# Modularity beyond rationality

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# Outline

# Rationality

Conformal field theory is quantum field theory with invariance under conformal (angle-preserving) transformations.

In two dimensions, local conformal transformations are (anti)analytic.

They give rise to two commuting copies of the Virasoro algebra.

The space of states H of the CFT is thus a Virasoro module.

More generally, H is a module of two commuting copies of the symmetry algebra of the CFT, a vertex operator algebra $V$.

## Definition

A CFT is rational if H is

- *semisimple* as a $V \otimes V$-module; and
- decomposes into a *finite* number of simple $V \otimes V$-modules.

# Examples and non-examples

1. The Ising model is rational with $V$ being the simple Virasoro VOA of central charge $\frac{1}{2}$ and

$$\mathsf{H} = (\mathsf{L}_0 \otimes \mathsf{L}_0) \oplus (\mathsf{L}_{1/16} \otimes \mathsf{L}_{1/16}) \oplus (\mathsf{L}_{1/2} \otimes \mathsf{L}_{1/2}),$$

where $\mathsf{L}_h$ is the irreducible highest-weight Virasoro module of conformal weight $h$.

2. The free boson is not rational with $V$ being the Heisenberg VOA of central charge $1$ and

$$\mathsf{H} = \int_{\mathbb{R}}^{\oplus} (\mathsf{F}_p \otimes \mathsf{F}_p) \, \mathrm{d}p,$$

where $\mathsf{F}_p$ is the Fock space of charge $p$. Whilst $\mathsf{H}$ is semisimple, it is composed of an uncountably-infinite number of simples.

3. The triplet model is not rational. It has four simple $V$-modules and the state space decomposes as

$$\mathsf{H} = (\mathsf{W}_{-1/8} \otimes \mathsf{W}_{-1/8}) \oplus (\mathsf{W}_{3/8} \otimes \mathsf{W}_{3/8}) \oplus \begin{bmatrix} \text{an indecomposable} \\ \text{with } 8 \text{ composition} \\ \text{factors built from} \\ \mathsf{W}_0 \text{ and } \mathsf{W}_1 \end{bmatrix}.$$

It is finite, but not semisimple.

4. The bosonic ghost system is not rational either. It has an uncountable infinity of simple $V$-modules $\mathsf{E}_\lambda^\ell$ and $\mathsf{G}^\ell$, $\ell \in \mathbb{Z}$ and $0 < \lambda < 1$, with

$$\mathsf{H} = \bigoplus_{\ell \in \mathbb{Z}} \oint_{(0,1)} (\mathsf{E}_\lambda^\ell \otimes \mathsf{E}_\lambda^\ell) \, \mathrm{d}\lambda \oplus \begin{bmatrix} \text{an indecomposable with} \\ \text{a countable infinity of} \\ \text{composition factors built} \\ \text{from the } \mathsf{G}^\ell \end{bmatrix}.$$

It is neither finite nor semisimple.

# Modularity

**Theorem [Huang]**

The modules of a rational CFT form a modular tensor category.

This means (among other things) that:

- the module category admits a tensor product (fusion);
- there is a "nice" action of the modular group $\mathsf{SL}_2(\mathbb{Z})$; and
- the fusion product and the modular group action are closely related.

This theorem was strongly motivated by the fact that the partition function (character of H) of a consistent CFT is modular-invariant.

**Definition**

A CFT is modular if its characters span a representation of $\mathsf{SL}_2(\mathbb{Z})$.

**Theorem [Zhu]**

If the CFT is rational, then it is modular.

Even better, Verlinde noticed that the fusion coefficients ${\mathcal{N}_{ij}}^{k}$, given by

$$\mathsf{M}_i \times \mathsf{M}_j \simeq \bigoplus_k {\mathcal{N}_{ij}}^{k}\, \mathsf{M}_k,$$

are determined by the modular S-transform of the characters,

$$\mathrm{ch}\big[\mathsf{M}_i\big] \xrightarrow{\mathcal{S}} \sum_j S_{ij}\, \mathrm{ch}\big[\mathsf{M}_j\big],$$

via the celebrated Verlinde formula:

$${\mathcal{N}_{ij}}^{k} = \sum_\ell \frac{S_{i\ell} S_{j\ell} S^*_{k\ell}}{S_{1\ell}} \qquad (\mathsf{M}_1 = V).$$

**Theorem [Huang]**

The Verlinde formula indeed holds for every rational CFT.

# Again with the examples...

1. The Ising model's three simples have characters given by

$$\mathrm{ch}\big[\mathsf{L}_0\big] = \tfrac{1}{2}\left(\sqrt{\tfrac{\vartheta_3(0,\tau)}{\eta(\tau)}} + \sqrt{\tfrac{\vartheta_4(0,\tau)}{\eta(\tau)}}\right), \qquad \mathrm{ch}\big[\mathsf{L}_{1/16}\big] = \sqrt{\tfrac{\vartheta_2(0,\tau)}{2\eta(\tau)}}.$$
$$\mathrm{ch}\big[\mathsf{L}_{1/2}\big] = \tfrac{1}{2}\left(\sqrt{\tfrac{\vartheta_3(0,\tau)}{\eta(\tau)}} - \sqrt{\tfrac{\vartheta_4(0,\tau)}{\eta(\tau)}}\right),$$

This CFT is modular with S- and T-matrices given by

$$\mathcal{S} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \quad \mathcal{T} = \mathsf{e}^{-\mathrm{i}\pi/24}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathsf{e}^{\mathrm{i}\pi/8} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

They generate a representation of $\mathsf{PSL}_2(\mathbb{Z})$: $\mathcal{S}^2 = (\mathcal{S}\mathcal{T})^3 = I$.

The Verlinde formula gives non-negative integers that agree with the fusion coefficients, as required.

2. The free boson's simple characters are

$$\mathrm{ch}\big[\mathsf{F}_p\big] = \frac{\mathsf{e}^{2\pi \mathfrak{i} p\zeta}\mathsf{e}^{\pi \mathfrak{i} p^2\tau}}{\eta(\tau)}, \quad p \in \mathbb{R}.$$

This CFT is also modular with S and T being represented by integral operators with kernels

$$\mathcal{S}(p,p') = \mathsf{e}^{-2\pi \mathfrak{i} pp'}, \quad \mathcal{T}(p,p') = \mathsf{e}^{-\pi \mathfrak{i}/12}\mathsf{e}^{\pi \mathfrak{i} p^2}\delta(p-p').$$

Note that S is just a Fourier transform!

This time, we have a representation of $\mathsf{SL}_2(\mathbb{Z})$: $\mathcal{S}^2 = (\mathcal{S}\mathcal{T})^3$, $\mathcal{S}^4 = I$.

Even though the free boson is not rational, the $(\sum \longrightarrow \int)$ Verlinde formula still gives the fusion coefficients.

3. The triplet model has four simples whose characters are

$$\mathrm{ch}\big[\mathsf{W}_0\big] = \frac{1}{2}\left(\frac{\vartheta_2(0,\tau)}{2\eta(\tau)} + \eta(\tau)^2\right), \quad \mathrm{ch}\big[\mathsf{W}_{-1/8}\big] = \frac{\vartheta_3(0,\tau)+\vartheta_4(0,\tau)}{2\eta(\tau)},$$

$$\mathrm{ch}\big[\mathsf{W}_1\big] = \frac{1}{2}\left(\frac{\vartheta_2(0,\tau)}{2\eta(\tau)} - \eta(\tau)^2\right), \quad \mathrm{ch}\big[\mathsf{W}_{3/8}\big] = \frac{\vartheta_3(0,\tau)-\vartheta_4(0,\tau)}{2\eta(\tau)}.$$

This CFT is not modular because of the S-transform of $\eta(\tau)^2$:

$$\eta(\tau)^2 \xrightarrow{\mathcal{S}} -\mathfrak{i}\tau\,\eta(\tau)^2.$$

Recall that $\mathsf{W}_0$ and $\mathsf{W}_1$ harbour the non-semisimplicity of the CFT. In particular, their projective covers are reducible but indecomposable.

$$\begin{tikzcd} & \mathsf{W}_0 \arrow[dl] \arrow[dr] & \\ \mathsf{W}_1 \arrow[dr] & \mathsf{P}_0 & \mathsf{W}_1 \arrow[dl] \\ & \mathsf{W}_0 & \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{W}_1 \arrow[dl] \arrow[dr] & \\ \mathsf{W}_0 \arrow[dr] & \mathsf{P}_1 & \mathsf{W}_0 \arrow[dl] \\ & \mathsf{W}_1 & \end{tikzcd}$$

We do have an action of $\mathsf{SL}_2(\mathbb{Z})$ on the projective characters, but they are not linearly independent: $\mathrm{ch}\big[\mathsf{P}_0\big] = \mathrm{ch}\big[\mathsf{P}_1\big]$.

This gives a modular-invariant partition function:

$$\tfrac{1}{2}\big|\mathrm{ch}\big[\mathsf{P}\big]\big|^2 + \big|\mathrm{ch}\big[\mathsf{W}_{-1/8}\big]\big|^2 + \big|\mathrm{ch}\big[\mathsf{W}_{3/8}\big]\big|^2.$$

However, the S-matrix is no good for Verlinde interpretations:

$$\mathcal{S} = \begin{pmatrix} 0 & 1 & -1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

However, the projectives carry a non-semisimple action of $L_0$, so one can introduce a pseudotrace that sees the non-trivial Jordan blocks.

Here, this augments the characters by the pseudocharacter $-\mathfrak{i}\tau\,\eta(\tau)^2$. Together, they span a $5$-dimensional $\mathsf{SL}_2(\mathbb{Z})$-module (but it is still no good for Verlinde games).

4. It seems that non-semisimplicity is bad for modularity. We therefore expect bosonic ghosts to behave badly.

But, the characters of the standard modules are modular:

$$\operatorname{ch}\left[\mathsf{E}_{\lambda}^{\ell}\right] = \frac{\mathsf{e}^{-\pi \mathsf{i} \ell(\ell-1)\tau}}{\eta(\tau)^2} \sum_{n \in \mathbb{Z}} \mathsf{e}^{2\pi \mathsf{i} n\lambda} \delta(\zeta + \ell\tau - n)$$

$$\Rightarrow \quad \mathcal{S}\left( {\ell \atop \lambda}, {\ell' \atop \lambda'} \right) = \mathsf{e}^{-2\pi\mathsf{i}(\ell\lambda' + \ell'\lambda - \frac{1}{2}\ell\ell')}, \quad (\ell, \ell' \in \mathbb{Z},\ 0 < \lambda, \lambda' < 1).$$

Moreover, the characters of the atypical modules are modular "almost everywhere" if we extend the standard characters to include $\lambda = 0$ and allow certain infinite-linear combinations:

$$\operatorname{ch}\left[\mathsf{G}^{\ell}\right] = \sum_{\ell' \geqslant 0} (-1)^{\ell'} \operatorname{ch}\left[\mathsf{E}_{0}^{\ell+\ell'+1}\right]$$

$$\Rightarrow \quad \mathcal{S}\left( {\ell \atop }, {\ell' \atop \lambda'} \right) = (-1)^{\ell+\ell'+1} \frac{\mathsf{e}^{-2\pi\mathsf{i}(\ell+\frac{1}{2})\lambda'}}{2\mathsf{i}\sin(\pi\lambda')}, \quad (\ell, \ell' \in \mathbb{Z},\ 0 < \lambda' < 1).$$

Most importantly (and surprisingly), the Verlinde formula works!

# Speculative conclusions

These examples suggest that we may expect modularity and Verlinde when the CFT is “semisimple almost everywhere”.

This observation is the basis for the standard module formalism.

It has been successfully tested on many examples, in particular on affine VOAs at admissible-levels.

It fails miserably for the triplet model where the non-semisimplicity is not confined to a set of measure zero (in parameter space).

We therefore expect this failure to persist for other finite but non-semisimple (log-rational) CFTs.

Unfortunately, we are no good at constructing log-rational examples...

# Why should we care?

One obvious reason is to find out what modularity really means.

Imposing finiteness is like studying theta functions without Poisson resummation and Fourier transforms.

Examples like the free boson and bosonic ghosts suggest that there must be nice classes of infinite MTCs.

Examples like the triplet model suggest that non-semisimple generalisations of MTCs may not be so easy to understand.

Finding and analysing more log-rational examples is necessary to determine which properties are natural.

Even if one is only concerned with (finite) MTCs, there are still good reasons to try expanding one's horizons.

Physicists have many constructions of rational CFTs, some of which are categorical and some of which start from non-rational models.

In particular, infinite simple current extensions often do this.

Example:

$$\begin{tikzcd}
c = 1 & \text{free boson} \arrow[r, shift left, "\text{infinite simple current extension}"] & \text{fermionic ghosts} \arrow[l, shift left, "\text{infinite orbifold}"] \arrow[d, shift left, "\text{square root}"] \\
c = \frac{1}{2} & \text{Ising model} \arrow[r, shift left, "\mathbb{Z}_2\text{ simple current extension}"] & \text{free fermion} \arrow[l, shift left, "\mathbb{Z}_2\text{-orbifold}"] \arrow[u, shift left, "\text{tensor square}"]
\end{tikzcd}$$

Many rational CFTs (*eg.* some W-algebras) are only known because they have been constructed from non-rational CFTs.

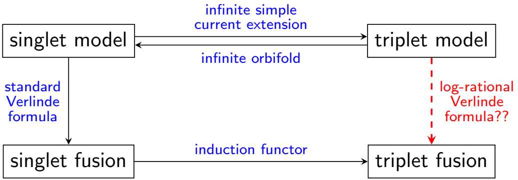
# Fixing the triplet

There have been several articles addressing modularity and the Verlinde formula for the triplet model, *eg.* [Fuchs-Hwang-Semikhatov-Tipunin, Gaberdiel-Runkel, Gainutdinov-Runkel, Creutzig-Gannon], but no general picture.

By contrast, there are now many successful works treating modularity and Verlinde for "semisimple almost everywhere" CFTs.

Can we learn something about the former from the latter successes?

The triplet has an automorphism of infinite order and the corresponding orbifold is the "semisimple almost everywhere" singlet model.



# The singlet CFT

The singlet model is very similar to the bosonic ghosts CFT.

In fact, the singlet VOA is the commutant, in the bosonic ghosts VOA, of a free boson sub-VOA.

It has a continuum of standard simples $\mathsf{T}_\lambda$, $\lambda \in \mathbb{R} - \mathbb{Z}$, and a discrete set of atypical simples $\mathsf{S}_\lambda$, $\lambda \in \mathbb{Z}$, with characters

$$\mathrm{ch}\big[\mathsf{T}_\lambda\big] = \frac{\mathsf{e}^{\pi \mathsf{i} (\lambda - \frac{1}{2})^2 \tau}}{\eta(\tau)}, \quad \mathrm{ch}\big[\mathsf{S}_\lambda\big] = \sum_{\ell \geqslant 0} (-1)^\ell \mathrm{ch}\big[\mathsf{T}_{\lambda - \ell}\big],$$

extending the standard characters to $\lambda \in \mathbb{Z}$. This is modular with

$$\mathcal{S}(\lambda, \lambda') = \begin{cases} \mathsf{e}^{-2\pi \mathsf{i} (\lambda - \frac{1}{2})(\lambda' - \frac{1}{2})} & \text{if } \lambda \notin \mathbb{Z}, \\ \dfrac{\mathsf{e}^{-2\pi \mathsf{i} \lambda (\lambda' - \frac{1}{2})}}{2\cos\big[\pi(\lambda' - \frac{1}{2})\big]} & \text{if } \lambda \in \mathbb{Z}. \end{cases}$$

The Verlinde formula works just fine!

# A Verlinde formula for the triplet

Given the standard Verlinde formula for the singlet, we can use induction to recover triplet fusion coefficients and then rewrite everything in terms of triplet modular data.

This is quite delicate (but possible). The result is a modified Verlinde formula that takes into account the partition of simples into:

atypical: $\{\mathsf{W}_0, \mathsf{W}_1\}$ typical: $\left\{\mathsf{W}_{-1/8}, \mathsf{W}_{3/8}\right\}$.

The triplet Verlinde formula:

$$\boxed{\mathcal{N}_{ij}{}^{k} = \sum_{\ell \in \text{typ.}} \frac{S_{i\ell} S_{j\ell} S^{-1}_{\ell k}}{S_{1\ell}} + \delta_{ijk} \sum_{\ell \in \text{atyp.}} \frac{S_{i\ell} S_{j\ell} S^{-1}_{\ell k}}{S_{1\ell}},}$$

where $\delta_{ijk} = 1$ if only $i \in$ atyp., only $j \in$ atyp. or $i, j, k \in$ atyp., and $\delta_{ijk} = 0$ otherwise.

Note that the S-matrix is that of the triplet with the factors of $\tau$!

# Conclusions

- Modularity is subtle for non-semisimple CFTs, *eg.* the triplet.
- We do not have an $\mathsf{SL}_2(\mathbb{Z})$-action on the characters in general and so the Verlinde formula does not seem to make sense.
- However, it appears that the standard module formalism provides a good organising principle for "semisimple almost everywhere" CFTs.
- Then, we have an $\mathsf{SL}_2(\mathbb{Z})$-action on the standard characters and the standard Verlinde formula seems to work.
- All known log-rational examples have an infinite order orbifold that is "semisimple almost everywhere".
- We can exploit this to compute (Grothendieck) fusion rules of log-rational theories using induction and restriction.
- We can also deduce log-rational versions of the standard Verlinde formula that apply directly to CFTs like the triplet model.
- We hope that these versions will one day (soon?) lead to a better understanding of modularity in general.