A tale of monstrous moonshine

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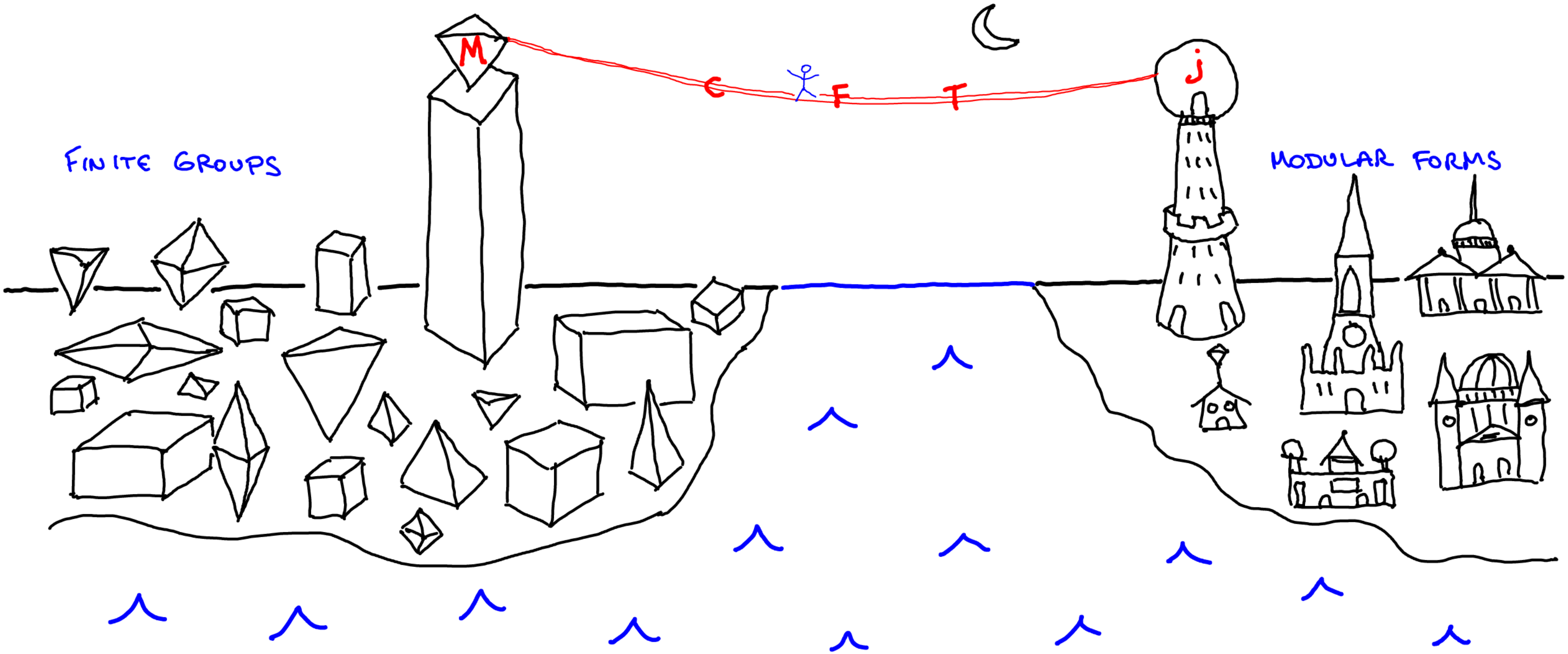


In 1978, John McKay noticed that $\boxed{196884 = 196883 + 1.}$

Together with Thompson, then Conway and Norton, this became the monstrous moonshine conjectures.

They connected the largest sporadic simple finite group (the monster) to certain modular forms (Hauptmoduls) arising in number theory.

These unexpected connections were proven by Richard Borcherds in the late 80's, which led to a Fields medal. His proof introduced vertex algebras, axiomatisations of the symmetries of mathematical physics models called conformal field theories.

Today, I'd like to outline the main players in this game:

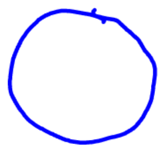
- Finite simple groups, eg. the monster $\mathbb{M}$.
- Modular forms, eg. the Hauptmodul j.
- Conformal field theory, the bridge between these concepts.

The aim is to explain why conformal field theory naturally produces modular forms. The link to the monster is somehow a special (and still poorly understood) example.

Finite simple groups

What's a group? A finite group?

What are they for? Symmetry



rotate θ
reflect ϕ

What's a simple finite group? atoms from which we build all finite groups

Do we know all the simple finite groups?

18 infinite families + 26 sporadic group

What is the monster? the biggest sporadic simple group

$$\left[\, |M| = 2^{46}\cdot 3^{20}\cdot 5^{9}\cdot 7^{6}\cdot 11^{2}\cdot 13^{3}\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 41\cdot 47\cdot 59\cdot 71 \approx 8\times 10^{53} \,\right]$$

In "the wild" (eg, in physics), groups are typically observed acting on something else (eg, a physical state space). This leads to representations.

What's a representation? replacement of gp elements by matrices

What's an irreducible representation? atoms from which we can build all representations...

What are the irreducible representations of the monster?

There are "only" 194 !

dimensions : 1, 196883, 21 296 876, 842 609 326, ...

Modular forms

What's a modular form / function? Modular form of wt k is a hol. function $f : \mathbb{H} \to \mathbb{C}$ $(\operatorname{Im} \tau > 0)$ satisfying

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \mu_{abcd} (c\tau+d)^k f(\tau), \quad \forall\, a,b,c,d \in \mathbb{Z} \text{ s.t. } ad-bc=1.$$

Modular function : meromorphic $f : \mathbb{H} \to \mathbb{C}$ with $\mu = 1$ and $k=0$.

Why would anyone care?

- Number theory — Sums of squares $\longrightarrow$ theta functions $\theta(x) = \sum_{n\in\mathbb{Z}} x^{n^2}$, $x = e^{\pi i \tau}$.
- Combinatorics — partition numbers $P(n)$ $\quad q^{-1/24} \sum_{n=0}^{\infty} P(n) q^n = \dfrac{1}{\prod_{i=1}^{\infty} (1-q^i) \cdot q^{1/24}}$ $(q = e^{2\pi i \tau})$
- Complex tori (elliptic curves) — parametrised by $\tau \in \mathbb{H}$ with two complex tori being the same iff their τ-parameters τ and τ' satisfy $\tau' = \dfrac{a\tau+b}{c\tau+d}$ for some $a,b,c,d \in \mathbb{Z}$ w/ $ad-bc=1$.
- Physics! (soon)

So what's this Hauptmodul j? Every modular function is $\frac{p(j(\tau))}{q(j(\tau))}$, p, q polys !!

Since $j(\tau) = j(\tau+1)$, j is 1-periodic $\longrightarrow$ Fourier series in $e^{2\pi i \tau}$:

$$j(\tau) = q^{-1} + \cancel{744} + 196884\, q + 21\,493\,760\, q^2 + 864\,299\,970\, q^3 + \cdots \qquad (q = e^{2\pi i \tau}).$$

So what's monstrous moonshine all about?

dimensions of irreducible Monster representations: 1, 196883, 21 296 876, 842 609 326, ...

$$\begin{aligned}
1 &= 1 \\
196\,884 &= 1 + 196\,883 \\
21\,493\,760 &= 1 + 196\,883 + 21\,296\,876 \\
864\,299\,970 &= 2\cdot 1 + 2\cdot 196\,883 + 21\,296\,876 + 842\,609\,326 \\
&\vdots
\end{aligned}$$

This instance of monstrous moonshine suggests that

- There is a somehow natural infinite-dimensional representation $V^\natural$ of the monster.
- This representation naturally splits into an infinite number of finite-dimensional subrepresentations.
- The dimensions of these subrepresentations are given by the coefficients of the q-expansion of $j(\tau) - 744$.

The key to finding $V^\natural$ lies in the modularity of j.

In the mid-80's, <u>physicists</u> had already been exploring models with infinite-dimensional representations and modular forms/functions!

Conformal field theory

What's a field theory? Classically, field = function on space/time.

Quantum : real (vector) - valued functions $\longrightarrow$ valued in operators on Hilbert space

Relativistic : Symmetries inc. Poincaré gp (translations + rotations / boosts)

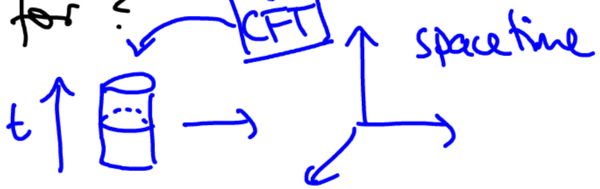
What's a conformal field theory (CFT)? ie length-invariant

Poincaré gp $\longrightarrow$ conformal gp (angle-preserving transformations)

ie Poincaré + rescalings (dilations) + ...

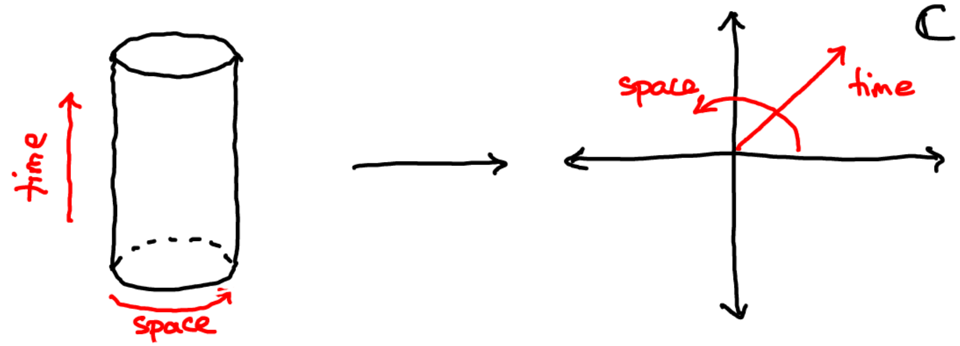
What are they good for? $\boxed{\text{2D CFT}}$

String theory —



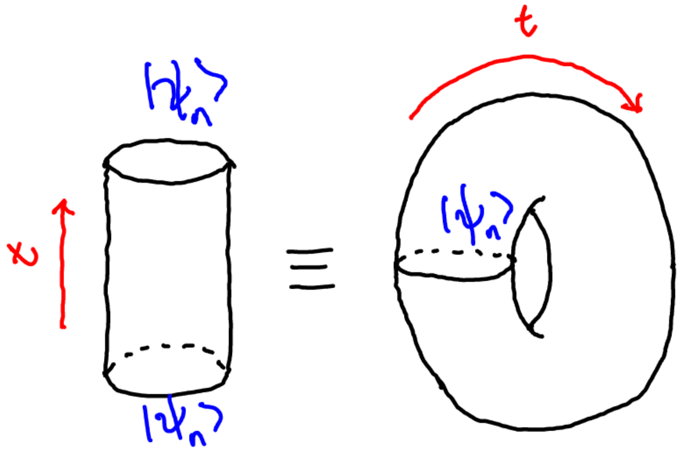
Phase transitions in statistical physics — critical pts have scale-inv. behaviour

Key fact: In 2 dimensions, the (infinitesimal) conformal transformations are the analytic/antianalytic transformations from complex analysis!



Today's mantra:

$\boxed{\text{The partition function } Z \text{ of a 2D CFT is a modular function!}}$



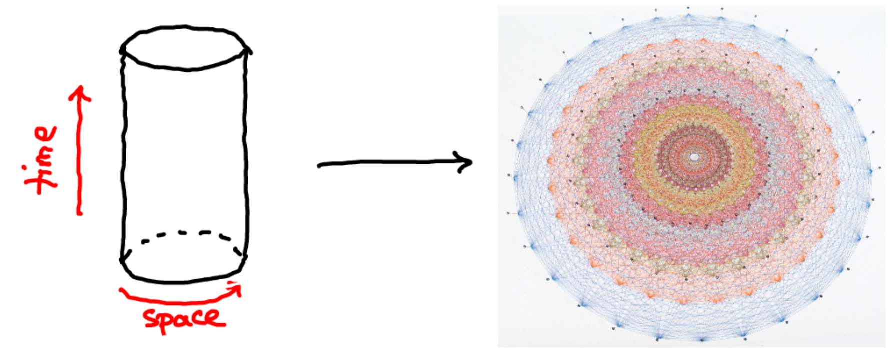
$$Z = \sum_n \underset{\substack{\uparrow \\ e^{-H/kT}}}{e^{-E_n/kT}} = \sum_n \underset{\substack{\uparrow \\ e^{-iHt/\hbar}}}{e^{-iE_n t/\hbar}} = \langle \psi_n | e^{-iHt/\hbar} | \psi_n \rangle$$

$\therefore$ Z is a function of τ, $e^{2\pi i \tau} = q = e^{-1/kT} = e^{-it/\hbar}$.

The Hilbert space of a 2D CFT is thus an excellent candidate for the natural representation $V^\natural$ of the monster $\mathbb{M}$.

One only needs to find the right CFT, one whose symmetries include $\mathbb{M}$ and whose partition function is $j(\tau)$.

Example [Witten '84]: Strings on the maximal torus of E_8. $Z(\tau)$ is



$$j(\tau)^{1/3} = q^{-1/3} + 248\,q^{2/3} + 4124\,q^{5/3} + 34752\,q^{8/3} + \cdots$$

$$1 = 1 \qquad 248 = 248$$

$$4124 = 1 + 248 + 3735$$

$$34752 = 1 + 2\cdot 248 + 3735 + 30380$$

$$\vdots$$

Actually, physicists were already very comfortable with the type of CFT needed, though not with the specific example.

In 1986, Frenkel, Lepowsky and Meurman nailed it.

$V^\natural$ is the Hilbert space of the 2D CFT describing string theory on (the $\mathbb{Z}_2$-orbifold of) a 24D torus built from the Leech lattice.

$\vdots \qquad \vdots \qquad \vdots$

$1 + 196883 + 21296876 \qquad E=2$

$1 + 196883 \qquad E=1$

$0 \qquad E=0$

$1 \qquad E=-1$

- $Z(\tau) = q^{-1} + 0 + 196884q + 21493760q^2 + \cdots = j(\tau)$.
- M acts on each energy level separately!
- In '92, Borcherds used $V^\natural$ (& "no ghost" theorem) to prove monstrous moonshine.

Outlook

- 2D CFTs continue to push the boundaries of modular forms research, revitalising many aspects (eg. Ramanujan's "mock" modular forms).
- Monstrous moonshine isn't over! There are still many unsolved problems, eg. is the monster CFT unique?
- Moonshine also works for some other sporadic groups, eg.
  - The Mathieu group M_{24} [Dong-Mason '94], the baby monster $\mathbb{B}$ [Höhn '02], the Conway group Co_1 [Duncan '05], etc...
  - Mathieu moonshine also arises in CFT on K3-surfaces [Eguchi-Ooguri-Tachikawa '10], see also umbral moonshine [Cheng-Duncan-Harvey '12].

Want further info?

- www.markronan.com/mathematics/symmetry-corner/
- Terry Gannon, Moonshine beyond the monster, Cambridge monographs on mathematical physics.
- Igor Frenkel, James Lepowsky and Arne Meurman, Vertex operator algebras and the monster, Academic press inc.
- Miranda Cheng, Mock modular forms and applications, MATRIX seminar Sept. 3, 2020 (google "matrix miranda cheng").