A introduction to logarithmic conformal field theory

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Outline

Motivation

CFT describes the continuum scaling limit of certain statistical models as well as the worldsheet dynamics of various types of string theory.

Rational CFT has been quite a success story for mathematical physics (not to mention pure maths).

2D CFT describes the quantum state space ("Hilbert space") in terms of a representation of two chiral algebras.

Rationality means this representation is a finite direct sum of irreducibles:

$$\mathsf{H} = \bigoplus_{i=1}^{n} \mathsf{M}_i \otimes \mathsf{M}_i.$$

But what if the theory requires reducible but indecomposable representations, *eg.* polymers, percolation? We need logarithmic CFT.

What does "logarithmic" mean?

Unlike rational CFTs, a logarithmic CFT has logarithmic singularities in some of its correlation functions [Rozansky-Saleur '92, Gurarie '93].

Typically, a logarithmic CFT has states and fields on which the hamiltonian $H = L_0 + \overline{L}_0$ acts non-diagonalisably.

The states on which H acts diagonalisably form a proper subrepresentation, *ie.* we have reducible but indecomposable reps.

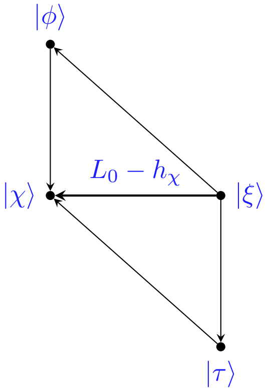
$L_0 + \overline{L}_0$ acts
non-diagonalisably

log-singularities
in correlators

reducible but
indecomposable reps

Chiral logCFT: a primer

At the chiral level, the hamiltonian is L_0. Then:



- $|\chi\rangle$ is singular, but not null:

$$\langle \chi(z)\xi(w)\rangle = \frac{\beta}{(z-w)^{2h_\chi}};$$

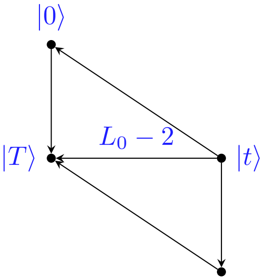
- $|\phi\rangle$, $|\tau\rangle$ and $|\chi\rangle$ each generate different proper subrepresentations;
- $|\xi\rangle$ is logarithmic:

$$\langle \xi(z)\xi(w)\rangle = \frac{\alpha - \beta\log(z-w)}{(z-w)^{2h_\chi}};$$

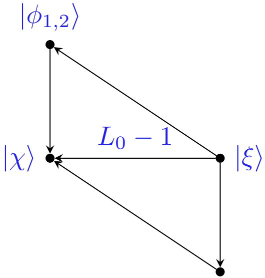
- β is an invariant of the representation (while α is not).

Example: percolation

Critical percolation has an infinite number of staggered (logarithmic) representations, *eg.*



$$\langle T(z) t(w) \rangle = \frac{-\frac{5}{8}}{(z-w)^4}$$



$$\langle \chi(z) \xi(w) \rangle = \frac{-\frac{1}{2}}{(z-w)^2}$$

Note: $\langle \phi_{1,2}(\infty)\phi_{1,2}(1)\phi_{1,2}(z)\phi_{1,2}(0) \rangle = \dfrac{3\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})^2} z^{1/3} {}_2F_1(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}; z)$ has no log-singularity [Cardy], but the 6-pt function does [Flores-Kleban-Simmons].

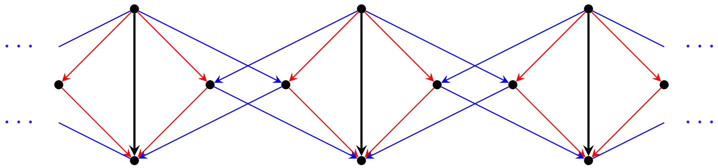
Bulk logCFT: a primer

The bulk state space of a logCFT splits into two sectors:

- A typical sector in which holomorphic factorisation continues to hold (as in rational theories):

$$\mathsf{H}_{\text{typ.}} = \bigoplus_i \mathsf{M}_i \otimes \mathsf{M}_i;$$

- An atypical sector in which holomorphic factorisation fails and is replaced by a complicated gluing of staggered modules:



The red/blue lines indicate the holomorphic/antiholomorphic action; the black lines denote the non-diagonalisable action of $H = L_0 + \overline{L}_0$.

How do we know this?

Analogy: similar indecomposable structures appear when studying non-compact Lie groups and certain associative algebras.

Classification: there is a wealth of mathematical results that limit the possibilities for these indecomposable structures.

Construction: fusion, à la Nahm and Gaberdiel-Kausch, lets us explicitly construct certain examples of chiral staggered modules.

Consistency: there are strong constraints on the spectrum *eg.* modular invariance and single-valuedness of bulk correlators.

Discretisation: one can study reducible but indecomposable representations for the diagram algebras and quantum groups that appear in statistical mechanics (loop models, spin chains, *etc.*).

- Observe non-diagonalisable hamiltonians.
- Predict non-completely reducible fusion rules.
- Approximate β numbers appearing in logarithmic correlators.

Why do we care?

- Logarithmic structure is generic, not pathological. We should expect it, or at least look for it, in all non-rational models.
- Looking beyond rationality provides new perspectives on old problems, *eg.* negative fusion coefficients for fractional-level WZW models.
- General CFTs already show tantalising deviations from rational behaviour, *eg.* modularity is much more subtle and the Verlinde formula needs significant modification.
- Applications!
  - (Mildly) non-local observables in statistical lattice models.
  - Superstrings on supermanifolds.
  - 3D chiral gravity duals.
  - AdS/CFT, *eg.* $\mathfrak{psl}(2|2) \sim \mathsf{AdS}_3$.
  - Quantum Hall transitions.
  - Non-equilibrium systems.
  - Higher-dimensional CFTs...

Which examples are under control?

Tractable examples of logarithmic CFTs are hard to find, but there are many candidates that should be worked out in detail.

Rational CFTs	Logarithmic CFTs
Compactified free bosons	Triplet models
Free fermions	Bosonic ghosts
Superconformal minimal models	Log-minimal models ($N \leqslant 2$)
Wess-Zumino-Witten models	Fractional-level WZW models
Parafermions	Log-parafermions
W-algebraic models	Logarithmic versions

Unfortunately, only "rank 1" (*ie.* $\mathfrak{sl}_2$-like) logarithmic models are understood to any degree.

To generalise to higher ranks, symmetry is expected to be helpful. The fractional-level WZW models are thus strong candidates for exploration.

Where might we go?

- We need more examples, especially of "log-rational" CFTs. These have reducible but indecomposable reps, but only finitely many irreducibles, *eg.* symplectic fermions and the triplet models.
- We also need more examples in higher ranks. Here, we are exploring fractional-level WZW models and their W-algebra reductions.
- Techniques are being developed for super-cases with a view to analysing the WZW models of $\mathfrak{sl}(2|1)$ and $\mathfrak{psl}(2|2)$.
- This is also opening the door to the $N > 2$ minimal models.
- Logarithmic know-how is aiding the identification of continuum scaling limits of spin chains and loop models. This helps to illuminate the question of what this limit actually is.
- The failure of holomorphic factorisation suggests asking why we expect chiral algebras in the first place. Already, there are proposals that some scaling limits might be usefully described by a non-factorisable symmetry algebra extending $\mathsf{Vir} \otimes \mathsf{Vir}$.
- ???

Fractional-level WZW models and $D > 2$ CFT

A hot topic in hep-th at the moment is CFT in higher dimensions and its relation to 2D CFTs, *eg.* that $N = 2$ 4D superCFTs have invariants that are computed in terms of some vertex operator algebra:

Schur index	Higgs branch	Coulomb branch	Schur w/ defect
VOA character	Associated variety	Ext-algebra	module character

For most known 4D CFTs, the VOA corresponds to a fractional-level WZW model or an associative W-algebra.

eg., The $N = 2$ superCFT describing a $D3$-brane at an F-theory singularity gives rise to VOAs from the Deligne exceptional series

$$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8$$

at level $k = -1 - \frac{\mathsf{h}^\vee}{6}$. [black = not realised; blue = admissible level; red = non-admissible.]

Similarly, Argyres-Douglas theories give non-unitary Virasoro minimal models, fractional-level $\mathfrak{sl}_2$-models and Feigin-Semikhatov W-algebras.

Fractional levels

The fractional-level WZW models correspond to the levels k for which the universal vacuum module of $\widehat{\mathfrak{g}}$ has a non-trivial singular vector:

$$k + \mathsf{h}^\vee = \frac{u}{v}, \quad u \in \mathbb{Z}_{\geqslant 2},\ v \in \mathbb{Z}_{\geqslant 1},\ \gcd\{u, v\} = 1.$$

We exclude the WZW models corresponding to $k \in \mathbb{Z}_{\geqslant 0}$.

Because there is a non-trivial singular vector, there are non-trivial constraints on the spectrum. It is hard to determine these constraints.

We know that highest-weight modules do not give a consistent spectrum. Relaxed highest-weight modules (+ spectral flow + extensions) are needed to have modular invariance and closure under fusion.

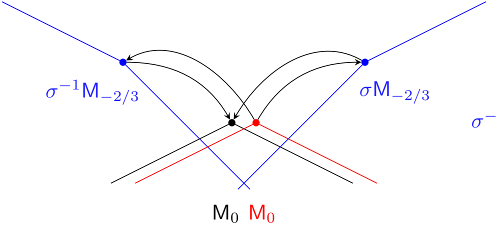
Relaxed highest-weight modules are representations generated by a state that only needs to be annihilated by $\widehat{\mathfrak{g}}$-modes with strictly positive indices.

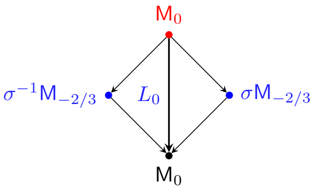
In general, a fractional-level model has finitely many highest-weight modules, but a continuum of relaxed ones.

Example: $\mathfrak{sl}_2$, $k = -\frac{4}{3}$

The $(A_1 A_3)$ AD model gives the first member of the Deligne series. This fractional-level WZW model was shown to be non-unitary, non-rational and logarithmic by Gaberdiel.

There are three highest-weight modules, the vacuum module M_0, $\mathsf{M}_{-2/3}$ and $\mathsf{M}_{-4/3}$. There are a countably infinite number of staggered modules, each with no minimal conformal dimension, *eg.*





There is also an uncountably infinite number of irreducible relaxed highest-weight modules, necessary for modular invariance.

Recent progress

The spectrum of fractional-level relaxed highest-weight modules is only known for $\mathfrak{sl}_2$ and $\mathfrak{osp}(1|2)$ (and implicitly for admissible-level $\mathfrak{sl}_3$).

Recently, Arakawa has determined the highest-weight spectrum for all simple (non-super) $\mathfrak{g}$ and all admissible levels.

There are very few results for non-admissible levels.

In this vein, we announce the following "dictum" (Kawasetsu-DR) whose proof will appear soon:

> *The classification of the relaxed highest-weight spectrum reduces to the classification of the highest-weight spectrum.*

ie., because Arakawa has classified the highest-weight modules, we can now classify the relaxed highest-weight modules at admissible levels.

We can also determine their characters using the ideas in [1803.01989].

ToDo

With this, we can explore modularity and fusion for admissible-level WZW models.

Quantum hamiltonian reduction will then yield information about the corresponding W-algebras.

This is expected to lead to many new archetypal examples of logarithmic (and perhaps some log-rational) CFTs.

We still need to understand the highest-weight spectrum for non-admissible levels, important for $D > 2$ CFT.

We also need to generalise to the super-case, important for AdS/CFT.

Question: are there defect configurations whose Schur indices correspond to staggered modules. Is the non-diagonalisability of L_0 relevant?