A Kazhdan-Lusztig correspondence for a vertex algebra associated to sl_3

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What do I mean by "Kazhdan-Lusztig correspondence"?

[KL'93-94] showed that two categories are braided-tensor equivalent:

- The parabolic category $\bar{\mathcal{O}}_k$ for $\hat{\mathfrak{g}}$ ($\mathfrak{g}$ simple) of level k ($k+h^\vee \notin \mathbb{Q}_{>0}$) in which D has finite-dimensional eigenspaces.
- The category of finite-dimensional $U_q(\mathfrak{g})$-modules for $q = e^{\pi i/\ell(k+h^\vee)}$.

The former is naturally interpreted in terms of the simple [Gorelik-Kac '06] affine vertex algebra $L_k(\mathfrak{g})$ as:

- The category of ordinary $L_k(\mathfrak{g})$-modules with $k+h^\vee \notin \mathbb{Q}_{>0}$.

KL's tensor product is the fusion product of CFT.

But, the nicest examples of $L_k(\mathfrak{g})$-module categories are for $k \in \mathbb{Z}_{\geq 0}$
[Witten '84, Gepner-Witten '84, Verlinde '88, Moore-Seiberg '88-89, Frenkel-Zhu '92, ...]

[Finkelberg '96] extended this Kazhdan-Lusztig correspondence to these levels*, but with an additional semisimplification on the quantum group side.

Why do such equivalences exist?

What's so natural about semisimplifications?

* Modulo a few cases that don't work, see [Huang '13].

Logarithmic Kazhdan-Lusztig correspondences

Semisimplification is awesome but perhaps there are more natural equivalences between non-semisimple quantum group and VOA module categories.

The first was proposed in [Feigin-Gainutdinov-Semikhatov-Tipunin '06-07] for $U_q(sl_2)$, $q = e^{\pi i/p}$, and the triplet models $W(p)$ of [Kausch '91].

(The precise identification of the quantum group took many years, see [Creutzig-Geer-Rupert '20].)

The idea here is that quantum group module categories are <u>much</u> easier to understand than VOA-module categories.

Our interest today lies with a special case of a logarithmic KL correspondence that differs from [FGST] in two ways:

1) The quantum group is related to $\mathfrak{sl}_3$ not $\mathfrak{sl}_2$.

2) The VOA is not a triplet model, but an $\mathfrak{sl}_3$-generalisation of Kausch's <u>singlet models</u> [Feigin-Tipunin '10].

We also specialise q to i, the simplest non-trivial root of unity.

Motivation:

[Semikhatov '11,13]

Octuplet model

$W_{A_2}(2)$

[Kawasetsu-DR-Wood '21]

Affine VOA

$L_{-3/2}(sl_3)$

simple current extension

T^2-orbifold

[Adamović-Milas-Wang '20]

parafermionic coset

[Creutzig-Kanade-Linshaw-DR '16]

[Costantino-Geer-Patureau-Mirand '14]

Unrolled restricted quantum group

$\bar{U}_i^H(sl_3)$

conjectural log

$\longleftrightarrow$

KL correspondence

sl_3 singlet VOA

$W^0_{A_2}(2)$

The unrolled restricted quantum group $\overline{U}_i^H(\mathfrak{sl}_3)$

Generators: $X_{\pm j}$, H_j $(j=1,2)$ and K_γ, $\gamma \in Q_{\mathfrak{sl}_3}$.

Relations: $K_0 = 1$, $K_{\gamma_1} K_{\gamma_2} = K_{\gamma_1+\gamma_2}$, $K_\gamma X_{\pm j} K_{-\gamma} = i^{\pm\langle \gamma, \alpha_j \rangle} X_{\pm j}$,

$$[X_j, X_{-j'}] = \delta_{j,j'} \frac{K_{\alpha_j} - K_{-\alpha_j}}{2i}, \qquad \boxed{X_{\pm j}^2 = 0, \quad (X_{\pm 1} X_{\pm 2})^2 = (X_{\pm 2} X_{\pm 1})^2},$$

$$[H_j, X_{\pm j'}] = \pm A_{jj'} X_{\pm j'}, \qquad [H_j, H_{j'}] = [H_j, K_\gamma] = 0.$$

Plus coproduct, counit, antipode...

(The q-Serre relations are tautologies when $q = i$.)

Category $\mathcal{C}$ of finite-dim. weight $\overline{U}_i^H(\mathfrak{sl}_3)$-modules

Weight means H_j acts semisimply and K_γ acts as $i^{\langle \gamma, - \rangle}$.

Irreducibles L_λ are highest-weight quotients of 8-dim. Vermas M_λ.

- Typical — M_λ is irreducible $\iff$ $\langle \lambda+\rho, \alpha_j \rangle \notin 2\mathbb{Z}+1$ $\forall j = 1,2,3$.
- Atypical — M_λ is reducible $\iff$ $\exists j$ s.t. $\langle \lambda+\rho, \alpha_j \rangle \in 2\mathbb{Z}+1$.

Projective cover P_λ of L_λ exists and BGG reciprocity holds.

$\mathcal{C}$ is braided ($\overline{\mathcal{U}}_i^H(\mathfrak{sl}_3)$ is quasitriangular). [Geer – Patureau-Mirand '11, Rupert '19]

Loewy diagrams

[swap $1 \leftrightarrow 2$ as needed]

	Degree-1 atypicality	Degree-2 atypicality	
$\langle \lambda+\rho, \alpha_1 \rangle$	$\in 2\mathbb{Z}+1$	$\in 2\mathbb{Z}+1$	$\in 2\mathbb{Z}+1$
$\langle \lambda+\rho, \alpha_2 \rangle$	$\notin \mathbb{Z}$	$\in 2\mathbb{Z}$	$\in 2\mathbb{Z}+1$
$\dim L_\lambda$	4	3	1

M_λ

$$\begin{tikzcd} L^{(4)}_{\lambda} \arrow[d] \\ L^{(4)}_{\lambda-\alpha_1} \end{tikzcd}$$

$$\begin{tikzcd} & L^{(3)}_{\lambda} \arrow[dl] \arrow[dr] & \\ L^{(1)}_{\lambda-\alpha_1} \arrow[dr] & & L^{(1)}_{\lambda-\alpha_1-2\alpha_2} \arrow[dl] \\ & L^{(3)}_{\lambda-\alpha_3} & \end{tikzcd}$$

$$\begin{tikzcd} & L^{(1)}_{\lambda} \arrow[dl] \arrow[dr] & \\ L^{(3)}_{\lambda-\alpha_1} \arrow[dr] & & L^{(3)}_{\lambda-\alpha_2} \arrow[dl] \\ & L^{(1)}_{\lambda-2\alpha_3} & \end{tikzcd}$$

Projective covers [self-dual & "rigid"]

16-dim.

$$\begin{array}{ccc} & L_{\lambda}^{(4)} & \\ L_{\lambda+\alpha_1}^{(4)} & & L_{\lambda-\alpha_1}^{(4)} \\ & L_{\lambda}^{(4)} & \end{array}$$

24-dim.

$$\begin{array}{cccc} & & L_{\lambda}^{(3)} & \\ & L_{\lambda+\alpha_1}^{(1)} & L_{\lambda-\alpha_1}^{(1)} & L_{\lambda-\alpha_1-2\alpha_2}^{(1)} \\ L_{\lambda+\alpha_3}^{(3)} & L_{\lambda+\alpha_1-\alpha_2}^{(3)} & L_{\lambda}^{(3)} & L_{\lambda-\alpha_3}^{(3)} \\ & L_{\lambda+\alpha_1}^{(1)} & L_{\lambda-\alpha_1}^{(1)} & L_{\lambda-\alpha_1-2\alpha_2}^{(1)} \\ & & L_{\lambda}^{(3)} & \end{array}$$

48-dim.

$$\begin{array}{ccccccc} & & & L_{\lambda}^{(1)} & & & \\ L_{\lambda+\alpha_1+2\alpha_2}^{(3)} & L_{\lambda+2\alpha_1+\alpha_2}^{(3)} & L_{\lambda+\alpha_2}^{(3)} & L_{\lambda+\alpha_1}^{(3)} & L_{\lambda-\alpha_1}^{(3)} & L_{\lambda-\alpha_2}^{(3)} & \\ L_{\lambda+2\alpha_3}^{(1)} & L_{\lambda+2\alpha_2}^{(1)} & L_{\lambda+2\alpha_1}^{(1)} & L_{\lambda}^{(1)} \oplus 4 & L_{\lambda-2\alpha_1}^{(1)} & L_{\lambda-2\alpha_2}^{(1)} & L_{\lambda-2\alpha_3}^{(1)} \\ L_{\lambda+\alpha_1+2\alpha_2}^{(3)} & L_{\lambda+2\alpha_1+\alpha_2}^{(3)} & L_{\lambda+\alpha_2}^{(3)} & L_{\lambda+\alpha_1}^{(3)} & L_{\lambda-\alpha_1}^{(3)} & L_{\lambda-\alpha_2}^{(3)} & \\ & & & L_{\lambda}^{(1)} & & & \end{array}$$

$$
\begin{tikzcd}
 & W_{A_2}(2) \arrow[dr, "\text{orbifold}"] & & L_{-3/2}(sl_3) \arrow[dl, "\text{coset}"'] \\
\overline{\mathcal{U}}^H_i(sl_3) \arrow[rr, leftrightarrow, "\text{KL}"] & & W^0_{A_2}(2) &
\end{tikzcd}
$$

- Irreducible weight $L_{-3/2}(sl_3)$-modules were classified in [Kawasetsu-DR-Wood '21].
- This suggests the explicit KL correspondence.

- Comparison of tensor products for $\overline{\mathcal{U}}^H_i(sl_3)$ and (Grothendieck) fusion products for $L_{-3/2}(sl_3)$ suggests that this is a tensor equivalence.
- Obtain conjectures for the structures of projective modules for $W^0_{A_2}(2)$, $W_{A_2}(2)$ and $L_{-3/2}(sl_3)$. First ever proposed structures for VOAs with "rank" greater than 1.

Conclusions

- We've detailed the first "higher-rank" logarithmic Kazhdan-Lusztig correspondence between a quantum group and a VOA.
- It conjectures an equivalence of braided tensor categories, with consequences that go far beyond what can be directly checked.
- It also suggests that more general higher-rank correspondences (yet to be detailed) will be found (and used).
- Unfortunately, it doesn't explain why such a correspondence exists. One can hope that the "magic" of CFT will eventually shed light on this (but it hasn't yet)...