Representations of affine vertex algebras: beyond category $\mathcal{O}$

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[Kawasetsu–DR, arXiv:1803.01989 and arXiv:1906.02935]

Outline

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Motivation

Vertex operator algebras (VOAs) are important in maths and physics.

Affine VOAs arguably form the most important class of all.

The rational ones (WZW models) are fundamental building blocks for rational examples on which much of our understanding rests.

Non-rational affine VOAs are crucial for understanding the non-rational (logarithmic) cases that arise in:

- statistical physics (percolation, polymers);
- string theory (sigma models with target superspaces);
- the 4D/2D correspondence (2D invariants of 4D gauge theories).

Mathematically, logarithmic VOAs give natural examples of mock-modular transformations and suggest non-semisimple generalisations of modular tensor categories [Moore–Seiberg, Turaev].

Affine VOAs

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra with fixed Cartan subalgebra $\mathfrak{h}$. Let $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ be its affinisation.

Let $\mathbb{C}_k$ denote the 1-dimensional module of $\mathfrak{g}[t] \oplus \mathbb{C}K$ on which $\mathfrak{g}[t]$ acts as 0 and K acts as $\mathsf{k} \cdot \mathbf{1}$ for some $\mathsf{k} \in \mathbb{C}$. Take $\mathsf{k} \neq -\mathsf{h}^\vee$.

Then, the $\widehat{\mathfrak{g}}$-module $\mathsf{V}_\mathsf{k}(\mathfrak{g}) = \mathrm{Ind}_{\mathfrak{g}[t]\oplus\mathbb{C}K}^{\widehat{\mathfrak{g}}} \mathbb{C}_\mathsf{k}$ carries a VOA structure. The modes of the generating fields, along with $\mathsf{k} \cdot \mathbf{1}$, span $\widehat{\mathfrak{g}}$.

Any quotient of a $\mathsf{V}_\mathsf{k}(\mathfrak{g})$ is called a level-k affine VOA [Frenkel–Zhu].

The level-k simple quotient will be denoted by $\mathsf{L}_\mathsf{k}(\mathfrak{g})$. This VOA is:

- rational (= semisimple rep theory), if $\mathsf{k} \in \mathbb{N}$;
- logarithmic (= non-semisimple rep theory), otherwise.

$\mathsf{V}_\mathsf{k}(\mathfrak{g})$ is always logarithmic.

Module categories

Physics needs partition functions (characters) so the natural module category is the weight category $\mathscr{W}_{\mathsf{k}}$ whose objects are (smooth) weight modules with finite-dimensional weight spaces.

Note that every module of a level-k affine VOA is a level-k $\widehat{\mathfrak{g}}$-module.

Here, a weight space is an eigenspace of $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K$ that is also a generalised eigenspace of the Virasoro zero mode L_0.

$\mathscr{W}_{\mathsf{k}}$ is hard to analyse in general...

For $\mathsf{k} \in \mathbb{N}$, $\mathscr{W}_{\mathsf{k}}$ reduces to the category of (finite direct sums of) integrable highest-weight modules. It is closed under:

- twisting by the automorphisms $\sigma = \mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g}) \ltimes \mathsf{P}^\vee$ of $\widehat{\mathfrak{g}}$ that preserve $\mathfrak{h}$.
- the VOA tensor product (fusion).

Moreover, $\mathscr{W}_{\mathsf{k}}$ is a modular tensor category [Huang].

Relaxed modules

For $\mathsf{k} \notin \mathbb{N}$. it follows from [Futorny-Tsylke] that $\mathscr{W}_{\mathsf{k}}$ reduces to the category $\mathscr{R}_{\mathsf{k}}^{\sigma}$ generated by σ-twists of relaxed highest-weight modules [DR–Wood].

- The relaxed triangular decomposition of $\widehat{\mathfrak{g}}$ is given by

$$\widehat{\mathfrak{g}} = t\mathfrak{g}[t] \oplus (\mathfrak{g} \oplus \mathbb{C}K) \oplus t^{-1}\mathfrak{g}[t^{-1}].$$

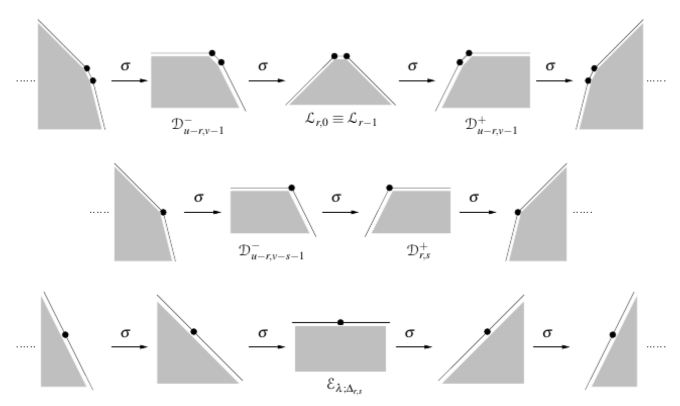
- A relaxed highest-weight vector is an eigenvector of $\widehat{\mathfrak{h}}$ and a generalised eigenvector of L_0 that is annihilated by $t\mathfrak{g}[t]$.
- A relaxed highest-weight module is a module that is generated by a single relaxed highest-weight vector.

Relaxed modules differ from parabolic highest-weight modules in that their “top spaces” need not be simple nor finite-dimensional.

Unlike category $\mathscr{O}_{\mathsf{k}}$ (highest-weight modules), $\mathscr{R}_{\mathsf{k}}^{\sigma}$ is believed to be closed under fusion and to have characters spanning a representation of $\mathsf{SL}_2(\mathbb{Z})$.

Example: $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ with $\mathsf{k}+2=\frac{u}{v}$, $u,v\in\mathbb{Z}_{\geqslant 2}$ and $(u,v)=1$.

- $u-1$ parabolic highest-weight modules;
- $(u-1)v$ highest-weight modules;
- $\frac{1}{2}(u-1)(v-1)$ 1-parameter families of (almost-always-simple) relaxed highest-weight modules;
- σ-twists of the above, parametrised by $\mathsf{P}^\vee\cong\mathbb{Z}$.



Classifying simple relaxed highest-weight modules

The simple objects of $\mathscr{R}_{\mathsf{k}}^{\sigma}$ are (σ-twists of) simple relaxed modules.

For $\mathsf{V}_{\mathsf{k}}(\mathfrak{g})$, the simple relaxed modules are in bijection [Zhu] with the simple weight $\mathfrak{g}$-modules with finite-dimensional weight spaces [Mathieu].

For $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$, the relaxed classification is presently unknown, although the highest-weight classification is known [Arakawa] when k is admissible.

Our new result is an algorithm for classifying the simple relaxed modules of $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$ (or any affine VOA), given the highest-weight classification.

Theorem (Kawasetsu-DR '19).

There is an explicit algorithm whose input is the classification of simple *highest-weight* $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$-modules and whose output is the classification of simple *relaxed highest-weight* $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$-modules.

In particular, the admissible-level relaxed classification is now in reach.

How does it work?

[Warning: white lie in effect!]

First, [Zhu] says that a simple relaxed $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-module may be reconstructed from its top space, *ie.* the subspace of minimal L_0-eigenvalue.

These top spaces are the simple $\mathfrak{g}$-modules that are annihilated by an ideal I_k of $\mathscr{U}(\mathfrak{g})$ [Frenkel–Zhu]. This ideal is difficult to compute.

Given the highest-weight classification, we know which simple highest-weight $\mathfrak{g}$-modules are annihilated by I_k.

Among these, some will be bounded: they are infinite-dimensional but the dimensions of their weight spaces are uniformly bounded.

Applying twisted localisation [Mathieu] to a simple bounded $\mathfrak{g}$-module gives (generically) simple dense $\mathfrak{g}$-modules (and irreducible coherent families).

Crucially, these dense modules are annihilated by I_k because the bounded module was [Kawasetsu–DR]. By [Zhu], these dense modules are the top spaces of the simple relaxed highest-weight $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-modules.

How does it really work?

[Note: the white lie has left the building!]

So this is morally how one classifies relaxed modules.

Alas, $\mathfrak{g}$ does not have simple dense modules unless $\mathfrak{g}$ is type A or C [Fernando]. And these only exhaust the simple relaxed modules for $\mathfrak{g} = \mathfrak{sl}_2$.

However, every simple weight $\mathfrak{g}$-module is a quotient of a parabolic induction of a simple dense $\mathfrak{l}$-module, where $\mathfrak{l}$ is the Levi factor of the chosen parabolic subalgebra of $\mathfrak{g}$ [Fernando].

We therefore have to refine our explanation to include restricting from $\mathfrak{g}$ to $\mathfrak{l}$ and then inducing from $\mathfrak{l}$ to $\mathfrak{g}$ (and taking simple quotients).

We introduce parabolic families of $\mathfrak{g}$-modules as "almost-simple" quotients of parabolic inductions of coherent families of $\mathfrak{l}$-modules.

These parabolic families allow us to complete the relaxed classification — for the gory details, see [Kawasetsu–DR, arXiv:1906.02935].

Algorithm (Kawasetsu-DR '19).

Assume that we know the simple highest-weight $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-modules.

- For each non-empty subset of the simple roots, check if the corresponding parabolic subalgebra $\mathfrak{p}$ has Levi factor $\mathfrak{l}$ with simple ideals $\mathfrak{s}_i$ of type A or C.
- If so, project the highest weight λ of the top space of each simple highest-weight $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-module H along each simple ideal $\mathfrak{s}_i$ of $\mathfrak{l}$. Check if *all* projections correspond to bounded $\mathfrak{s}_i$-modules.
- If so, twisted localisation gives an irreducible coherent family of $\mathfrak{s}_i$-modules containing the bounded $\mathfrak{s}_i$-module, for all i.
- Tensoring them together, along with a (uniquely determined) $\mathfrak{z}(\mathfrak{l})$-module, then inducing to $\mathfrak{g}$, gives an irreducible parabolic family whose Zhu-induction contains H.
- The direct summands of this induction are $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-modules and all simple relaxed highest-weight $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-modules (that are not already highest-weight) arise in this way, up to isomorphism.

Example: Relaxed $\mathsf{L}_{-2}(\mathfrak{so}_8)$-modules.

$\mathsf{L}_{-2}(\mathfrak{so}_8)$ has one simple ordinary module L_0 and four non-ordinary highest-weight modules L_i, $i = 1, 2, 3, 4$. L_0 is invariant under W-twists while L_2 gives 24 twists and the others give 8 each.

The algorithm now gives

simple root subset	$\mathfrak{l}$	# parabolic families
$\{1\}, \{3\}, \{4\}$	$\mathfrak{sl}_2 \oplus \mathfrak{gl}_1^{\oplus 3}$	24 each
$\{2\}$	$\mathfrak{sl}_2 \oplus \mathfrak{gl}_1^{\oplus 3}$	72
$\{1,2\}, \{2,3\}, \{2,4\}$	$\mathfrak{sl}_3 \oplus \mathfrak{gl}_1^{\oplus 2}$	32 each
$\{1,3\}, \{1,4\}, \{3,4\}$	$\mathfrak{sl}_2^{\oplus 2} \oplus \mathfrak{gl}_1^{\oplus 2}$	no families
$\{1,3,4\}$	$\mathfrak{sl}_2^{\oplus 3} \oplus \mathfrak{gl}_1$	no families
$\{1,2,3\}, \{1,2,4\}, \{2,3,4\}$	$\mathfrak{sl}_4 \oplus \mathfrak{gl}_1$	8 each

and each of these families gives either a 1-, 2- or 3-parameter family of (almost-always-simple) relaxed $\mathsf{L}_{-2}(\mathfrak{so}_8)$-modules.

Non-simple relaxed highest-weight modules

Motivated by ideas for constructing projective covers, we also want to study non-simple indecomposable relaxed $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-modules.

Say that a weight $\mathfrak{l}$-module is α-bijective, for some given root α, if the root vector $e^\alpha \in \mathfrak{l}$ acts bijectively.

Theorem (Kawasetsu-DR '19).

Let:

- $\mathfrak{l}$ be the Levi factor of some parabolic subalgebra of $\mathfrak{g}$ with simple ideals of type A or C;
- H be a simple bounded highest-weight $\mathfrak{l}$-module such that the simple quotient of its parabolic induction is the top space of an $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-module;
- $\mathsf{C} \supset \mathsf{H}$ be an irreducible α-bijective coherent family of $\mathfrak{l}$-modules.

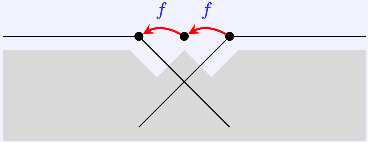
Then, every subquotient of the parabolic family P induced from C is the top space of an $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-module.

Example: Non-semisimple highest-weight $\mathsf{L}_{-2}(\mathfrak{so}_8)$-modules.

Recall that $\mathsf{L}_{-2}(\mathfrak{so}_8)$ has one simple ordinary module L_0 and four non-ordinary highest-weight modules L_i, $i = 1, 2, 3, 4$.

For the simple root subset $\{1\}$, $\mathfrak{l} \cong \mathfrak{sl}_2 \oplus \mathfrak{gl}_1^{\oplus 3}$ and the relevant coherent family of $\mathfrak{l}$-modules has direct summands with three composition factors.

The corresponding parabolic family of $\mathfrak{so}_8$-modules then does too:



The maximal proper submodule then Zhu-induces to a non-split extension of L_2 by L_1:

$$0 \longrightarrow \mathsf{L}_1 \longrightarrow \mathsf{M} \longrightarrow \mathsf{L}_2 \longrightarrow 0.$$

The previous theorem guarantees that M is an $\mathsf{L}_{-2}(\mathfrak{so}_8)$-module.

Outlook

- These results allow us to explore relaxed module theory in general. In particular, all admissible levels are now accessible (in principle).
- Character calculations indicate that relaxed $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$-modules are inextricably linked to highest-weight (and maybe relaxed) modules of the W-algebras obtained by quantum hamiltonian reduction.
- We therefore need to simultaneously investigate W-algebra representation theory [*cf.* Zac's talk].
- Non-semisimple relaxed modules (and their σ-twists) are expected to be crucial in studying projective covers.
- An ultimate goal would be to establish a vertex tensor category structure on $\mathscr{R}_{\mathsf{k}}^{\sigma}$.
- The future of affine vertex algebras is looking good...

"*Only those who attempt the absurd will achieve the impossible.*"

— M C Escher

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