highest-weight modules.
- Conjecturally, we then get all irreducible weight modules for the VOA by applying spectral flow.
- Our procedure reproduces the known results for fractional-level $\mathfrak{sl}_2$, $\mathfrak{osp}(1|2)$ and $\mathfrak{sl}_3$ models, whilst dramatically simplifying the proofs.
- For $\mathfrak{sl}_3$, our methods also suggest a powerful and general organising principle. Extending the results to general $\mathfrak{g}$ now appears feasible.

ToDo...

- We need to work out the details of the classification argument for other $\mathfrak{g}$, *eg.* $\mathfrak{sp}_4$, $\mathfrak{g}_2$, ...
- It would be physically interesting to extend the results to simple basic classical Lie superalgebras, *eg.* $\mathfrak{sl}(2|1)$, $\mathfrak{psl}(2|2)$ and $\mathfrak{d}(2|1;\alpha)$.
- For almost all $\mathfrak{g}$, the characters of the irreducible relaxed highest-weight $\mathfrak{g}$-modules remain unknown (but see Kazuya's talk).
- To get linearly independent characters, necessary for modular shenanigans, we need to understand characters that account for all Casimirs, not just the quadratic one (L_0).
- Then, there are certain (non-admissible) fractional levels for which Arakawa's result on highest-weight modules fails.
- And of course, there are cosets, orbifold and quantum hamiltonian reductions to explore... sounds like a good grant application!

"*Only those who attempt the absurd will achieve the impossible.*"

- M C Escher