The spectral theorem

(and what it's good for!)

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highlights of mathematical physics

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1 Introduction

Systems of linear 1^{st}-order ODEs

$$\underset{\sim}{x}'(t) = A\,\underset{\sim}{x}(t)$$

$$\Rightarrow \quad \underset{\sim}{x}(t) = e^{At}\,x(0).$$

Matrix exponential: $e^{A} = \sum_{j=0}^{\infty} \frac{A^j}{j!}$ (always converges)

Finding eigenvalues: $(A - \lambda I)^{-1} = -\frac{1}{\lambda}\frac{1}{I - \lambda^{-1}A} = -\sum_{j=0}^{\infty} \lambda^{-j-1} A^j$

(converges if $\|A\| < \lambda$)

(operator norm)

Weaknesses

- What if we want $|\lambda| < \|A\|$?
- What if we want a non-analytic function of A ?
- What if $\|A\| = \infty$? Quantum mechanics example:

$$\hat{p} = \frac{\hbar}{i}\frac{d}{dx} \quad \text{on} \quad L^2(\mathbb{R}) \quad \text{has} \quad \|p\| = \infty .$$

Translation operator $\quad T_{x'}\psi(x) = \psi(x - x'), \quad T_{x'} = e^{-i\hat{p}x'/\hbar}.$

Evolution Operator $\quad T_{t'}\psi(x,t) = \psi(x, t-t') \quad T_{t'} = e^{-i\hat{H}t'/\hbar}.$

② In finite dimensions, diagonalise!

Assume A self-adjoint : $A^\dagger = A$ so evals are real & evects are orthogonal.

Spectral thm (self-adj./ finite-dims): A has a basis of eigenvectors. Equiv. A is diagonalisable.

$$A = SDS^{-1} \qquad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix} \qquad S^\dagger = S^{-1}$$

$$A^n = SD^nS^{-1} \qquad D = \begin{pmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_d^n \end{pmatrix} \qquad \underline{\underline{f(A)}} = Sf(D)S^{-1} \quad f(D) = \begin{pmatrix} f(\lambda_1) & & 0 \\ & \vdots & \\ 0 & & f(\lambda_d) \end{pmatrix}.$$

any f !!

(3) Does this work in infinite dimensions? Can we diagonalise?

A matrix representative of a linear operator is diagonalisable iff a basis of eigenvectors. But, $\hat{x}$ on $L^2(\mathbb{R})$ is self-adjoint but has no eigenvalues $\longrightarrow$ can't diagonalise.

$e_n(x) = A_n H_n(x) e^{-x^2/2}$ orthonormal basis

$$\hat{x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & \sqrt{2} & & \\ & \sqrt{2} & 0 & \sqrt{3} & \\ & & \sqrt{3} & 0 & \\ & & & & \ddots \end{pmatrix}.$$

4. Back to the (finite-dimensional) drawing board (A self-adjoint)

$$A = SDS^{-1} \qquad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix} = \lambda_1 \underbrace{\begin{pmatrix} 1 & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}}_{E_1} + \cdots + \lambda_d \underbrace{\begin{pmatrix} 0 & & & 0 \\ & \ddots & & \\ & & 0 & \\ 0 & & & 1 \end{pmatrix}}_{E_d}$$

<u>Note</u>: $E_i^2 = E_i$, $E_i^\dagger = E_i$,

$E_i E_j = E_j E_i = 0 \quad \forall\, i \neq j$.

ie the E_i are orthogonal projections.

Set $P_i = S E_i S^{-1} \implies P_i^2 = P_i$, $P_i^\dagger = P_i$, $P_i P_j = P_j P_i = 0$. ($i \neq j$)

$\implies P_i$ <u>also</u> orthogonal projections. They project onto the eigenspaces!

$$D = \sum_{i=1}^{d} \lambda_i E_i \quad \Rightarrow \quad \underline{A = \sum_{i=1}^{d} \lambda_i P_i} .$$

definition!

Just as good for making functions: $f(A) = \sum_{i=1}^{d} f(\lambda_i) P_i$.

This is the diagonalisation-free version of the spectral thm (for self-adj. / finite dims).

<u>Note</u>: $$I = \sum_{i=1}^{d} P_i$$ resolution of identity (completeness of basis).

(5) A conundrum with the continuum

Self-adjoint operators have pure-point (eigenvalues) and continuous (not eigenvalues) spectra, a subset of $\mathbb{R}$.

- $\hat{x}$, $\hat{p}$ has pp spec $\emptyset$ and cont. spec. $\mathbb{R}$.
- Harmonic oscillator $N = \frac{1}{2}\left(\hat{x}^2 - \frac{d^2}{dx^2} - I\right)$ has pp spec $\mathbb{Z}_{\geq 0}$ and cont. spec. $\emptyset$.
- Hydrogen atom hamiltonian

No cont. spec. $\Rightarrow$ basis of eigenvects $\Rightarrow$ diagonalisation.

How to cope with a cont. spectrum? $\sum \longmapsto \int$

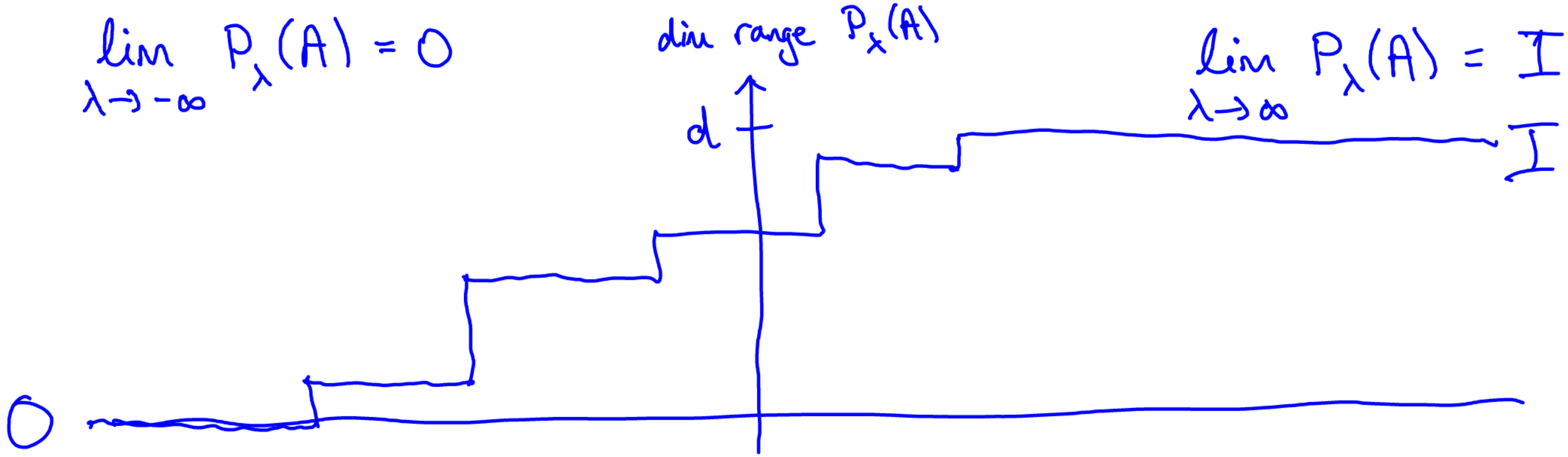
Trick: add up the projections P_i. Let

Heaviside

$$P_\lambda(x) = H(\lambda - x) = \begin{cases} 1 & \lambda > x \\ 0 & \lambda < x \end{cases} \Rightarrow P_\lambda(A) = \sum_{i=1}^{d} P_\lambda(\lambda_i) P_i = \sum_{\lambda_i < \lambda} P_i$$

$P_\lambda(A)$ is also a projection and it "increases" with λ:

$$\lim_{\lambda \to -\infty} P_\lambda(A) = 0 \qquad \lim_{\lambda \to \infty} P_\lambda(A) = I$$



Recall $\frac{\partial}{\partial \lambda} H(\lambda - x) = \delta(\lambda - x)$.

$$\therefore \frac{\partial}{\partial \lambda} P_\lambda(A) = \sum_{i=1}^{d} \frac{\partial}{\partial \lambda} H(\lambda - \lambda_i) P_i = \sum_{i=1}^{d} \delta(\lambda - \lambda_i) P_i$$

$$\Rightarrow \quad A = \sum_{i=1}^{d} \lambda_i P_i = \int_{\mathbb{R}} \lambda \sum_{i=1}^{d} \delta(\lambda - \lambda_i) P_i \, d\lambda = \int_{\mathbb{R}} \lambda \frac{\partial}{\partial \lambda} P_\lambda(A) \, d\lambda$$

$$\boxed{A = \int_{\mathbb{R}} \lambda \, dP_\lambda(A).} \quad \text{(measure)}$$

This is the formula that generalises to the continuous spectrum!

<u>Spectral thm for self-adj. A</u>: $\exists$ a weakly increasing family $\{P_\lambda(A)\}_{\lambda \in \mathbb{R}}$ of projections with $\lim_{\lambda \to -\infty} P_\lambda(A) = 0$, $\lim_{\lambda \to \infty} P_\lambda(A) = I$ and $A = \int_{\mathbb{R}} \lambda \, dP_\lambda(A)$.

⑥ <u>Example</u>

$$\hat{x} = \int_{\mathbb{R}} \lambda \, dH(\lambda - \hat{x}) \Rightarrow f(\hat{x}) = \int_{\mathbb{R}} f(\lambda) \, dH(\lambda - \hat{x})$$

$$\Rightarrow \psi(x) = I\psi(x) = \int_{\mathbb{R}} dH(\lambda - \hat{x})\psi(x) = \int_{\mathbb{R}} dH(\lambda - x)\psi(x)$$

$$= \int_{\mathbb{R}} \frac{d}{d\lambda} H(\lambda - x)\, \psi(x)\, d\lambda = \int_{\mathbb{R}} \delta(\lambda - x)\, \psi(x)\, d\lambda$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \delta(y-x)^* \psi(y)\, dy \;\; \delta(\lambda - x)\, d\lambda \longleftarrow$$

not eigenfunctions

$$\text{"="} \int_{\mathbb{R}} \langle \delta_x, \psi \rangle \; \delta_x \; d\lambda \longrightarrow \underbrace{\int |\delta_x(\lambda)\rangle\langle\delta_x(\lambda)| \, d\lambda}_{\text{resolution of identity}} \, |\psi\rangle$$

$$\boxed{P_\mu(x) \sim \int_{-\infty}^{\mu} |\delta_x(\lambda)\rangle\langle\delta_x(\lambda)| \, d\lambda}$$