A higher-rank fractional-level Wess-Zumino-Witten model

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Outline

Motivation

I like to study 2D logarithmic CFTs.

This means that the quantum state space ("Hilbert space") will include reducible but indecomposable representations.

Such CFTs generally have, unlike rational CFTs, logarithmic singularities in certain correlation functions [Rozansky-Saleur '92, Gurarie '93].

Historically, applications have come from statistical physics, *eg.* polymers and percolation, and string theories with supersymmetric target spaces, *cf.* bosonic ghosts.

Recently, certain invariants (Schur indices, Higgs/Coulomb branches) of 4D $N = 2$ super-gauge field theories have been constructed in terms of data from (typically logarithmic) 2D CFTs [Beem *et al.* '15].

Many rational CFTs may be studied by constructing them from "building blocks", especially the Wess-Zumino-Witten models.

These models are particularly tractable (and beautiful) because their symmetries are affine Kac-Moody algebras at positive levels.

I like to study the fractional-level WZW models because they seem to play a similar role as "building blocks" for logarithmic CFTs.

They also provide the starting point for constructing some rational CFTs, *eg.* the W-algebras obtained by quantum hamiltonian reduction.

One example is the $\mathbb{Z}_2$-orbifold of the bosonic ghost CFT: its symmetry algebra is $\widehat{\mathfrak{sl}}_2$ with $k = -\frac{1}{2}$.

The fractional-level models for the Deligne exceptional series

$$\mathfrak{sl}_2 \subset \mathfrak{sl}_3 \subset \mathfrak{g}_2 \subset \mathfrak{so}_8 \subset \mathfrak{f}_4 \subset \mathfrak{e}_6 \subset \mathfrak{e}_7 \subset \mathfrak{e}_8$$

with $k = -\frac{\mathsf{h}^\vee}{6} - 1$ have also appeared as 4D $N = 2$ invariants.

Fractional-level WZW models

"Fractional-level" means that the level k of the affine algebra satisfies

$$k + \mathsf{h}^\vee = \frac{u}{v}, \quad u \in \mathbb{Z}_{\geqslant 2},\ v \in \mathbb{Z}_{\geqslant 1},\ \gcd\{u, v\} = 1$$

and $k \notin \mathbb{Z}_{\geqslant 0}$ [Gorelik-Kac '06].

For rational WZW models ($k \in \mathbb{Z}_{\geqslant 0}$), the CFT is built from irreducible highest-weight modules of $\widehat{\mathfrak{g}}$. For fractional-level models, the modules need not be highest-weight (nor irreducible).

For $\mathfrak{g} = \mathfrak{sl}_2$, we need relaxed highest-weight modules (plus spectral flow and extensions). We know characters, modular transforms and (Grothendieck) fusion [Creutzig-DR '13, Adamović '17, Kawasetsu-DR '18].

For $\mathfrak{g} = \mathfrak{osp}(1|2)$, we have similar results [DR-Snadden-Wood '17, Wood '18, Creutzig-Kanade-Liu-DR '18]. I expect that this will generalise "nicely" to higher-rank $\mathfrak{g}$. Let's check the "simplest" case..

A case study: $\mathfrak{g} = \mathfrak{sl}_3$ and $k = -\frac{3}{2}$

We choose this higher-rank fractional-level WZW model because:

- k is admissible [Kac-Wakimoto '88] (highest-weight characters);
- it is "small" (relatively few modules);
- it belongs to the Deligne exceptional series (good for 4D $N = 2$ stuff);
- its minimal and principal W-algebras are $\mathbb{C}$ and 0 (relaxed characters);
- it's related to an $N = 4$ CFT and higher-rank triplets [Adamović '16].

Also, [Perše '07] has already computed the "Zhu ideal" and determined the irreducible highest-weight modules as $\widehat{\mathfrak{sl}}_3$-modules.

Corollary: The ground states of every irreducible weight module form an irreducible representation of $\mathfrak{sl}_3$ whose multiplicities are all 1.

This multiplicity-1 result fails for all other levels (except $k = -1, 0, 1$).

How to relax

An irreducible relaxed highest-weight $\widehat{\mathfrak{g}}$-module is a weight module whose conformal dimensions are bounded below (so it has ground states).

They are characterised by their ground states which form an irreducible weight module of $\mathfrak{g}$ that may be finite- or infinite-dimensional.

For $\mathfrak{sl}_3$, there are four types of irreducible weight modules:

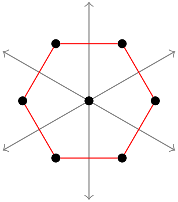
1. finite-dimensional modules, highest-weight wrt any Borel;
2. infinite-dimensional highest-weight modules wrt some Borel;
3. dense modules;
4. parabolic inductions of dense $\mathfrak{sl}_2$-modules.

A dense module is a weight module whose set of weights is a translate of the root lattice.

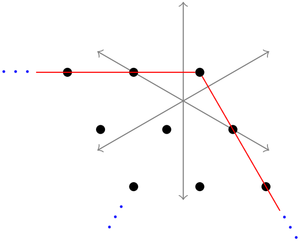
Parabolic induction means take a dense $\mathfrak{sl}_2$-module and induce it along a regular embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_3$ (sending root vectors to root vectors).

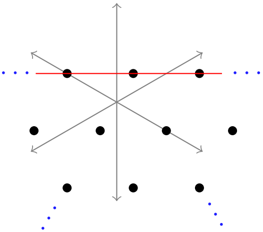
Irreducible weight $\mathfrak{sl}_3$-modules

finite-dimensional

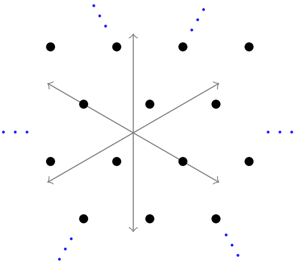


infinite-dimensional highest-weight





parabolically induced



dense

Classification for $\mathfrak{g} = \mathfrak{sl}_3$ and $k = -\frac{3}{2}$

Theorem [Kawasetsu-DR-Wood '19]

Up to twists by $\mathsf{D}_6 = \mathsf{S}_3 \times \mathbb{Z}_2$, the fractional-level WZW model has irreducible relaxed highest-weight $\widehat{\mathfrak{sl}}_3$-modules whose ground states are:

1. finite-dimensional — only the trivial one-dimensional $\mathfrak{sl}_3$-module;
2. infinite-dimensional highest-weight — one of three irreducible highest-weight $\mathfrak{sl}_3$-modules, namely those whose highest weights are $-\frac{3}{2}\omega_1$, $-\frac{3}{2}\omega_2$ and $-\frac{1}{2}\rho$ [Perše '07].
3. parabolic inductions — a one-parameter family of irreducible $\mathfrak{sl}_3$-modules, namely those whose "edge weights" λ satisfy $\langle\lambda|\omega_2\rangle = -\frac{1}{2}$ and whose quadratic Casimir eigenvalue is $-\frac{3}{2}$.
4. dense — a two-parameter family of irreducible $\mathfrak{sl}_3$-modules, namely those whose quadratic Casimir eigenvalue is $-\frac{3}{2}$ and whose cubic Casimir eigenvalue is 0.

Notes:

i. The vacuum module of the CFT is the unique "type-1" $\widehat{\mathfrak{sl}}_3$-module, *ie.* with finite-dimensional ground states.

ii. The type-1 modules may be obtained from the type-2 modules by twisting by spectral flow automorphisms.

iii. The type-3 modules are reducible for a codimension-1 set of parameters and the composition factors are type-2 modules.

iv. Every type-2 module is a composition factor of some reducible type-3 module.

v. The type-4 modules are reducible for a codimension-1 set of parameters and the composition factors are type-3 modules. For a codimension-2 set, these reduce to type-2 modules.

vi. Every type-3 module is a composition factor of some reducible type-4 module.

ii, iii, iv and v hold for general fractional levels; i and vi do not.

Characters

[Kac-Wakimoto '88] gave character formulae for the highest-weight modules. However, these are linearly dependent when spectral flow is used, so give incorrect modularity results [DR '08].

The methods of [Kawasetsu-DR '18] give the type-4 relaxed characters:

$$\mathrm{ch}_\lambda = \frac{1}{\eta(q)^4} \sum_{\mu \in \lambda + \mathsf{Q}} z^\mu .$$

The codimension-1 and -2 degenerations then imply resolutions for type-3 and type-2 modules, hence alternating sum formulae for their characters.

Spectral flow gives the type-1 (vacuum) character.

For general k, expect that 1 is replaced by a minimal W-character. May also need principal W-characters for the additional type-3 characters.

Modularity

For modular transforms, we use the type-4 modules (the standard modules [Creutzig-DR '13]) and their spectral flows.

Spectral flow automorphisms are parametrised by the coweight lattice $\mathsf{P}^\vee$ of $\mathfrak{g}$. For $\mathfrak{g}$ simply-laced (*eg.* $\mathfrak{sl}_3$), we have $\mathsf{P}^\vee = \mathsf{P}$.

The S-transform of the spectrally flowed type-4 characters is

$$\mathrm{ch}_{(\lambda,\omega)}\left(\tfrac{\zeta}{\tau}\middle|-\tfrac{1}{\tau}\right) = \sum_{\omega' \in \mathsf{P}^\vee} \int_{\mathfrak{h}^*/\mathsf{Q}} S_{(\lambda,\omega)(\lambda',\omega')} \mathrm{ch}_{(\lambda',\omega')}(\zeta|\tau)\, \mathrm{d}\lambda',$$

$$S_{(\lambda,\omega)(\lambda',\omega')} = \exp\bigl(-2\pi\mathrm{i}\bigl[\langle\lambda|\omega'\rangle + \langle\lambda'|\omega\rangle + k\langle\omega|\omega'\rangle\bigr]\bigr).$$

For general k, expect a factor corresponding to S-matrices of minimal and principal W-characters.

Resolutions give the S-transforms of the characters of the remaining modules (the atypicals) including the vacuum module.

Grothendieck fusion

Fractional-level WZW models are logarithmic, but characters cannot see reducible but indecomposable structure.

The standard Verlinde formula [Creutzig-DR '13] thus computes the Grothendieck fusion rules, not the actual fusion rules:

$$\mathrm{ch}\big[\mathsf{M}\big] \times \mathrm{ch}\big[\mathsf{N}\big] = \sum_{\omega \in \mathsf{P}^\vee} \int_{\mathfrak{h}^*/\mathsf{Q}} \mathcal{N}_{\mathsf{M}\mathsf{N}}{}^{(\lambda,\omega)} \mathrm{ch}_{(\lambda,\omega)} \, \mathrm{d}\lambda,$$

$$\mathcal{N}_{\mathsf{M}\mathsf{N}}{}^{(\lambda,\omega)} = \sum_{\omega' \in \mathsf{P}^\vee} \int_{\mathfrak{h}^*/\mathsf{Q}} \frac{S_{\mathsf{M}(\lambda',\omega')} S_{\mathsf{N}(\lambda',\omega')} S^*_{(\lambda,\omega)(\lambda',\omega')}}{S_{\mathsf{Vac}(\lambda',\omega')}} \, \mathrm{d}\lambda'.$$

Here, M and N are arbitrary but the third module must be standard (type-4).

The standard by standard Grothendieck fusion product gives a sum of eight standard modules.

Standard fusion rules [Kawasetsu-DR-Wood '19]

$$\mathrm{ch}_{(\lambda,\omega)} \times \mathrm{ch}_{(\lambda',\omega')} = \mathrm{ch}_{(\lambda+\lambda'+\frac{3}{2}\omega_2,\omega+\omega'+\omega_2)} + \mathrm{ch}_{(\lambda+\lambda'+\frac{1}{2}(\omega_1+\omega_2),\omega+\omega'+\frac{1}{2}(\omega_1-\omega_2))} + \mathrm{ch}_{(\lambda+\lambda'-\frac{3}{2}\omega_1,\omega+\omega'-\omega_1)} + 2\,\mathrm{ch}_{(\lambda+\lambda',\omega+\omega')} + \mathrm{ch}_{(\lambda+\lambda'+\frac{3}{2}\omega_1,\omega+\omega'+\omega_1)} + \mathrm{ch}_{(\lambda+\lambda'-\frac{1}{2}(\omega_1+\omega_2),\omega+\omega'-\frac{1}{2}(\omega_1-\omega_2))} + \mathrm{ch}_{(\lambda+\lambda'-\frac{3}{2}\omega_2,\omega+\omega'-\omega_2)}$$

This "adjoint-module" structure to the standard fusion is also observed for $\mathfrak{g} = \mathfrak{sl}_2$ and $k = -\frac{4}{3}$. (This is in the Deligne exceptional series too...)

The other Grothendieck fusion rules are also easily computed.

Outlook

I would like to generalise this methodology to all fractional-level models. But, explicit computation of the "Zhu ideal" (à la Perše) is not feasible.

However, geometric methods have been used to classify the highest-weight modules for all admissible-level models [Arakawa '12].

Combining this with localisation [Mathieu '00] allows us to (algorithmically) classify all relaxed modules in principle [Kawasetsu-DR '19].

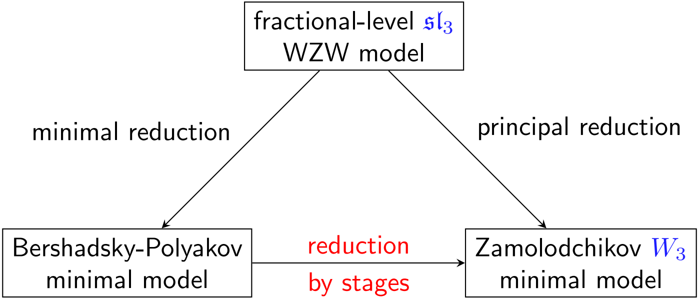
Fleshing out this strategy for $\mathfrak{g} = \mathfrak{sl}_3$ suggests a beautiful link between geometry (nilpotent orbits) and weight modules of all types.

Nilpotent orbit	hw. $\mathfrak{sl}_3$-mods	$\mathfrak{sl}_3$-families	$\widehat{\mathfrak{sl}}_3$-mods
zero: $\begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}$	finite-dim.	finite-dim.	ordinary hw.
minimal: $\begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix}$	bounded	dense	relaxed hw.
principal: $\begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$	unbounded	parabolic	"semi-relaxed" hw.

A complete picture for admissible-level $\mathfrak{sl}_3$ WZW models thus seems within reach.

An understanding of the characters and their modularity will require an understanding of those of the minimal and principal W-algebras, which haven't been worked out yet [Adamović-Fehily-Kawasetsu-DR '19].

The right setting for understanding these relationships is an affine version of "hamiltonian reduction by stages" [Marsden *et al.* '07]:



Conclusions

- Fractional-level WZW models are promising candidates for tractable examples of higher-rank logarithmic CFTs.
- We can use Arakawa's geometric highest-weight classification to deduce the classification of irreducible relaxed highest-weight modules.
- Characters can be computed using Mathieu's localisation construction or Semikhatov's inverse hamiltonian reduction [Semikhatov '94].
- Modular transforms and Grothendieck fusion rules may then be computed (and checked for consistency).
- Results so far suggest that the world of relaxed modules is the right setting to understand Arakawa's link to geometry (nilpotent orbits).
- It also may be the right setting to understand W-algebras and quantum hamiltonian reduction.

"*Only those who attempt the absurd will achieve the impossible.*"

— M C Escher