# Relaxed modules and logarithmic CFT

[an overview via an example]

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[Quantum Field Theories and Quantum Topology Beyond Semisimplicity]

# Outline

1. Motivation
2. $\mathsf{L}_k(\mathfrak{sl}_2)$ — Some (ancient) history
3. $\mathsf{L}_k(\mathfrak{sl}_2)$ — Relaxed highest-weight modules
4. $\mathsf{L}_k(\mathfrak{sl}_2)$ — Characters and modularity
5. A bigger picture
6. Outlook

# Motivation

I want to understand conformal field theory...

rational — factorisable — non-rational

[lattice bosons]
[free fermions]
[$bc$-ghosts]
[minimal models]
[compact WZW models]
[a few W-algebras]

[free bosons]
[Liouville]
[non-compact WZW?]

discrete - - - - continuous

[symplectic fermions]
[triplet models]
[polymers, percolation?]
[log minimal models?]
[SLE?]

[$\beta\gamma$-ghosts]
[supergroup WZW]
[Nappi–Witten]
[fractional-level WZW]
[most W-algebras]
[spin chains?]

log-rational — logarithmic — generic

A rational CFT has a VOA module category that is

- semisimple: modules are completely reducible,
- finite: there are finitely many irreducibles (up to $\cong$),
- $\mathsf{q}$-finite: modules have $\mathsf{q}$-characters ($\operatorname{tr} \mathsf{q}^{L_0 - \mathsf{c}/24}$).

Generalising to the log-rational setting (*ie.* when the VOA is lisse), we lose semisimplicity but keep both finiteness conditions.

However, there aren't many easily accessible lisse examples beyond symplectic fermions (and friends).

Lie-theoretic VOAs usually have even weight-$1$ fields. In nonrational cases, these typically break $C_2$-cofiniteness (*cf.* the free boson).

One is therefore led to explore accessible examples of CFTs with nonsemisimple and nonfinite VOA module categories.

Today: the admissible-level CFTs associated with $\mathfrak{sl}_2$...

# Ancient history

Our story begins in 1986, with the celebrated coset construction of the unitary Virasoro minimal models [Goddard–Kent–Olive]:

$$\mathsf{M}(\mathsf{k}+2,\mathsf{k}+3) \cong \frac{\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2) \otimes \mathsf{L}_1(\mathfrak{sl}_2)}{\mathsf{L}_{\mathsf{k}+1}(\mathfrak{sl}_2)}, \quad \mathsf{k} \in \mathbb{N}.$$

Kent asked: does this extend to nonunitary models?

For $\mathsf{M}(\mathsf{u},\mathsf{u}+\mathsf{v})$, $\mathsf{v} > 1$, this would require making sense of

$$\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2) \quad \text{with} \quad \mathsf{k}+2 = \frac{\mathsf{u}}{\mathsf{v}}.$$

These are the admissible levels of [Kac–Wakimoto '88]. For these levels, category $\mathscr{O}_\mathsf{k}$ for $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ is semisimple and finite (but not $q$-finite).

Moreover, the irreducible characters are vector-valued modular forms, suggesting that these admissible-level models are rational CFTs.

In [Verlinde '88], a formula for the fusion coefficients of a rational CFT in terms of the S-matrix was proposed:

$$\mathcal{N}_{ij}{}^{k} = \sum_{\ell} \frac{S_{i\ell} S_{j\ell} S^{*}_{k\ell}}{S_{0\ell}} \in \mathbb{N}.$$

Subsequently, [Moore–Seiberg '88] showed (modulo assumptions) that Verlinde's formula follows from self-consistency of the CFT.

It should therefore hold for all rational CFTs — eventually proven [Huang '04].

But, it doesn't hold for the admissible-level $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ theories (with $\mathsf{k} \notin \mathbb{N}$): there are always negative fusion coefficients [Koh–Sorba '88].

Much work ensued [BF '90,MW '90,AY '92,R '93,FM '93,A '95,PRY '96,FGP '96,. . . ] but with no resolution. [Di Francesco–Mathieu–Sénéchal '97] refer to these "fractional-level WZW models" as being "physically sick".

# A nonrational CFT

Of course, these admissible-level models are just not rational. At the level of modules, this was already established in [Adamović–Milas '95], where $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$, $\mathsf{k} \notin \mathbb{N}$, was shown to admit infinitely many irreducible modules.

[Feigin–Semikhatov–Tipunin '97] rediscovered this infinitude in relation to Kazama–Suzuki for $N = 2$ minimal models. They dubbed them relaxed highest-weight modules and added their spectral flows to the mix.

[Maldacena–Ooguri '00] made relaxed modules and spectral flows the centrepiece of their proposal for the $\mathsf{SL}(2,\mathbb{R})$ WZW model spectrum.

[Gaberdiel '01] proved that for $\mathsf{k} = -\frac{4}{3}$, the category $\mathscr{O}_\mathsf{k}$ is not closed under fusion and that a physically consistent category must include relaxed modules, spectral flows and logarithmic modules.

[DR '10] extended this to $\mathsf{k} = -\frac{1}{2}$, motivated by links to the $\mathsf{c} = -2$ singlet and triplet models:

$$\mathsf{L}_{-1/2}(\mathfrak{sl}_2) \xrightarrow[\text{coset}]{\text{parafermion}} \mathsf{Sing}(1,2) \xrightarrow[\text{extension}]{\text{simple current}} \mathsf{Trip}(1,2).$$

# Relaxed highest-weight modules

So what is a relaxed highest-weight $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-module?

Let $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ in $\mathfrak{sl}_2$.

Let $\widehat{\mathfrak{sl}}_2 = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ and let $j_n = j \otimes t^n$, for all $j \in \mathfrak{sl}_2$.

Then, $\widehat{\mathfrak{sl}}_2$ has a generalised triangular decomposition:

$$\widehat{\mathfrak{sl}}_2 = \underbrace{\widehat{\mathfrak{sl}}_2^{<}}_{=\mathfrak{sl}_2 \otimes t^{-1}\mathbb{C}[t^{-1}]} \oplus \underbrace{\widehat{\mathfrak{sl}}_2^{0}}_{=\mathfrak{sl}_2 \oplus \mathbb{C}K} \oplus \underbrace{\widehat{\mathfrak{sl}}_2^{>}}_{=\mathfrak{sl}_2 \otimes t\mathbb{C}[t]}.$$

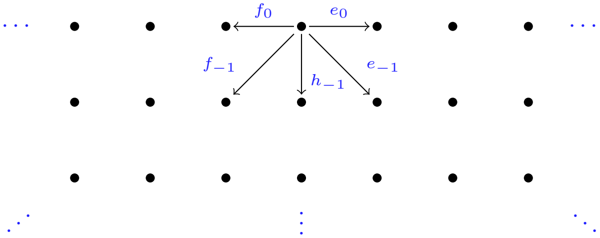
A relaxed highest-weight $\widehat{\mathfrak{sl}}_2$-vector is then a simultaneous eigenvector of $h_0$ and $K$ that is annihilated by $\widehat{\mathfrak{sl}}_2^{>}$.

A relaxed highest-weight $\widehat{\mathfrak{sl}}_2$-module is then a module generated by a single relaxed highest-weight vector.

This restricts to $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-modules ($K$ acts as multiplication by $\mathsf{k}$).

This notion includes the highest-weight modules ($e_0$ also annihilates), their "conjugates" ($f_0$ also annihilates) and more.

In particular, one can have a "top space" without a highest-weight or lowest-weight $\mathfrak{sl}_2$-vector.




Note that relaxed highest-weight modules are positive-energy. The irreducible ones may therefore be classified using Zhu algebra methods [Zhu '96].

# The irreducible spectrum of $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$

For $\mathsf{u}, \mathsf{v} \in \mathbb{Z}_{\geqslant 2}$, let $\mathsf{k} + 2 = \dfrac{\mathsf{u}}{\mathsf{v}}$,

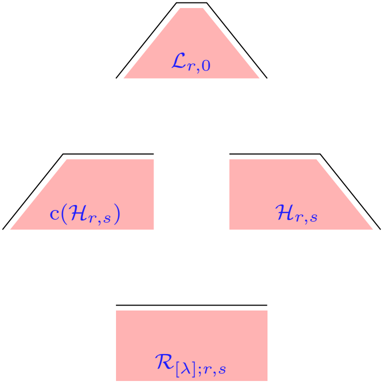
$$\lambda_{r,s} = r - 1 - \frac{\mathsf{u}}{\mathsf{v}} s \quad \text{and} \quad \Delta_{r,s} = \frac{(\mathsf{v}r - \mathsf{u}s)^2 - \mathsf{v}^2}{4\mathsf{u}\mathsf{v}}.$$

For $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$, the classification of irreducible relaxed highest-weight modules is then as follows [Adamović–Milas '95, DR–Wood '15]:

- The $\mathsf{q}$-finite highest-weight modules $\mathcal{L}_{r,0}$, $r = 1, \dots, \mathsf{u} - 1$, of highest weight $\lambda_{\mathsf{u}-r,\mathsf{v}-1}\omega_0 + \lambda_{r,0}\omega_1$ and conformal weight $\Delta_{r,0}$.
- The highest-weight modules $\mathcal{H}_{r,s}$, $r = 1, \dots, \mathsf{u} - 1$ and $s = 1, \dots, \mathsf{v} - 1$, of highest weight $\lambda_{\mathsf{u}-r,\mathsf{v}-1-s}\omega_0 + \lambda_{r,s}\omega_1$ and conformal weight $\Delta_{r,s}$.
- The conjugates $\mathrm{c}(\mathcal{H}_{r,s})$ of the $\mathcal{H}_{r,s}$ (the $\mathcal{L}_{r,0}$ are self-conjugate).
- The non-highest-weight modules $\mathcal{R}_{[\lambda];r,s}$, $r = 1, \dots, \mathsf{u} - 1$, $s = 1, \dots, \mathsf{v} - 1$ and $[\lambda] \in (\mathbb{C}/2\mathbb{Z}) \setminus \{[\lambda_{r,s}], [\lambda_{\mathsf{u}-r,\mathsf{v}-s}]\}$, with $h_0$-eigenvalues $[\lambda]$ and conformal weight $\Delta_{r,s}$.

These are mutually nonisomorphic except that $\mathcal{R}_{[\lambda];r,s} \cong \mathcal{R}_{[\lambda];\mathsf{u}-r,\mathsf{v}-s}$.

The vacuum module is $\mathcal{L}_{1,0}$.




| | | |
|---|---|---|
| q-finite | $\mathcal{L}_{r,0}$ | $\mathsf{u}-1$ irreps |
| (conjugate) highest-weight | $c(\mathcal{H}_{r,s})$, $\mathcal{H}_{r,s}$ | $(\mathsf{u}-1)(\mathsf{v}-1)$ irreps each |
| relaxed | $\mathcal{R}_{[\lambda];r,s}$ | $\frac{1}{2}(\mathsf{u}-1)(\mathsf{v}-1)$ $1$-parameter families |

Clearly, the non-highest-weight relaxed modules $\mathcal{R}_{[\lambda];r,s}$ dominate the (irreducible, positive-energy) spectrum of $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$.

# Automorphisms and twists

The conjugate of an $\widehat{\mathfrak{sl}}_2$-module is obtained by twisting the action with the conjugation automorphism $\mathrm{c}$, defined by

$$\mathrm{c}(e_n) = f_n, \quad \mathrm{c}(h_n) = -h_n, \quad \mathrm{c}(f_n) = e_n, \quad \mathrm{c}(K) = K.$$

Conjugation also preserves the Virasoro zero-mode: $\mathrm{c}(L_0) = L_0$.

Spectral flow refers to another family of automorphisms $\sigma^\ell$, $\ell \in \mathbb{Z}$, defined by

$$\sigma^\ell(e_n) = e_{n-\ell}, \quad \sigma^\ell(h_n) = h_n - \delta_{n,0}\ell K, \quad \sigma^\ell(f_n) = f_{n+\ell}, \quad \sigma^\ell(K) = K.$$

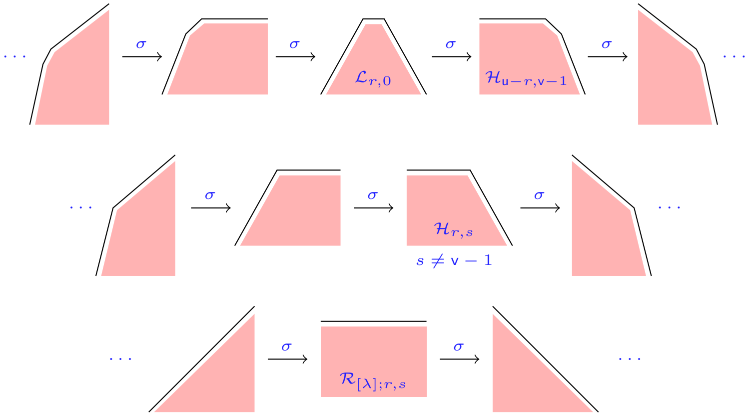
We also have $\sigma^\ell(L_0) = L_0 - \frac{1}{2}\ell h_0 + \frac{1}{4}\ell^2 K$.

Since these automorphisms preserve the Cartan subalgebra $\mathbb{C}h_0 \oplus \mathbb{C}K$, twisting by them defines invertible functors on the category of weight modules of $\widehat{\mathfrak{sl}}_2$.

These functors map a $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-module to another: $\mathcal{M} \xrightarrow{\sigma^\ell} \mathcal{M}^\ell \equiv \sigma^\ell(M)$.

They do not (for $\mathsf{k} \notin \mathbb{N}$) preserve the property of being (relaxed) highest-weight.

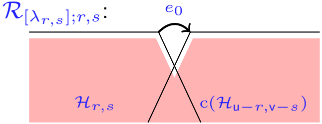
We depict the spectral flow orbits of the irreducible $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-modules:




It follows from [Futorny–Tsylke '01] that these exhaust the irreducibles in the category $\mathscr{W}_\mathsf{k}$ of weight $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-modules.

# Nonsemisimplicity

The category $\mathscr{W}_{\mathsf{k}}$ of weight $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-modules is not semisimple (for $k \notin \mathbb{N}$), *eg.*




$$\mathrm{Ext}^1\big(\mathcal{H}_{r,s}, \mathrm{c}(\mathcal{H}_{\mathsf{u}-r,\mathsf{v}-s})\big) \cong \mathbb{C}.$$

However, it is "almost semisimple": the irreducible $\mathcal{R}^{\ell}_{[\lambda];r,s}$, *ie.* those with $[\lambda] \neq [\lambda_{r,s}], [\lambda_{\mathsf{u}-r,\mathsf{v}-s}]$, are projective and injective.

$\mathscr{W}_{\mathsf{k}}$ therefore breaks up into an uncountably infinite number of semisimple blocks and a finite number of nonsemisimple blocks.

As per [Kac '77], we call these blocks (and their constituent modules) typical (semisimple) and atypical (nonsemisimple).

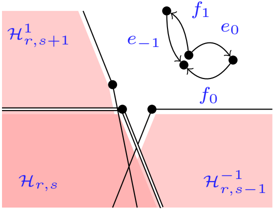
The typical $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-modules are thus finite direct sums of the $\mathcal{R}^{\ell}_{[\lambda];r,s}$, with $[\lambda] \neq [\lambda_{r,s}], [\lambda_{\mathsf{u}-r,\mathsf{v}-s}]$. The vacuum module $\mathcal{L}_{1,0}$ is atypical.

The atypical blocks of $\mathscr{W}_{\mathsf{k}}$ are naturally harder to understand.

There exist [Gaberdiel '01, Adamović–Milas '09, DR '10, Adamović '17] logarithmic $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-modules $\mathcal{P}_{r,s}$, $r = 1, \dots, \mathsf{u} - 1$ and $s = 0, \dots, \mathsf{v} - 1$, in $\mathscr{W}_{\mathsf{k}}$.

$\mathcal{P}_{r,s}$:

$$\begin{tikzcd}
& \mathcal{H}_{r,s} \arrow[dl] \arrow[dr] & \\
\mathcal{H}^{-1}_{r,s-1} \arrow[dr] & & \mathcal{H}^{1}_{r,s+1} \arrow[dl] \\
& \mathcal{H}_{r,s} &
\end{tikzcd}$$




On $\mathcal{P}_{r,s}$, $L_0$ acts with rank-$2$ Jordan blocks (but $h_0$ acts semisimply).

$\mathcal{P}_{r,s}$ is conjectured to be the projective cover (and injective hull) of $\mathcal{H}_{r,s}$.

# A BGG category

Finally, note that $\mathcal{P}_{r,s}$ is an indecomposable sum of two spectrally flowed reducible relaxed modules, each of which is an indecomposable sum of two spectrally flowed irreducible highest-weight modules:

$$\begin{tikzcd}
& \mathcal{H}_{r,s} \arrow[dl] \arrow[dr] & \\
\mathcal{H}^{-1}_{r,s-1} \arrow[dr] & & \mathcal{H}^{1}_{r,s+1} \arrow[dl] \\
& \mathcal{H}_{r,s} &
\end{tikzcd}
\qquad\qquad
\mathcal{R}^1_{[\lambda_{r,s+1}];r,s+1} \hookrightarrow \mathcal{P}_{r,s} \twoheadrightarrow \mathcal{R}_{[\lambda_{r,s}];r,s}.$$

This suggests regarding the $\mathcal{R}^{\ell}_{[\lambda];r,s}$ as the standard modules of $\mathscr{W}_{\mathsf{k}}$. Their contragredient duals are then the costandard modules.

The $\mathcal{P}_{r,s}$ are now tilting modules obeying BGG reciprocity:

$$\begin{matrix} \text{multiplicity in } \mathcal{P}_{r,s} \\ \text{of } \mathcal{R}^{\ell}_{[\lambda_{r',s'}];r',s'} \end{matrix} \quad = \quad \begin{matrix} \text{multiplicity in} \\ \mathcal{R}^{\ell}_{[\lambda_{r',s'}];r',s'} \text{ of } \mathcal{H}_{r,s} \end{matrix}.$$

# Modularity

But do we need this continuum of relaxed highest-weight modules and their spectral flows to construct a consistent CFT from $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-modules?

The answer is yes and the reason is that the partition function needs to be invariant under the action of the modular group $\mathsf{SL}(2;\mathbb{Z})$.

Recall that affine characters are decorated by an additional variable:

$$\mathrm{ch}\big[\mathcal{M}\big] = \mathrm{tr}_{\mathcal{M}}\, \mathsf{z}^{h_0} \mathsf{q}^{L_0 - \mathsf{c}/24}, \quad \mathsf{z} = \mathsf{e}^{2\pi \mathsf{i} \zeta},\ \mathsf{q} = \mathsf{e}^{2\pi \mathsf{i} \tau}.$$

In a rational CFT, modularity manifests as the characters (or one-point functions) spanning a finite-dimensional representation of $\mathsf{SL}(2;\mathbb{Z})$:

$$\begin{aligned} \mathcal{U} \cdot \mathrm{ch}\big[\mathcal{M}_i\big]\,(\zeta \,|\, \tau) &= \mathrm{ch}\big[\mathcal{M}_i\big]\,(\mathcal{U} \cdot \zeta \,|\, \mathcal{U} \cdot \tau) \\ &= \sum_j \mathcal{U}_{ij} \mathrm{ch}\big[\mathcal{M}_j\big]\,(\zeta \,|\, \tau)\,, \quad \mathcal{U} \in \mathsf{SL}(2;\mathbb{Z}). \end{aligned}$$

In our case, a finite sum will not suffice.

We need to be careful about this action because Kac–Wakimoto's modularity results for highest-weight modules led to the failure of the Verlinde formula.

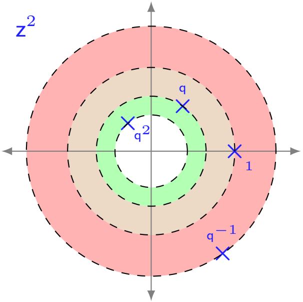
Let's see why: First, the Kac–Wakimoto character formula is

$$\mathrm{ch}\big[\mathcal{H}_{r,s}\big] = \frac{\Big[\sum_{m\in\mathbb{Z}+\frac{\mathsf{v}r-\mathsf{u}s}{2\mathsf{u}\mathsf{v}}} - \sum_{m\in\mathbb{Z}-\frac{\mathsf{v}r+\mathsf{u}s}{2\mathsf{u}\mathsf{v}}}\Big]\mathsf{z}^{2\mathsf{u}m}\mathsf{q}^{\mathsf{u}\mathsf{v}m^2}}{\Big[\sum_{m\in\mathbb{Z}+\frac{1}{4}} - \sum_{m\in\mathbb{Z}-\frac{1}{4}}\Big]\mathsf{z}^{4m}\mathsf{q}^{2m^2}}.$$

This has zeroes in the denominator when $\mathsf{z}^2 = \mathsf{q}^i$, $i \in \mathbb{Z}$, but the zeroes of the numerator only cancel every $\mathsf{v}$-th denominator zero.

For $\mathsf{v} \neq 1$ (*ie.* $\mathsf{k} \notin \mathbb{N}$), this formula is therefore not holomorphic in $\mathsf{z}$.

It is meromorphic, but this means that one must specify the correct annulus of convergence to expand in [Lesage–Mathieu–Rasmussen–Saleur '02, DR '08].



Second, the modular S-transform

$$\mathcal{S}\colon\ (\zeta\,|\,\tau) \mapsto \left(\frac{\zeta}{\tau}\,\middle|\,-\frac{1}{\tau}\right)$$

does not preserve these annuli of convergence.

To S-transform the Kac–Wakimoto characters, one must thus forget about convergence in z and use meromorphically continued formulae [LMRS '02].

Third, the characters of the spectral flows of the $\mathcal{H}_{r,s}$ are unfortunately only distinguished by these annuli of convergence. The meromorphically continued Kac–Wakimoto characters are not linearly dependent.

In particular, the meromorphic Kac–Wakimoto characters give

$$\operatorname{ch}\left[\mathcal{H}_{r,s}\right] = -\operatorname{ch}\left[\mathcal{H}^{-1}_{r,s-1}\right] \quad \Rightarrow \quad \operatorname{ch}\left[\mathcal{R}_{[\lambda_{r,s}];r,s}\right] = 0.$$

These meromorphic continuations are clearly unusable [DR '10].

# Enter the distribution

All good analysts know that comparing functions with disjoint convergence regions requires care. One way to be careful is to treat them as distributions.

A well known, but salient, example is

$$\sum_{j=0}^{\infty} \mathsf{z}^j = \frac{1}{1-\mathsf{z}}, \quad |z| < 1, \quad \text{and} \quad \sum_{j=-\infty}^{-1} \mathsf{z}^j = -\frac{1}{1-\mathsf{z}}, \quad |z| > 1.$$

The meromorphic continuations of these series sum to $0$, but the series themselves sum to a Dirac comb:

$$\sum_{j=-\infty}^{\infty} \mathsf{z}^j = \sum_{j\in\mathbb{Z}} \mathsf{e}^{2\pi i \zeta j} = \sum_{n\in\mathbb{Z}} \delta(\zeta - n).$$

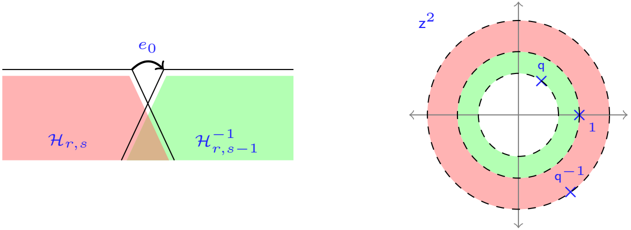
Note that this distribution is supported at $\zeta \in \mathbb{Z}$, *ie.* $\mathsf{z} = 1$: the pole of the meromorphic continuations.

The "right" way to sum these series therefore results in zero everywhere except at this pole, where a distribution lurks unseen by meromorphic continuations.

And so it is with Kac–Wakimoto characters [Creutzig–DR '13]:

$$\mathrm{ch}\big[\mathcal{R}_{[\lambda_{r,s}];r,s}\big] = \mathrm{ch}\big[\mathcal{H}_{r,s}\big] + \mathrm{ch}\big[\mathcal{H}^{-1}_{r,s-1}\big] = \frac{\mathsf{z}^{\lambda_{r,s}} \chi^{\text{Vir.}}_{r,s}(\tau)}{\eta(\tau)^2} \sum_{n\in\mathbb{Z}} \delta(2\zeta - n).$$

The presence of the Virasoro minimal model character $\chi^{\text{Vir.}}_{r,s}$ is neatly explained by inverse quantum hamiltonian reduction [Adamović '17].




The resulting distribution is supported at $\mathsf{z}^2 = 1$. To shift the support to another pole, simply apply spectral flow.

# The standard S-transform

These distributional character formulae generalise to typical modules [Kawasetsu–DR '18]:

$$\mathrm{ch}\big[\mathcal{R}^{\ell}_{[\lambda];r,s}\big] = \frac{\mathsf{z}^{\ell\mathsf{k}}\mathsf{q}^{\ell^2\mathsf{k}/4}\chi^{\mathsf{Vir.}}_{r,s}(\tau)}{\eta(\tau)^2}\sum_{n\in\mathbb{Z}}\mathsf{e}^{\pi i n\lambda}\delta(2\zeta+\tau-n).$$

Moreover, the standard $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-characters span a representation of $\mathsf{SL}(2;\mathbb{Z})$ [Creutzig–DR '13]:

$$\mathrm{ch}\big[\mathcal{R}^{\ell}_{[\lambda];r,s}\big]\Big|_{\mathcal{S}} = A(\zeta\,|\,\tau)\sum_{\ell'\in\mathbb{Z}}\sum_{(r',s')}{}' \int_{\mathbb{R}/\mathbb{Z}} \mathcal{S}^{\ell';[\lambda'];r',s'}_{\ell;[\lambda];r,s}\,\mathrm{ch}\big[\mathcal{R}^{\ell'}_{[\lambda'];r',s'}\big]\,\mathrm{d}[\lambda],$$

$$\mathcal{S}^{\ell';[\lambda'];r',s'}_{\ell;[\lambda];r,s} = \frac{1}{2}\mathsf{e}^{-\pi\mathsf{i}(\mathsf{k}\ell\ell'+\ell\lambda'+\lambda\ell')}\mathcal{S}^{\mathsf{Vir.}}_{(r,s)(r',s')}.$$

By inspection, the S-matrix elements are symmetric and the S-transform is unitary and squares to conjugation.

This suggests that the category $\mathscr{W}_{\mathsf{k}}$ of weight $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-modules is modular.

# The atypical S-transform

But the typical $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-modules do not exhaust the spectrum of irreducible weight modules (when $\mathsf{k} \notin \mathbb{N}$). What about the $\mathcal{H}^{\ell}_{r,s}$?

Every irreducible weight module admits an infinite resolution by standard modules. [But, these resolutions technically only converge if $\mathsf{k} < 0$.]

There is thus a completion of the Grothendieck group of $\mathscr{W}_{\mathsf{k}}$ in which the atypical irreducibles are represented by infinite alternating sums of standards.

The choice of resolution / completion is not unique, but all choices lead to the same results. In particular, we can now compute atypical S-transforms, *eg.*

$$\operatorname{ch}\left[\mathcal{L}^{\ell}_{r,0}\right]\Big|_{\mathcal{S}} = A\,(\zeta\,|\,\tau) \sum_{\ell' \in \mathbb{Z}} \sum_{(r',s')}{}' \int_{\mathbb{R}/\mathbb{Z}} \mathcal{S}^{\ell';[\lambda'];r',s'}_{\ell;r,0} \operatorname{ch}\left[\mathcal{R}^{\ell'}_{[\lambda'];r',s'}\right] \mathrm{d}[\lambda],$$

$$\mathcal{S}^{\ell';[\lambda'];r',s'}_{\ell;r,0} = \frac{1}{2} \frac{\mathsf{e}^{-\pi\mathsf{i}(\mathsf{k}\ell\ell' + \ell\lambda' + (r-1)\ell')}}{2\cos(\pi\lambda') + (-1)^{r'} 2\cos(\pi\mathsf{k}s')} \mathcal{S}^{\mathsf{Vir.}}_{(r,1)(r',s')}.$$

The vacuum S-transform is obtained by setting $\ell = 0$ and $r = 1$.

# Grothendieck fusion

$\mathscr{W}_{\mathsf{k}}$ is closed under fusion [Nahm '94]. Assume that fusing with any module in $\mathscr{W}_{\mathsf{k}}$ is exact, so the Grothendieck group inherits a ring structure.

We substitute our explicit S-transforms into the Verlinde formula, more specifically that obtained from the rational formula by the replacement

$$\sum \longmapsto \sum_{\ell' \in \mathbb{Z}} \sum_{(r',s')}{}' \int_{\mathbb{R}/\mathbb{Z}} -\mathrm{d}[\lambda'].$$

The result is nonnegative-integer Grothendieck fusion coefficients, *eg.* the fusion rule for two standard $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-modules is

$$\begin{aligned} \big[\mathcal{R}^{\ell}_{[\lambda];r,s}\big] \boxtimes \big[\mathcal{R}^{\ell'}_{[\lambda'];r',s'}\big] &= \sum_{(r'',s'')}{}' \mathcal{N}^{\text{Vir.}\ (r'',s'')}_{(r,s)(r',s')} \Big( \big[\mathcal{R}^{\ell+\ell'+1}_{[\lambda+\lambda'-\mathsf{k}];r'',s''}\big] + \big[\mathcal{R}^{\ell+\ell'-1}_{[\lambda+\lambda'+\mathsf{k}];r'',s''}\big] \Big) \\ &\qquad + \sum_{(r'',s'')}{}' \Big( \mathcal{N}^{\text{Vir.}\ (r'',s'')}_{(r,s)(r',s'-1)} + \mathcal{N}^{\text{Vir.}\ (r'',s'')}_{(r,s)(r',s'+1)} \Big) \big[\mathcal{R}^{\ell+\ell'}_{[\lambda+\lambda'];r'',s''}\big]. \end{aligned}$$

$\mathscr{W}_{\mathsf{k}}$ is thus modular in the same way as module categories of rational CFTs.

# A bigger picture

We've seen that the category $\mathscr{W}_\mathsf{k}$ of weight $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-modules enjoys many properties not shared by its better-studied subcategories:

- It is almost semisimple, *ie.* almost all of its blocks are semisimple.
- It admits a set of standard modules such that the irreducibles, standards and (conjecturally) projectives satisfy BGG reciprocity.
- A completion of its Grothendieck group carries an action of the modular group. The standard modules give a distinguished basis of this group.
- An obvious continuous generalisation of the Verlinde formula returns nonnegative-integer Grothendieck fusion coefficients.

One might expect such beautiful outcomes, given the well known joys of affine symmetry. However, there are many other examples of VOAs for which the weight category behaves similarly.

Relatively trivial examples are the (suitably) rational VOAs.

Another familiar example is the free boson, where the standard modules are the Fock spaces (with real weights).

Logarithmic examples explored to date include:

- Level-agnostic affine VOAs: $\mathsf{L}(\mathfrak{gl}(1|1))$, its "Takiffisation", Nappi–Witten.
- Admissible-level affine VOAs: $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$, $\mathsf{L}_\mathsf{k}(\mathfrak{osp}(1|2))$, $\mathsf{L}_\mathsf{k}(\mathfrak{sl}(2|1))$, $\mathsf{L}_\mathsf{k}(\mathfrak{sl}(3))$.
- Universal VOAs: Virasoro, $N = 1$.
- W-algebras: Bershadsky–Polyakov, type-A subregulars.
- Screening kernels: $\mathfrak{sl}_2$ singlet algebras $\mathsf{Sing}(p, p')$.
- Cosets: bosonic ghosts, $N = 2$ minimal models, $\mathfrak{sl}_2$ parafermions.

Together, these suggest a standard module formalism that covers an enormous range of CFTs [Creutzig–DR '13, DR–Wood '14].

There are of course non-examples for which this formalism doesn't apply: the log-rational (non-rational lisse) CFTs, *eg.* symplectic fermions, triplets.

Here, the atypical blocks describe a set of nonzero measure. Nevertheless, one can rework the Verlinde formula to give the correct Grothendieck fusion rules.

In fact, all (?) known examples are simple current extensions of examples to which the standard module formalism does apply.

# Outlook

So... where to now?

- Obviously, we need a better handle on the categories in the examples we have, *eg.* projectivity, rigidity.
- Ideally, we'd like a general theory that proves the standard Verlinde formula.
- This will surely require many more examples, particularly higher-rank ones, to glean further insights into the general structure.
- One issue with higher-rank examples is the plethora of classes of irreducibles between highest-weight and fully relaxed. Their role in the standard module formalism needs to be pinned down.
- In affine and W-algebraic examples, the structure of the weight category should be controlled by nilpotent orbit classifications.
- Finally, I'd really like to know if there are any log-rational examples with finite automorphism groups (so they can't be realised as simple current extensions of models to which the standard module formalism applies).

"*Only one who attempts the absurd is capable of achieving the impossible.*"

— Miguel de Unamuno

# Thank you!