Representations of affine vertex algebras

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Outline

Motivation

"... it is a highly nontrivial problem to construct essentially any example of a vertex operator algebra."

"A significant feature of the theory is that the construction of modules for a vertex operator algebra is more subtle than the construction of the algebra itself."

— Jim and Haisheng.

Affine VOAs arguably form the most important class of all.

The rational ones (WZW models) are widely regarded as fundamental (and beautiful) building blocks on which much of our understanding rests.

What about non-rational affine VOAs? They should also be crucial in understanding non-rational (logarithmic) cases. And they should be beautiful. It's a pity we don't understand them at all...

Affine things

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra with fixed Cartan subalgebra $\mathfrak{h}$.

Let $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ be its affinisation.

Let $\mathbb{C}_k$ denote the 1-dimensional module of $\mathfrak{g}[t] \oplus \mathbb{C}K$ on which $\mathfrak{g}[t]$ acts as 0 and K acts as $\mathsf{k} \cdot \mathbf{1}$ for some $\mathsf{k} \in \mathbb{C}$.

Then, $\mathsf{V}_{\mathsf{k}}(\mathfrak{g}) = \mathrm{Ind}_{\mathfrak{g}[t] \oplus \mathbb{C}K}^{\widehat{\mathfrak{g}}} \mathbb{C}_{\mathsf{k}}$ carries the structure of a vertex algebra of level k, in fact a vertex operator algebra if $\mathsf{k} \neq -\mathsf{h}^{\vee}$.

Any quotient of a $\mathsf{V}_{\mathsf{k}}(\mathfrak{g})$ is called an affine vertex algebra.

The level-k simple quotient will be denoted by $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$.

Module categories

Conformal field theory seems to need characters because the partition function must be invariant under the standard action of $\mathsf{SL}_2(\mathbb{Z})$.

Consider, for some affine vertex algebra, the category $\mathscr{W}_{\mathsf{k}}$ of smooth weight modules (with finite-dimensional weight spaces).[1]

This is hard to analyse, but we can try to find consistent full subcategories instead.

For $\mathsf{k} \in \mathbb{N}$, the category $\mathscr{KL}_{\mathsf{k}}$ of ordinary $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$-modules is consistent:

- It is closed under twisting by the automorphisms $\sigma = \mathrm{Aut}(\mathfrak{g}) \ltimes \mathsf{P}^\vee$ of $\widehat{\mathfrak{g}}$ that preserve $\mathfrak{h}$.
- It is closed under the VOA tensor product (fusion).
- Its characters form a vector-valued modular form, so the usual sesquilinear combination (partition function) is modular-invariant.

[1] [To save space, let "weight" mean "weight with finite-dimensional weight spaces".]

Take $\mathsf{k} \notin \mathbb{N}$. Then, $\mathscr{KL}_{\mathsf{k}}$ is not closed under σ-twists and modularity also fails. It is, however, closed under fusion.

Try the BGG category $\mathscr{O}_{\mathsf{k}}$ whose simples are highest-weight $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$-modules. Not closed under σ-twists or fusion; modularity fails.

Try $\mathscr{O}_{\mathsf{k}}^{\sigma}$ whose simples are σ-twists of highest-weight $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$-modules. Not closed under fusion; modularity still fails.

Better try $\mathscr{R}_{\mathsf{k}}^{\sigma}$ whose simples are σ-twists of relaxed $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$-modules.

Conjecture: $\mathscr{R}_{\mathsf{k}}^{\sigma}$ is closed under fusion and the usual modular transformations preserve the vector space spanned by the characters.

Question: Is $\mathscr{R}_{\mathsf{k}}^{\sigma} = \mathscr{W}_{\mathsf{k}}$?

> "*It is known from the representation theory of affine Lie algebras and the Virasoro algebra that one needs to use all modules to obtain a modular invariant...*"
>
> — Yi-Zhi.

How to relax

Relaxation is good for the soul:

- The relaxed triangular decomposition of $\widehat{\mathfrak{g}}$ is given by

$$\widehat{\mathfrak{g}} = t\mathfrak{g}[t] \oplus (\mathfrak{g} \oplus \mathbb{C}K) \oplus t^{-1}\mathfrak{g}[t^{-1}].$$

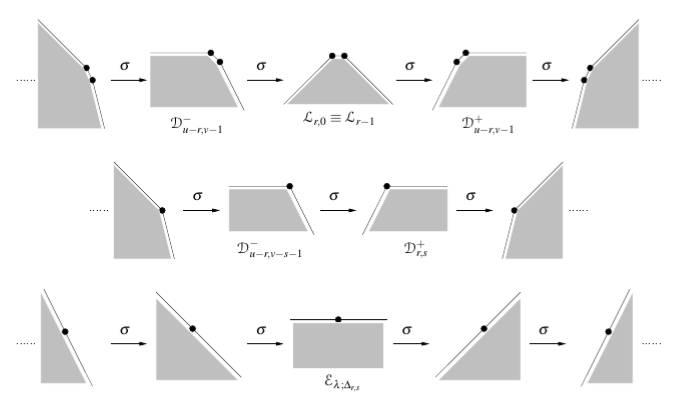
- A relaxed highest-weight vector is
  - an eigenvector of $\mathfrak{h}$;
  - a generalised eigenvector of L_0 (via Sugawara);
  - annihilated by $t\mathfrak{g}[t]$.
- A relaxed highest-weight module[2] is a module that is generated by a single relaxed highest-weight vector.

Relaxed modules differ from parabolic highest-weight modules in that their top spaces (Zhu images) need not be simple nor finite-dimensional.

[2] [We say "relaxed module" for short.]

Example: $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ with $\mathsf{k}+2=\frac{u}{v}$, $u,v\in\mathbb{Z}_{\geqslant 2}$ and $(u,v)=1$.

- $u-1$ ordinary modules;
- $(u-1)v$ highest-weight modules;
- $\frac{1}{2}(u-1)(v-1)$ 1-parameter families of (almost-always-simple) relaxed modules;
- spectral flows of the above, parametrised by $\mathsf{P}^\vee \cong \mathbb{Z}$.



Modularity

Intermediate between the simples and the (indecomposable) projectives are the standard modules.

For $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$, the standard modules are the relaxed modules with dense top spaces and their σ-twists.

Standard characters are quite interesting!

- $\operatorname{tr} z^{h_0} q^{L_0 - \mathsf{c}/24}$ diverges everywhere as a function, but converges as a formal power series (distribution) in z whose coefficients are analytic functions of q on the punctured disc $0 < |q| < 1$.
- Each family of standard modules is parametrised by a weight coset $\lambda \in \mathfrak{h}^*/\mathsf{Q}$ and the characters have the form

$$\operatorname{ch}\left[\mathsf{E}^{\omega}_{\lambda;\dots}\right](z;q) = z^{\mathsf{k}\omega} q^{\mathsf{k}\|\omega\|^2/2} \frac{\operatorname{ch}\left[\mathsf{W}\dots\right]}{\eta(q)^{\#}} \sum_{\mu \in \lambda} z^{\mu} q^{(\mu,\omega)}.$$

- They form a (topological) basis for the space of characters.

Modular transforms involve (all?) W-algebras:

$$\mathcal{S}\{\mathrm{ch}[\mathsf{M}]\} = \sum_{\mu;\cdots} \int_{\mathfrak{h}^*/\mathsf{Q}} \mathcal{S}[\mathsf{M} \to \mathsf{E}^{\xi}_{\mu;\dots}] \, \mathrm{ch}[\mathsf{E}^{\xi}_{\mu;\dots}] \, \mathrm{d}\mu,$$

$$\mathcal{S}[\mathsf{E}^{\omega}_{\lambda;\dots} \to \mathsf{E}^{\xi}_{\mu;\dots}] = \mathsf{e}^{-2\pi\mathfrak{i}[\mathsf{k}(\omega,\xi)+(\lambda,\xi)+(\mu,\omega)]} \mathcal{S}^{\mathsf{W}}_{\cdots,\cdots}.$$

Moreover, $\mathcal{S}$ is symmetric, unitary and squares to conjugation.

The standard Verlinde formula seems to give non-negative integers:

$$\mathcal{N}_{\mathsf{MN}}{}^{\mathsf{E}^{\chi}_{\nu;\cdots}} = \sum_{\psi,\cdots} \int_{\mathfrak{h}^*/\mathsf{Q}} \frac{\mathcal{S}\Big[\mathsf{M} \to \mathsf{E}^{\psi}_{\rho;\dots}\Big] \mathcal{S}\Big[\mathsf{N} \to \mathsf{E}^{\psi}_{\rho;\dots}\Big] \mathcal{S}\Big[\mathsf{E}^{\chi}_{\nu;\dots} \to \mathsf{E}^{\psi}_{\rho;\dots}\Big]^*}{\mathcal{S}\Big[\mathsf{L}_{\mathsf{k}}(\mathfrak{g}) \to \mathsf{E}^{\psi}_{\rho;\dots}\Big]} \, \mathrm{d}\rho.$$

These are (conjecturally) the Grothendieck fusion coefficients:

$$[\mathsf{M}] \times [\mathsf{N}] = \sum_{\chi,\cdots} \int_{\mathfrak{h}^*/\mathsf{Q}} \mathcal{N}_{\mathsf{MN}}{}^{\mathsf{E}^{\chi}_{\nu;\cdots}} [\mathsf{E}^{\chi}_{\nu;\dots}] \, \mathrm{d}\nu.$$

Projectives

For almost all parameter values $\lambda \in \mathfrak{h}^*/\mathrm{Q}$, the standard modules $\mathsf{E}^{\omega}_{\lambda;\dots}$ are simple. We expect that these simple standard modules are projective.

We say that simple standards are typical. The non-standard simple modules then arise as composition factors of the atypical standards.

The projective covers of the atypical simples are believed to have filtrations by standard (and costandard) modules and to satisfy a generalised BGG reciprocity principle:

$$\left[\mathsf{P}^{\omega}_{\lambda;\dots} : \mathsf{E}^{\xi}_{\mu;\dots}\right] = \left[\mathsf{E}^{\xi}_{\mu;\dots} : \mathsf{L}^{\omega}_{\lambda;\dots}\right].$$

The projective indecomposables are also expected to be the building blocks of the bulk state space (coend) of the conformal field theory:

$$\mathsf{H} \cong \bigoplus_{\omega,\cdots} \oint_{\mathsf{typ.}} \mathsf{E}^{\omega}_{\lambda;\dots} \otimes \mathsf{E}^{\omega}_{\lambda;\dots} \, \mathrm{d}\lambda \oplus \text{“} \bigoplus_{\substack{\mathsf{atyp.}\lambda \\ \omega,\cdots}} \mathsf{P}^{\omega}_{\lambda;\dots} \otimes \mathsf{L}^{\omega}_{\lambda;\dots} \text{”}.$$

Example: $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ with $\mathsf{k}+2=\frac{u}{v}$, $u,v\in\mathbb{Z}_{\geqslant 2}$ and $(u,v)=1$.

The standards come in families labelled by $r=1,\dots,u-1$ and $s=1,\dots,v-1$, with “Kac-symmetry” $(r,s)\leftrightarrow(u-r,v-s)$.

In the (r,s) family, the atypical standards break up into simple highest-weight modules $\mathsf{L}^{\omega}_{r,s}$ and $\mathsf{L}^{\xi}_{u-r,v-s}$.

The projective covers are conjectured to have Loewy diagrams

$$\begin{tikzcd} & \mathsf{L}^{\omega}_{r,s} \arrow[dl] \arrow[dr] & \\ \mathsf{L}^{\omega-\Lambda}_{r,s-1} \arrow[dr] & \mathsf{P}^{\omega}_{r,s} & \mathsf{L}^{\omega+\Lambda}_{r,s+1} \arrow[dl] \\ & \mathsf{L}^{\omega}_{r,s} & \end{tikzcd}.$$

[These diagrams will be much more complicated for $\mathfrak{g}\neq\mathfrak{sl}_2$ because there will be different “degrees” of atypicality.]

Adding rigour — where to start?

Much of what has been discussed is at the level of empirical observations. We aim to build and prove a (hopefully) widely applicable formalism.

However, we first have to figure out what to prove: need more examples.

In particular, we need a much better understanding of the possible types of relaxed modules.

Unfortunately, there is not much literature:

- The simple relaxed modules have only been classified for $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ [AM,RW], $\mathsf{L}_{\mathsf{k}}(\mathfrak{osp}(1|2))$ [W,CKLR] and $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_3)$ [AFR].
- The characters of these simples have only been computed for $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ and $\mathsf{L}_{\mathsf{k}}(\mathfrak{osp}(1|2))$ [A,KR].
- Only the fusion rules of the ordinary modules have been proven [CHY,C].
- In no case do we know that the atypical projectives are projective.
- Proving that $\mathscr{R}_{\mathsf{k}}^{\sigma}$ is a rigid braided tensor category is still a dream.

Classifications

We aim to classify the simple relaxed $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-modules. (But the method works for any affine vertex algebra!)

As usual, the tool to do so is Zhu's algebra, *eg.*

$$\mathsf{Zhu}[\mathsf{V}_\mathsf{k}(\mathfrak{g})] \cong \mathscr{U}(\mathfrak{g}), \quad \mathsf{Zhu}[\mathsf{L}_\mathsf{k}(\mathfrak{g})] \cong \frac{\mathscr{U}(\mathfrak{g})}{I_\mathsf{k}}.$$

For $\mathsf{V}_\mathsf{k}(\mathfrak{g})$, the simple Zhu-modules are then the simple weight $\mathfrak{g}$-modules.

For $\mathsf{L}_\mathsf{k}(\mathfrak{g})$, our job is then to determine which of these simple weight $\mathfrak{g}$-modules are annihilated by I_k.

But this presupposes that we have a classification of simple weight $\mathfrak{g}$-modules. Luckily, Mathieu completed this classification relatively recently, building on work of Fernando (and others).

Recall that a weight $\mathfrak{g}$-module is dense if its support is $\lambda + \mathrm{Q}$, for some $\lambda \in \mathfrak{h}^*$. For simple weight modules, dense = cuspidal = torsion-free.

Recall also that every parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ contains a Borel, hence a set of simple root vectors e^{α_i}.

The Levi factor of $\mathfrak{p}$ is the subalgebra $\mathfrak{l}$ generated by $\mathfrak{h}$ and the $e^{\pm\alpha_i}$ for which $e^{-\alpha_i} \in \mathfrak{p}$. Levi factors are always reductive.

Recall that parabolic induction means extending an $\mathfrak{l}$-module to a $\mathfrak{p}$-module, by letting all root vectors in $\mathfrak{p} \setminus \mathfrak{l}$ act as 0, then inducing to a $\mathfrak{g}$-module.

Theorem (Fernando '90).

- Every simple weight $\mathfrak{g}$-module is the simple quotient of a parabolic induction of a simple dense $\mathfrak{l}$-module.
- Simple dense $\mathfrak{l}$-modules exist only if all simple ideals of $\mathfrak{l}$ have AC-type.

We say that a $\mathfrak{g}$-module is bounded if it is infinite-dimensional and the dimensions of its weight spaces are uniformly bounded.

Theorem (Mathieu '00).

- Simple dense $\mathfrak{g}$-modules exist iff $\mathfrak{g}$ has AC-type.
- Every simple dense $\mathfrak{g}$-module is the direct summand of a unique irreducible semisimple coherent family of $\mathfrak{g}$-modules.
- A semisimple coherent family of $\mathfrak{g}$-modules always contains a bounded highest-weight submodule.
- Any such bounded highest-weight submodule completely characterises the coherent family (up to isomorphism).

Mathieu also explicitly determined the conditions for a highest-weight $\mathfrak{g}$-module to be bounded.

This completes the classification of simple weight $\mathfrak{g}$-modules. The classification of simple relaxed $\mathsf{V}_{\mathsf{k}}(\mathfrak{g})$-modules now follows.

However, we want to classify simple relaxed modules for $\mathsf{L}_k(\mathfrak{g})$ (or some other quotient).

Theorem (Kawasetsu-DR '19).

The classification of simple relaxed $\mathsf{L}_k(\mathfrak{g})$-modules may be obtained algorithmically from the classification of simple *highest-weight* $\mathsf{L}_k(\mathfrak{g})$-modules.

More precisely, a simple relaxed $\widehat{\mathfrak{g}}$-module, that is *not* highest-weight wrt any Borel subalgebra, is an $\mathsf{L}_k(\mathfrak{g})$-module iff its Zhu-image is a submodule of an irreducible semisimple parabolic family of $\mathfrak{g}$-modules that has a simple $\mathfrak{l}$-bounded highest-weight submodule whose Zhu-induction is an $\mathsf{L}_k(\mathfrak{g})$-module.

Here,

- a parabolic family of $\mathfrak{g}$-modules is the "almost-simple" quotient of the parabolic induction of some coherent family of $\mathfrak{l}$-modules;
- an $\mathfrak{l}$-bounded $\mathfrak{g}$-module is the "almost-simple" quotient of the parabolic induction of some bounded $\mathfrak{l}$-module.

Algorithm (Kawasetsu-DR '19).

Assume that we know the simple highest-weight $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$-modules.

- For each (non-empty) subset of the simple roots, check if the corresponding parabolic subalgebra $\mathfrak{p}$ has AC-type.
- If so, project the highest weight λ of the Zhu-image of each simple highest-weight $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$-module H along each simple ideal $\mathfrak{s}_i$ of $\mathfrak{l}$. Check if *all* projections correspond to bounded $\mathfrak{s}_i$-modules.
- If so, one has an irreducible semisimple coherent family of $\mathfrak{s}_i$-modules containing the bounded $\mathfrak{s}_i$-module, for all i. Tensoring them together, along with an appropriate $\mathfrak{z}(\mathfrak{l})$-module, gives an irreducible semisimple standard parabolic family whose Zhu-induction contains H.
- The direct summands of this induction are all $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$-modules.

Along with the simple highest-weight $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$-modules, the direct summands found with this algorithm form a complete set, up to isomorphism, of simple relaxed $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$-modules.

Example: Relaxed $\mathsf{L}_{-2}(\mathfrak{so}_8)$ modules.

$\mathsf{L}_{-2}(\mathfrak{so}_8)$ has one simple ordinary module L_0 and four non-ordinary highest-weight modules L_i, $i = 1, 2, 3, 4$. L_0 is invariant under W-twists while L_2 gives 24 twists and the others give 8 each.

The algorithm now gives

simple root subset	$\mathfrak{l}$	# parabolic families
$\{1\}, \{3\}, \{4\}$	$\mathfrak{sl}_2 \oplus \mathfrak{gl}_1^{\oplus 3}$	24 each
$\{2\}$	$\mathfrak{sl}_2 \oplus \mathfrak{gl}_1^{\oplus 3}$	72
$\{1,2\}, \{2,3\}, \{2,4\}$	$\mathfrak{sl}_3 \oplus \mathfrak{gl}_1^{\oplus 2}$	32 each
$\{1,3\}, \{1,4\}, \{3,4\}$	$\mathfrak{sl}_2^{\oplus 2} \oplus \mathfrak{gl}_1^{\oplus 2}$	no families
$\{1,3,4\}$	$\mathfrak{sl}_2^{\oplus 3} \oplus \mathfrak{gl}_1$	no families
$\{1,2,3\}, \{1,2,4\}, \{2,3,4\}$	$\mathfrak{sl}_4 \oplus \mathfrak{gl}_1$	8 each

and each of these families gives either a 1-, 2- or 3-parameter family of (almost-always-simple) relaxed $\mathsf{L}_{-2}(\mathfrak{so}_8)$-modules.

Outlook

- This result allows us to explore relaxed module theory in general. In particular, all admissible levels are now accessible (in principle) because of Arakawa's highest-weight classification.
- For example, we intend to study examples to isolate the role of W-algebras (and nilpotent orbits) in the characters, modularity and fusion rules of $\mathsf{L}_\mathsf{k}(\mathfrak{g})$.
- This will require the concurrent investigation of the relaxed modules of these W-algebras.
- We also have a theorem that concludes the existence of certain families of *non-semisimple* parabolic families of $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-modules which we expect to be crucial in studying atypical projectives.
- Combining this with methods of [S], [A], [AW], we hope to establish a vertex tensor category structure on $\mathscr{R}_\mathsf{k}^\sigma$.
- The future of affine vertex algebras is looking good...

"*Only those who attempt the absurd will achieve the impossible.*"

— M C Escher

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