Inverse quantum hamiltonian reduction

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[the art of mathematical physics]

Outline

1. Motivation
2. Quantum hamiltonian reduction
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4. Conclusions
5. Outlook

Motivation

I want to understand conformal field theory...

rational — factorisable — non-rational

[lattice bosons] [free fermions] [bc-ghosts] [minimal models] [compact WZW models] [a few W-algebras]	[free bosons] [Liouville] [non-compact WZW?]

discrete - - - - - continuous

[symplectic fermions] [triplet models] [polymers, percolation?] [log minimal models?] [SLE?]	[$\beta\gamma$-ghosts] [supergroup WZW] [Nappi–Witten] [fractional-level WZW] [most W-algebras] [spin chains?]

log-rational — logarithmic — generic

Quantum hamiltonian reduction

There are many ways to construct new chiral algebras from old ones:

- Tensoring, *eg.* two free fermions = one compactified boson.
- Simple current extensions, *eg.* Ising $\rightarrow$ free fermion.
- Group orbifolds, *eg.* free fermion $\rightarrow$ Ising.
- Cosets (commutants), *eg.* $\mathbb{Z}_k$-parafermions $= \dfrac{\widehat{\mathfrak{sl}}(2)_k}{\widehat{\mathfrak{h}}}$.
- Quantum hamiltonian reduction, *eg.* $\widehat{\mathfrak{sl}}(2)_k \mapsto \mathfrak{Vir}_k$.

In conformal field theory, it's important to also be able to construct representations of the new chiral algebra from those of the old!

Sometimes this is easy, sometimes it is hard...

How to do it

Quantum hamiltonian reduction converts an affine chiral algebra $\widehat{\mathfrak{g}}_k$ into a W-algebra $\mathfrak{W}_k(\mathfrak{g})$ by gauging the action of the positive root fields.

- First, tensor (the vacuum module of) $\widehat{\mathfrak{g}}_k$ with pairs of bc-ghosts, one for each positive root of $\mathfrak{g}$.
- Construct a fermionic field with conformal dimension 1 and ghost number 1:

$$d(z) = \sum_{\alpha} [e^{\alpha}(z) - \delta_{\alpha,\text{simple}}] c^{\alpha}(z) + [\text{cubic term in } b^{\alpha},\ c^{\alpha}].$$

- Its zero mode d_0 is a differential and the subspaces $C^{(n)}$ of $\widehat{\mathfrak{g}}_k \otimes (bc)^{\#}$ with constant ghost number n define a differential complex:

$$\cdots \xrightarrow{d_0} C^{(-2)} \xrightarrow{d_0} C^{(-1)} \xrightarrow{d_0} C^{(0)} \xrightarrow{d_0} C^{(1)} \xrightarrow{d_0} C^{(2)} \xrightarrow{d_0} \cdots$$

- The cohomology $H_k^{(n)}$ of this complex is 0 for all $n \neq 0$.
- The regular/principal W-algebra $\mathfrak{W}_k(\mathfrak{g})$ is $H_k^{(0)}$.

Generalisations

This generalises: given any nilpotent $f \in \mathfrak{g}$, there is a quantum hamiltonian reduction taking $\widehat{\mathfrak{g}}_k$ to a W-algebra $\mathfrak{W}_k^f(\mathfrak{g})$.

- Complete f to an $\mathfrak{sl}(2)$-triple $\{f, h, e\}$.
- Tensor $\widehat{\mathfrak{g}}_k$ with pairs of bc-ghosts, as before, but now also tensor with $\beta\gamma$-ghosts, one for each root with $\alpha(h) = 1$.
- Construct a fermionic field with conformal dimension 1 and (fermionic) ghost number 1:

$$d(z) = \sum_{\alpha} \big[e^{\alpha}(z) - \langle f | e^{\alpha} \rangle \big] c^{\alpha}(z) + \text{[terms in } b^{\alpha},\ c^{\alpha},\ \beta^{\alpha},\ \gamma^{\alpha}\text{]}.$$

- Its zero mode d_0 is a differential, the ghost-number subspaces of $\widehat{\mathfrak{g}}_k \otimes (bc)^{\#_1} \otimes (\beta\gamma)^{\#_2}$ define a differential complex, and the non-zero cohomology vanishes (at least conjecturally).
- The W-algebra $\mathfrak{W}_k^f(\mathfrak{g})$ associated to f is again $H_k^{(0)}$.

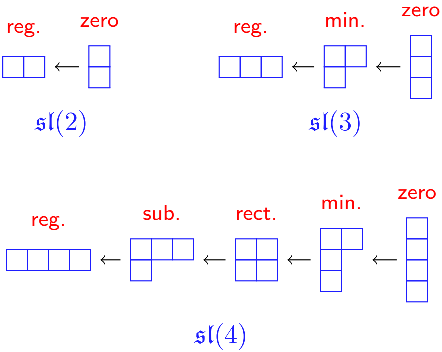
This also works for modules: replace $\widehat{\mathfrak{g}}_k$ by a $\widehat{\mathfrak{g}}_k$-module in the above and the cohomology $H_k^{(0)}$ is a $\mathfrak{W}_k^f(\mathfrak{g})$-module!

Examples

- Taking $f = 0$ results in $\mathfrak{W}^f_k(\mathfrak{g}) = \widehat{\mathfrak{g}}_k$, *ie.* reduction does nothing.
- Taking $f = \sum_{\alpha \text{ simple}} f^\alpha$ gives the regular W-algebra: $\mathfrak{W}^{\text{reg.}}_k(\mathfrak{g}) = \mathfrak{W}_k(\mathfrak{g})$.
- Taking $f = f^\theta$ gives the minimal W-algebra $\mathfrak{W}^{\text{min.}}_k(\mathfrak{g})$.
- $\mathfrak{W}^{\text{reg.}}_k(\mathfrak{sl}(2)) = \mathfrak{W}^{\text{min.}}_k(\mathfrak{sl}(2))$ is the Virasoro algebra $\mathfrak{Vir}_k$.
- $\mathfrak{W}^{\text{reg.}}_k(\mathfrak{sl}(3))$ is the Zamolodchikov algebra $\mathfrak{W}_{3,k}$.
- $\mathfrak{W}^{\text{min.}}_k(\mathfrak{sl}(3))$ is the Bershadsky–Polyakov algebra $\mathfrak{W}^{(2)}_{3,k}$.
- $\mathfrak{W}^{\text{reg.}}_k(\mathfrak{sl}(n))$ is a Casimir algebra of type $(2, 3, 4, \dots, n)$.
- $\mathfrak{W}^{\text{min.}}_k(\mathfrak{sl}(n))$ is a W-algebra of type $(1^{(n-2)^2}, (\frac{3}{2})^{2(n-2)}, 2)$.
- $\mathfrak{W}^{\text{min.}}_k(\mathfrak{osp}(1|2))$ is the $N = 1$ superconformal algebra $\mathfrak{N} = 1_k$.
- $\mathfrak{W}^{\text{min.}}_k(\mathfrak{sl}(2|1))$ is the $N = 2$ superconformal algebra $\mathfrak{N} = 2_k$.
- $\mathfrak{W}^{\text{min.}}_k(\mathfrak{osp}(3|2))$ is the (small) $N = 3$ superconformal algebra.
- $\mathfrak{W}^{\text{min.}}_k(\mathfrak{psl}(2|2))$ is the (small) $N = 4$ superconformal algebra.
- $\mathfrak{W}^{\text{min.}}_k(\mathfrak{d}(2|1;\alpha))$ is the (big) $N = 4$ superconformal algebra.

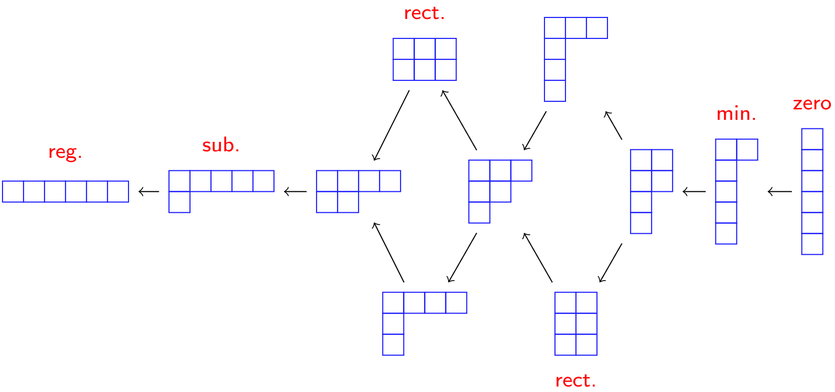
But wait, there's more!

In higher ranks, there's more than just regular and minimal W-algebras. For $\mathfrak{g} = \mathfrak{sl}(n)$, the possibilities are classified by partitions of n.



Sometimes these W-algebras are rational, but usually they're logarithmic.

So many W-algebras...



$\mathfrak{sl}(6)$

Inversion by example

For $\widehat{\mathfrak{sl}}(2)_k \mapsto \mathfrak{Vir}_k$, take k admissible but non-integral:

$$k + 2 = \frac{u}{v}, \quad u, v \in \mathbb{Z}_{\geqslant 2},\ \gcd\{u, v\} = 1.$$

Then, $\widehat{\mathfrak{sl}}(2)_k$ is logarithmic but $\mathfrak{Vir}_k$ is rational.

What can we learn about representations of $\widehat{\mathfrak{sl}}(2)_k$ from those of $\mathfrak{Vir}_k$?

$\mathfrak{Vir}_k$-mod	$\widehat{\mathfrak{sl}}(2)_k$-mod
[ordinary]	[ordinary]
	[highest-weight]
	[conjugate highest-weight]
	[relaxed highest-weight]
	[staggered]
	[spectral flows]
	[Whittaker]
	[others...]

Free-field realisations suggest a path:

- Feigin–Fuchs say $\mathfrak{Vir}^k \hookrightarrow \widehat{\mathfrak{h}}$. [Superscript k means "universal".]
- And Wakimoto says $\widehat{\mathfrak{sl}}(2)^k \hookrightarrow \widehat{\mathfrak{h}} \otimes \beta\gamma$.
- Now, Friedan–Martinec–Shenker bosonise the ghosts: $\beta\gamma \hookrightarrow \Pi$.
- But, Semikhatov notices that one can trade FF for FMS:

$$\widehat{\mathfrak{sl}}(2)^k \hookrightarrow \mathfrak{Vir}^k \otimes \Pi.$$

- Finally, Adamović proves that $\widehat{\mathfrak{sl}}(2)_k \hookrightarrow \mathfrak{Vir}_k \otimes \Pi$ iff $k \notin \mathbb{N}$.

Thus, every $M \in \mathfrak{Vir}_k$-mod and $N \in \Pi$-mod yield a representation

$$M \otimes N \in \widehat{\mathfrak{sl}}(2)_k\text{-mod},$$

by restriction (for $k \notin \mathbb{N}$).

What sort of representations can we get?

Life of Π

Take k admissible but non-integral, so $\mathfrak{Vir}_k$ only has ordinary representations $\widehat{\mathscr{L}_\lambda}$. Any extraordinary ones must then come from Π.

Π is a partial compactification of 2 free bosons of indefinite signature:

$$\Pi = \left\langle a(z), b(z), \mathsf{e}^{na(z)} \ : \ n \in \mathbb{Z} \right\rangle,$$

$$a(z)a(w) \sim b(z)b(w) \sim 0, \qquad a(z)b(w) \sim \frac{1}{(z-w)^2}.$$

To make the embedding $\widehat{\mathfrak{sl}}(2)_k \hookrightarrow \mathfrak{Vir}_k \otimes \Pi$ conformal, the dimension of $\mathsf{e}^{na(z)}$ must be linear in n:

$\ddots$

e^{-2a}

e^{-a}

$\ddots$

$a\mathsf{e}^{-a}, b\mathsf{e}^{-a}$

1

a, b

$\vdots$

e^{a}

$a\mathsf{e}^{a}, b\mathsf{e}^{a}$

e^{2a}

$\ddots$

$\ddots$

charge

dimension

Inverse quantum hamiltonian reduction

Π is thus the spectral flow of a relaxed highest-weight module! In fact, this is true for all the irreducibles $\Pi_\ell(\mu)$ ($\ell \in \mathbb{Z}$, $\mu \in \mathbb{C}/\mathbb{Z}$) of Π.

$\widehat{\mathscr{L}_\lambda} \otimes \Pi_\ell(\mu)$ is then a relaxed highest-weight $\widehat{\mathfrak{sl}}(2)_k$-module.

- Amazingly, it is generically irreducible. [Adamović]
  [Proof: compare character with that computed by Creutzig–DR / Kawasetsu–DR.]
- This explains why relaxed $\widehat{\mathfrak{sl}}(2)_k$ characters are $\propto$ to $\mathfrak{Vir}_k$ characters.
- Happily, this also gives all irreducible relaxed modules.
  [Proof: compare with classification of Adamović–Milas / DR–Wood.]

The functors

$$\mathfrak{Vir}_k\text{-mod} \to \widehat{\mathfrak{sl}}(2)_k\text{-mod}, \qquad \widehat{\mathscr{L}_\lambda} \mapsto \widehat{\mathscr{L}_\lambda} \otimes \Pi_\ell(\mu),$$

are what we call inverse quantum hamiltonian reduction (for $\mathfrak{sl}(2)$).

Relaxed highest-weight modules might sound exotic, but their spectral flows are the standard modules of $\widehat{\mathfrak{sl}}(2)_k$. [Creutzig–DR, DR–Wood]

History:	[Kac] Kac modules typical / atypical [Lie superalgebras]	⟶	[Saleur–Schomerus] Kac modules typical / atypical [supergroup WZW]	⟶	[Creutzig–DR] standard modules typical / atypical [log CFTs]	.

Being the standard modules means that:

- They are generically irreducible and projective.
- Every irreducible weight module can be obtained as a quotient.
  $\Rightarrow$ Irreducible weight modules can be resolved by standards.
- Their characters carry a representation of $\mathsf{SL}(2; \mathbb{Z})$.
  $\Rightarrow$ The Verlinde formula gives (Grothendieck) fusion coefficients.

Because inverse reduction constructs the standard modules, every irreducible highest-weight module is accessible via quotients/resolutions.

Beyond $\mathfrak{sl}(2)$

Other examples have been / are being worked out:

- The inverse reduction embedding for $\mathfrak{osp}(1|2)$ takes the form [Adamović]

$$\widehat{\mathfrak{osp}}(1|2)_k \hookrightarrow (\mathfrak{N} = \mathfrak{1})_k \otimes \mathfrak{F} \otimes \Pi^{1/2},$$

assuming that k is admissible but non-integral:

$$k + \frac{3}{2} = \frac{u}{2v}, \qquad u, v \in \mathbb{Z}_{\geqslant 2}, \ \frac{u-v}{2} \in \mathbb{Z}, \ \gcd\{\frac{u-v}{2}, v\} = 1.$$

The inverse reduction functors amount to tensoring an ordinary $\mathfrak{N} = \mathfrak{1}_k$-module with either $NS \otimes \Pi_\ell^{1/2}(\mu)$ or $R \otimes \Pi_\ell^{1/2}(\mu)$.

The results reproduce the standard modules of [Creutzig–Kanade–Liu–DR] and perfectly explain why $\mathfrak{N} = \mathfrak{1}_K$ (super)characters appear in the relaxed $\widehat{\mathfrak{osp}}(1|2)_k$ characters [Kawasetsu–DR].

- $\mathfrak{sl}(3)$ is the first case with different regular and minimal W-algebras. Which is relevant to inverse reduction?

  The relaxed $\widehat{\mathfrak{sl}}(3)_k$ characters turn out to be proportional to the minimal (Bershadsky–Polyakov) characters. [Kawasetsu]

  Inverse reduction should take $\mathfrak{W}_k^{\text{min.}}(\mathfrak{sl}(3))$-mod to $\widehat{\mathfrak{sl}}(3)_k$-mod.

  But, Bershadsky–Polyakov has relaxed modules. [Fehily–Kawasetsu–DR] Are their characters proportional to regular (Zamolodchikov W_k^3) ones?

  Yes! An inverse reduction embedding exists, [Adamović–Kawasetsu–DR]

$$\mathfrak{W}_k^{\text{min.}}(\mathfrak{sl}(3)) \hookrightarrow \mathfrak{W}_k^{\text{reg.}}(\mathfrak{sl}(3)) \otimes \Pi,$$

  iff k is admissible but non-degenerate:

$$k+3 = \frac{u}{v}, \qquad u, v \geqslant 3, \ \gcd\{u, v\} = 1.$$

  The inverse reduction functors are again tensoring with $\Pi_\ell(\mu)$.

- This generalises: there is an inverse reduction embedding, [Fehily]

$$\mathfrak{W}_k^{\text{sub.}}(\mathfrak{sl}(n)) \hookrightarrow \mathfrak{W}_k^{\text{reg.}}(\mathfrak{sl}(n)) \otimes \Pi,$$

  iff k is admissible but non-degenerate:

$$k + n = \frac{u}{v}, \qquad u, v \geqslant n,\ \gcd\{u, v\} = 1.$$

  The inverse reduction functors are still just tensoring with $\Pi_\ell(\mu)$.
- The story is similar for the regular and subregular W-algebras of $\mathfrak{sp}(4)$.
- Work is progressing on connecting $\mathfrak{W}_k^{\text{min.}}(\mathfrak{sl}(3))$ and $\widehat{\mathfrak{sl}}(3)_k$.

There is clearly a lot still to do...

The big picture

It seems that the right way to analyse W-algebra CFTs is:

- Start with the regular W-algebra at an admissible but non-degenerate level. These are rational with known representation theories!
- Use inverse reduction to construct the standard modules of the subregular W-algebra. Get the other irreducibles as quotients.
- Repeat, working your way up the lattice of nilpotents until the representation theory of the desired W-algebra is known!

If the level is admissible but degenerate, don't despair: start instead with a rational exceptional W-algebra. [Arakawa–van Ekeren]

- For $k + h^\vee = \dfrac{u}{v}$, the degenerate denominator $v = 1$ means that the exceptional W-algebra is $\widehat{\mathfrak{g}}_k$ (which is rational).
- For $\mathfrak{g} = \mathfrak{sl}(3)$, $u \geqslant 3$ and $v = 2$ is degenerate-admissible and the exceptional is Bershadsky–Polyakov (which is rational).
- For $\mathfrak{g} = \mathfrak{sl}(n)$, $u \geqslant n$ and $v = n - 1$, the subregular is rational.

Outlook

- Inverse quantum hamiltonian reduction is a very promising means to analyse logarithmic CFTs with W-algebra symmetry.
- It allows us to classify standard modules, hence irreducible weight modules, compute modular transformations and (Gr) fusion rules.
- There is also potential to explicitly construct projective covers.
- We may also be able to determine the fusion rules themselves.
- It is said that WZW models are the building blocks of rational CFT. If the same is true for admissible-level WZW models and log CFT, then we can expect these methods to generalise widely!
- Either way, the future of these CFTs is looking good...

"*Only those who attempt the absurd will achieve the impossible.*"

— M C Escher

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