

A Kazhdan-Lusztig correspondence  
for a vertex algebra associated to  $sl_3$

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What do I mean by "Kazhdan-Lusztig correspondence"?

[KL'93-94] showed that two categories are braided-tensor equivalent:

- The parabolic category  $\bar{\mathcal{O}}_k$  for  $\hat{\mathfrak{g}}$  ( $\mathfrak{g}$  simple) of level  $k$  ( $k+h^\vee \notin \mathbb{Q}_{>0}$ ) in which  $D$  has finite-dimensional eigenspaces.
- The category of finite-dimensional  $U_q(\mathfrak{g})$ -modules for  $q = e^{\pi i / l(k+h^\vee)}$ .

The former is naturally interpreted in terms of the simple [Gorelik-Kac '06] affine vertex algebra  $L_k(\mathfrak{g})$  as:

- The category of ordinary  $L_k(\mathfrak{g})$ -modules with  $k+h^\vee \notin \mathbb{Q}_{>0}$ .

KL's tensor product is the fusion product of CFT.

But, the nicest examples of  $L_k(\mathfrak{g})$ -module categories are for  $k \in \mathbb{Z}_{\geq 0}$   
[Witten '84, Gepner-Witten '84, Verlinde '88, Moore-Seiberg '88-89, Frenkel-Zhu '92, ...]  
[Finkelberg '96] extended this Kazhdan-Lusztig correspondence to these levels,\*  
but with an additional semisimplification on the quantum group side.

Why do such equivalences exist?

What's so natural about semisimplifications?

\* Modulo a few cases that don't work, see [Huang '13].

## Logarithmic Kazhdan-Lusztig correspondences

Semisimplification is awesome but perhaps there are more natural equivalences between non-semisimple quantum group and VOA module categories.

The first was proposed in [Feigin-Gaiutdinov-Semikhatov-Tipunin '06-07] for  $U_q(\mathfrak{sl}_2)$ ,  $q = e^{\pi i/p}$ , and the triplet models  $W(p)$  of [Kausch '91].

(The precise identification of the quantum group took many years, see [Creutzig-Geer-Rupert '20].)

The idea here is that quantum group module categories are much easier to understand than VOA-module categories.

Our interest today lies with a special case of a logarithmic KL correspondence that differs from [FGST] in two ways:

- 1) The quantum group is related to  $sl_3$  not  $sl_2$ .
- 2) The VOA is not a triplet model, but an  $sl_3$ -generalisation of Kausch's singlet models [Feigin-Tipunin '10].

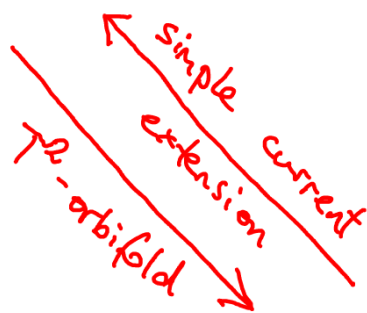
We also specialise  $q$  to  $i$ , the simplest non-trivial root of unity.

Motivation :

[Semikhatov '11,13]

Octuplet model

$W_{A_2}(2)$



[Costantino-Geer-Patureau-Mirand '14]

Unrolled restricted quantum group

$\bar{U}_i^H(\mathfrak{sl}_3)$



$\mathfrak{sl}_3$  singlet VOA

$W_{A_2}^0(2)$

[Kawasetsu-DR-Wood '21]

Affine VOA

$L_{-3/2}(\mathfrak{sl}_3)$



The unrolled restricted quantum group  $\bar{U}_i^\hbar(\mathfrak{sl}_3)$

Generators:  $X_{\pm j}, H_j$  ( $j=1,2$ ) and  $K_\gamma$ ,  $\gamma \in Q_{\mathfrak{sl}_3}$ .

Relations:  $K_0 = 1$ ,  $K_{\gamma_1} K_{\gamma_2} = K_{\gamma_1 + \gamma_2}$ ,  $K_\gamma X_{\pm j} K_{-\gamma} = i^{\pm \langle \gamma, \alpha_j \rangle} X_{\pm j}$ ,

$$[X_j, X_{-j'}] = \delta_{j,j'} \frac{K_{\alpha_j} - K_{-\alpha_j}}{2i}, \quad \boxed{X_{\pm j}^2 = 0, \quad (X_{\pm 1} X_{\pm 2})^2 = (X_{\pm 2} X_{\pm 1})^2,}$$

$$[H_j, X_{\pm j'}] = \pm A_{jj'} X_{\pm j'}, \quad [H_j, H_{j'}] = [H_j, K_\gamma] = 0.$$

Plus coproduct, counit, antipode...

(The  $q$ -Serre relations are tautologies when  $q=i$ .)

Category  $\mathcal{L}$  of finite-dim. weight  $\overline{U}_i^H(\mathfrak{sl}_3)$ -modules

Weight means  $H_j$  acts semisimply and  $K_\gamma$  acts as  $i^{\langle \gamma, - \rangle}$ .

Irreducibles  $L_\lambda$  are highest-weight quotients of  $\mathfrak{sl}_3$ -dim. Vermas  $M_\lambda$ .

- Typical —  $M_\lambda$  is irreducible  $\Leftrightarrow \langle \lambda + \rho, \alpha_j \rangle \notin 2\mathbb{Z} + 1 \quad \forall j = 1, 2, 3$ .
- Atypical —  $M_\lambda$  is reducible  $\Leftrightarrow \exists j$  s.t.  $\langle \lambda + \rho, \alpha_j \rangle \in 2\mathbb{Z} + 1$ .

Projective cover  $P_\lambda$  of  $L_\lambda$  exists and BGG reciprocity holds.

$\mathcal{L}$  is braided ( $\overline{U}_i^H(\mathfrak{sl}_3)$  is quasitriangular). [ Geer - Patureau-Mirand '11  
Rupert '19 ]



# Loewy diagrams

[swap 1 ↔ 2 as needed]

Degree-1 atypicality

Degree-2 atypicality

$$\langle \lambda + \rho, \alpha_1 \rangle$$

$$\in 2\mathbb{Z} + 1$$

$$\in 2\mathbb{Z} + 1$$

$$\in 2\mathbb{Z} + 1$$

$$\langle \lambda + \rho, \alpha_2 \rangle$$

$$\notin 2$$

$$\in 2\mathbb{Z}$$

$$\in 2\mathbb{Z} + 1$$

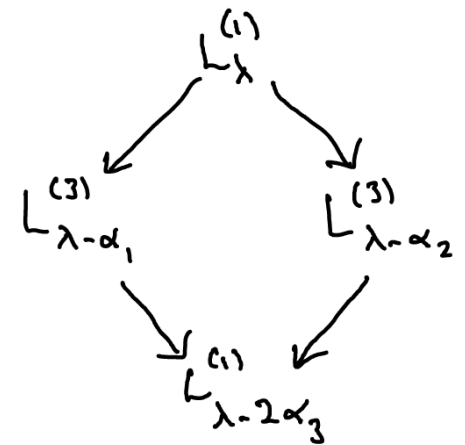
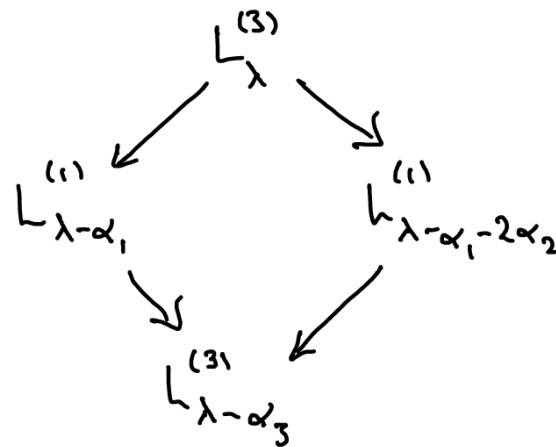
dim  $L_\lambda$

4

3

1

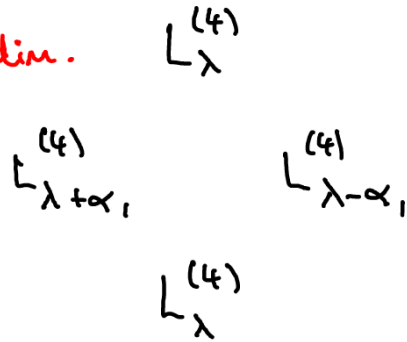
$M_\lambda$



Projective covers

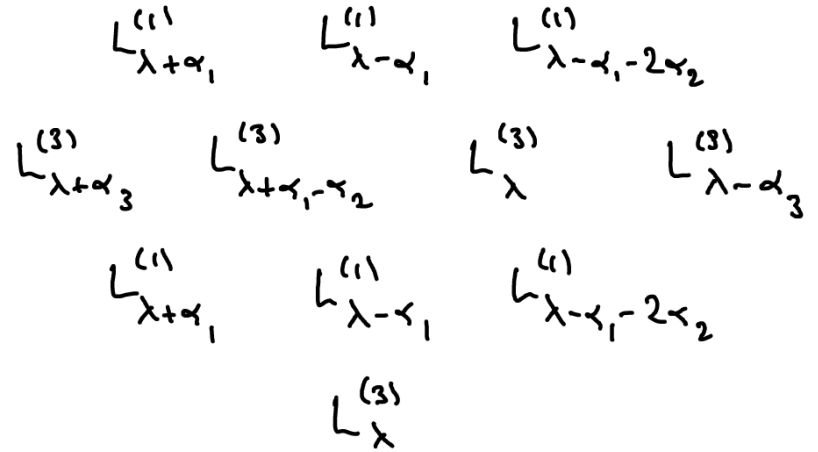
[self-dual & "rigid"]

16-dim.



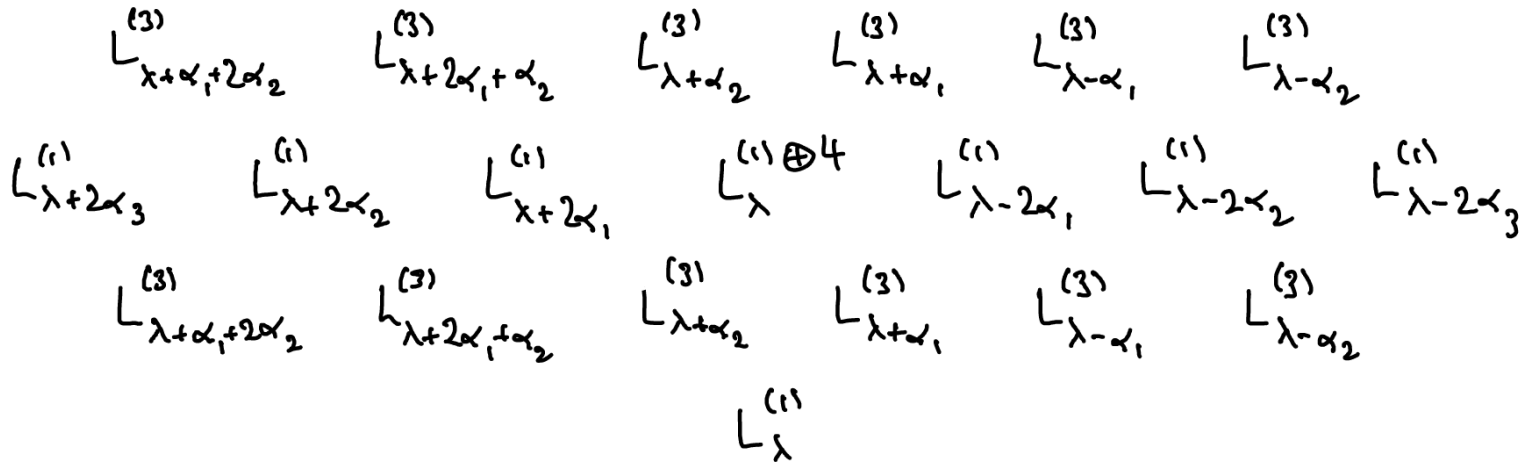
$\textcircled{3}$   
 $\downarrow$   
 $\lambda$

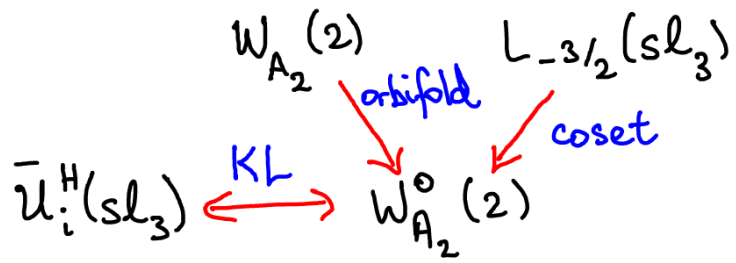
24-dim.



$\textcircled{11}$   
 $\downarrow$   
 $\lambda$

48-dim.





- Irreducible weight  $L_{-3/2}(sl_3)$ -modules were classified in [Kawasetsu-DR-Wood '21].
- This suggests the explicit KL correspondence.
- Comparison of tensor products for  $\bar{U}_i^H(sl_3)$  and (Grothendieck) fusion products for  $L_{-3/2}(sl_3)$  suggests that this is a tensor equivalence.
- Obtain conjectures for the structures of projective modules for  $W_{A_2}^\circ(2)$ ,  $W_{A_2}(2)$  and  $L_{-3/2}(sl_3)$ . First ever proposed structures for VOAs with "rank" greater than 1.

## Conclusions

- We've detailed the first "higher-rank" logarithmic Kazhdan-Lusztig correspondence between a quantum group and a VOA.
- It conjectures an equivalence of braided tensor categories, with consequences that go far beyond what can be directly checked.
- It also suggests that more general higher-rank correspondences (yet to be detailed) will be found (and used).
- Unfortunately, it doesn't explain why such a correspondence exists.

One can hope that the "magic" of CFT will eventually shed light on this (but it hasn't yet)...