Vertex algebras & modularity

$\mathfrak{g} = \mathfrak{gl}_1$ or simple complex Lie algebra (eg. $\mathfrak{sl}_2$).

$\Rightarrow \exists$ a nondeg. inv. symm. bilin. form!

$$\kappa([x,y],z) = \kappa(x,[y,z])$$

$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K$ affine Kac-Moody alg.

Set $x_n = x \otimes t^n$ so $[x_n, K] = 0$

& $[x_m, y_n] = [x,y]_{m+n} + m\,\kappa(x,y)\,\delta_{m+n,0} K$.

Δ-decompositions : creation / annihilation ops.

$$\hat{\mathfrak{g}}^{>} = \mathfrak{g} \otimes t\mathbb{C}[t] = \mathrm{sp}_{\mathbb{C}}\{x_n : x \in \mathfrak{g},\ n>0\}$$

$$\hat{\mathfrak{g}}^{0} = \mathfrak{g} \oplus \mathbb{C}K$$

$$\hat{\mathfrak{g}}^{<} = \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}] = \mathrm{sp}_{\mathbb{C}}\{x_n : x \in \mathfrak{g},\ n<0\}.$$

Level-k universal vacuum module $V_k(\mathfrak{g})$:

$\mathbb{C}\Omega$ = trivial $\mathfrak{g}$-module

$\rightarrow \hat{\mathfrak{g}}^{0}$-module : $K\Omega = k\Omega$, $k \in \mathbb{C}$.

$\rightarrow (\hat{\mathfrak{g}}^{>} \oplus \hat{\mathfrak{g}}^{0})$-module : $x_n \Omega = 0$, $n>0$.

$$V_k(\mathfrak{g}) = \mathrm{Ind}_{\hat{\mathfrak{g}}^{\geq}}^{\hat{\mathfrak{g}}}(\mathbb{C}\Omega).$$

Examples:

- $\mathfrak{g} = \mathfrak{gl}_1 = \mathrm{sp}_{\mathbb{C}}\{a\}$, $\hat{\mathfrak{gl}}_1$: $[a_m, a_n] = m\delta_{m+n,0} K$.

  Universal vacuum module $V_k(\mathfrak{gl}_1)$:

$$\begin{array}{c} \Omega \\ \\ a_{-1}\Omega \\ \\ a_{-1}^2\Omega \quad a_{-2}\Omega \\ \\ a_{-1}^3\Omega \quad a_{-2}a_{-1}\Omega \quad a_{-3}\Omega \\ \vdots \end{array}$$

  Simple $\forall\, k \neq 0$.

- $\mathfrak{g} = \mathfrak{sl}_2 = \mathrm{sp}_{\mathbb{C}}\{e, h, f\}$,

  $\hat{\mathfrak{sl}}_2$: $[e_m, e_n] = 0 = [f_m, f_n]$,

  $[h_m, e_n] = 2e_{m+n}$, $[h_m, f_n] = -2f_{m+n}$,

  $[h_m, h_n] = 2m\delta_{m+n,0}K$,

  $[e_m, f_n] = h_{m+n} + m\delta_{m+n,0}K$.

  Universal vacuum module $V_k(\mathfrak{sl}_2)$:

$$\begin{array}{ccccc} & & \Omega & & \\ & f_{-1}\Omega & h_{-1}\Omega & e_{-1}\Omega & \\ & f_{-2}\Omega & h_{-2}\Omega & e_{-2}\Omega & \\ f_{-1}^2\Omega & & h_{-1}^2\Omega & & e_{-1}^2\Omega \\ & f_{-1}h_{-1}\Omega & & h_{-1}e_{-1}\Omega & \\ \iddots & & f_{-1}e_{-1}\Omega & & \ddots \\ & & \vdots & & \end{array}$$

  Simple <u>iff</u> $k \neq -2$ <u>OR</u> $k+2 = \frac{u}{v}$, $u \in \mathbb{Z}_{\geq 2}$, $v \in \mathbb{Z}_{\geq 1}$, $(u,v)=1$.

  Vacuum module $L_k(\mathfrak{sl}_2)$ = simple quotient of $V_k(\mathfrak{sl}_2)$.

The vacuum modules $V_k(\mathfrak{g})$ and $L_k(\mathfrak{g})$ are VOAs $\forall\, k \neq -h^\vee$. What does this mean?

- $\exists$ a vacuum state $\Omega \in V$
- $\exists$ a conformal vector $T = \frac{1}{2(k+h^\vee)} \sum_i u^i_{-1} v^i_{-1} \Omega$

  ($\{u^i\}$ and $\{v^i\}$ dual bases of $\mathfrak{g}$ wrt κ)
- $\exists$ a state-field correspondence

$$V \longrightarrow \mathrm{End}(V)[\![z, z^{-1}]\!]$$

$$v \longmapsto v(z)$$

eg.

$$\Omega \longmapsto \mathrm{id}_V$$

$$T \longmapsto T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

s.t.

- $v(z)\Omega\big|_{z=0} = v$
- $[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\, c_k\, \mathrm{id}_V$

  (Virasoro algebra with $c_k = \frac{k \dim \mathfrak{g}}{k+h^\vee}$)
- $[L_0, v_{-n}] = n v_{-n}$ (L_0 grades V)
- $v \longmapsto v(z) \Rightarrow L_{-1}v \longmapsto \partial v(z)$ ($\partial \equiv \partial_z$)
- $(z-w)^N [u(z), v(w)] = 0 \quad \forall\, N \gg 0$.

$\Rightarrow$ V has an infinite number of products wrapped into an operator product expansion:

$$u(z)v(w) = \sum_{n\in\mathbb{Z}} \phi_n(w)(z-w)^{-n-\Delta_u}.$$

$$\left(L_0 u = \Delta_u u, \qquad u(z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-\Delta_u}\right)$$

Example: $V = V_k(\mathfrak{gl}_1)$:

$$a \equiv a_{-1}\Omega \longmapsto a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$$

$$a(z)a(w) = \sum_{n\in\mathbb{Z}} \phi_n(w)(z-w)^{-n-1}$$

L_0-eig.

$$\Omega$$

$$a_{-1}\Omega$$

$$T = \frac{1}{2k} a_{-1}^2 \Omega$$

$$\vdots$$

$$\Rightarrow \quad a(z)a(w)\Omega\Big|_{w=0} = \sum_{n\in\mathbb{Z}}' \phi_n(w)(z-w)^{-n-1}\Omega\Big|_{w=0}$$

$$\Rightarrow \quad \sum_{n\in\mathbb{Z}}' a_n z^{-n-1} a = \sum_{n\in\mathbb{Z}}' \phi_n z^{-n-1}$$

$$\Rightarrow \quad \phi_n = a_n a = a_n a_{-1}\Omega .$$

Use $[a_m, a_n] = m\delta_{m+n,0}\, k\cdot \mathrm{id}_V$:

$$\phi_0 = a_0 a_{-1}\Omega = a_{-1}a_0\Omega = 0,$$

$$\phi_1 = a_1 a_{-1}\Omega = (a_{-1}a_1 + k)\Omega = k\Omega \longleftrightarrow k\cdot \mathrm{id}_V,$$

$$\phi_n = a_n a_{-1}\Omega = a_{-1}a_n\Omega = 0 \qquad (n>1).$$

$$\therefore \quad a(z)a(w) = \frac{k\cdot \mathrm{id}}{(z-w)^2} + O(1) \qquad [\text{OPE}].$$

The OPE is the operation of a VOA.

Its singular terms are equivalent to the commutation relations:

$$[a_m, a_n] \overset{!}{=} \oint_0 \oint_w a(z)a(w) z^m w^n \frac{dz}{2\pi i}\frac{dw}{2\pi i}$$

$$= \oint_0 \oint_w \frac{z^m w^n \cdot k\,\mathrm{id}}{(z-w)^2} \frac{dz}{2\pi i}\frac{dw}{2\pi i}$$

$$= \oint_w mk\, w^{m+n-1}\, \mathrm{id}\, \frac{dw}{2\pi i}$$

$$= m\delta_{m+n,0}\, k\, \mathrm{id}. \quad \checkmark$$

For $V_k(\mathfrak{g})$ or $L_k(\mathfrak{g})$, the OPEs are only slightly more complicated:

$$v \mapsto v(z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \qquad (v \in \mathfrak{g})$$

$$\Rightarrow \quad u(z)v(w) = \frac{\kappa(u,v)\, k\cdot \mathrm{id}}{(z-w)^2} + \frac{[u,v](w)}{z-w} + O(1).$$

And there are <u>many</u> other classes of VOAs, mostly with more complicated OPEs.

VOAs are algebras $\rightsquigarrow$ representation theory?

VOA elements: $v \mapsto v(z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-\Delta_v}$.

VOA module : vector space M on which the v_n act subject to relations (OPEs + more).

Important restriction — annihilation condition:

$$\forall m \in M, \ \exists N \text{ s.t. } v_n m = 0 \ \ \forall n > N.$$

$$\text{ie} \quad v(z)m \in M[[z, z^{-1}]].$$

Examples:

- Every $\hat{\mathfrak{g}}$-module M with $K \mapsto k\cdot\mathrm{id}_M$ that satisfies the annihilation condition is a $V_k(\mathfrak{g})$-module. [OPEs $\leftrightarrow$ commutators!]
- The Verma modules $\mathcal{F}_p$ of $\hat{\mathfrak{gl}}_1$ are $V_k(\mathfrak{gl}_1)$-mods.

$$\mathcal{F}_p = \mathrm{Ind}_{\hat{\mathfrak{gl}}_1^{\geq}}^{\hat{\mathfrak{gl}}_1} (\mathbb{C}\Omega_p), \qquad a_0\Omega_p = p\Omega_p, \ K\Omega_p = k\Omega_p, \ a_n\Omega_p = 0 \ \forall n > 0.$$

- $L_1(\mathfrak{sl}_2) = V_1(\mathfrak{sl}_2)/\langle e_{-1}^2\Omega\rangle \leftarrow$ extra relations!

$$e_{-1}^2\Omega \longmapsto :ee:(z) = \sum_{n\in\mathbb{Z}} \underbrace{\left[\sum_{r\leq -1} e_r e_{n-r} + \sum_{r\geq 0} e_{n-r} e_r\right]}_{:ee:_n} z^{-n-2}$$

If $V_n m = 0$ $\forall n>0$, then

$$\begin{array}{ccc} \cdots & m & \cdots \\ e_{-1}m & h_{-1}m & f_{-1}m \\ \vdots & \vdots & \vdots \end{array}$$

$$0 = :ee:_0 m = e_0^2 m.$$

So the only (irreducible highest-weight) $L_1(\mathfrak{sl}_2)$-mods are the Weyl modules $\mathcal{L}_0 (\simeq L_1(\mathfrak{sl}_2))$ and $\mathcal{L}_1$:

$$\begin{array}{ccc} & \Omega & \\ f_{-1}\Omega & h_{-1}\Omega & e_{-1}\Omega \\ \vdots & \vdots & \vdots \\ & \mathcal{L}_0 & \end{array} \qquad \begin{array}{cc} \Omega_{-1} & \Omega_1 \\ h_{-1}\Omega_{-1} & h_{-1}\Omega_1 \\ \vdots & \vdots \\ \multicolumn{2}{c}{\mathcal{L}_1} \end{array}$$

Categories:

- $L_1(\mathfrak{sl}_2)$-mod is semisimple with finitely many irreducibles (up to $\simeq$).

  It is also tensor (fusion product), braided and rigid!

$$\mathcal{L}_0 \otimes \mathcal{L}_0 \simeq \mathcal{L}_0, \quad \mathcal{L}_0 \otimes \mathcal{L}_1 \simeq \mathcal{L}_1, \quad \mathcal{L}_1 \otimes \mathcal{L}_1 \simeq \mathcal{L}_0.$$

  It is also <u>modular</u>:

  * characters $\mathrm{ch}[\mathcal{L}_\lambda] = \mathrm{tr}_{\mathcal{L}_\lambda} e^{2\pi i \zeta h_0} e^{2\pi i \tau (L_0 - c/24)}$

    are known explicitly.

* $(\mathrm{ch}[\mathcal{L}_0], \mathrm{ch}[\mathcal{L}_1])$ is a vector-valued modular form:

$$\mathrm{ch}[\mathcal{L}_\lambda](\zeta/\tau \,|\, {-1/\tau}) = \sum_\mu \boxed{S_{\lambda\mu}}\, \mathrm{ch}[\mathcal{L}_\mu](\zeta|\tau)$$
$$\mathrm{ch}[\mathcal{L}_\lambda](\zeta \,|\, \tau+1) = \sum_\mu \boxed{T_{\lambda\mu}}\, \mathrm{ch}[\mathcal{L}_\mu](\zeta|\tau)$$

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} e^{-\pi i/12} & 0 \\ 0 & e^{-5\pi i/12} \end{pmatrix}.$$

* The multiplicities of the fusion product are given by the Verlinde formula:

$$N_{11}{}^{0} = \sum_\lambda \frac{S_{1\lambda} S_{1\lambda} S_{0\lambda}^*}{S_{0\lambda}} = S_{10}^2 + S_{11}^2 = \frac{1}{2} + \frac{1}{2} = 1,$$
$$N_{11}{}^{1} = \sum_\lambda \frac{S_{1\lambda} S_{1\lambda} S_{1\lambda}^*}{S_{0\lambda}} = \frac{S_{10}^3}{S_{00}} + \frac{S_{11}^3}{S_{01}} = \frac{1}{2} - \frac{1}{2} = 0$$

$$\Rightarrow \mathcal{L}_1 \otimes \mathcal{L}_1 = \bigoplus_\lambda N_{11}{}^{\lambda} \mathcal{L}_\lambda = \mathcal{L}_0. \checkmark$$

- $V_k(\mathfrak{gl}_1)-\mathrm{wtmod}_{\mathbb{R}}$ is semisimple with an uncountably infinite number of irreducibles:

$$\mathcal{F}_p, \; p \in \mathbb{R} \qquad \begin{array}{ccc} & \Omega_p & \\ & a_{-1}\Omega_p & \\ a_{-1}^2\Omega_p & & a_{-2}\Omega_p \\ \vdots & & \vdots \end{array}$$

It is nevertheless tensor, braided and rigid:

$$\mathcal{F}_p \otimes \mathcal{F}_q \simeq \mathcal{F}_{p+q}.$$

It is also modular!

$*$ $$\mathrm{ch}[\mathcal{F}_p] = \mathrm{tr}_{\mathcal{F}_p} e^{2\pi i \zeta a_0} e^{2\pi i \tau (L_0 - 1/24)}$$
$$= \frac{e^{2\pi i \zeta p} e^{\pi i \tau p^2}}{\eta(\tau)}.$$

$*$ $$\mathrm{ch}[\mathcal{F}_p](\zeta/\tau \,|\, -1/\tau) = \int_{\mathbb{R}} S_{pq}\, \mathrm{ch}[\mathcal{F}_q](\zeta|\tau)\, dq$$
$$\mathrm{ch}[\mathcal{F}_p](\zeta|\tau+1) = \int_{\mathbb{R}} T_{pq}\, \mathrm{ch}[\mathcal{F}_q](\zeta|\tau)\, dq$$

$$S_{pq} = e^{-2\pi i pq}, \quad T_{pq} = e^{\pi i (p^2 - 1/12)} \delta(p-q).$$

$*$ $$N_{pq}{}^{r} = \int_{\mathbb{R}} \frac{S_{p\sigma} S_{q\sigma} S^*_{r\sigma}}{S_{0\sigma}}\, d\sigma$$
$$= \int_{\mathbb{R}} e^{2\pi i (r - p - q)\sigma}\, d\sigma$$
$$= \delta(r-p-q)$$

$$\Rightarrow \mathcal{F}_p \otimes \mathcal{F}_q \cong \oint_{\mathbb{R}} N_{pq}{}^{r}\, \mathcal{F}_r\, dr = \mathcal{F}_{p+q}. \checkmark$$

My research:

I study $L_k(\mathfrak{g})\text{-wtmod}_{\mathbb{R}}$ (and related examples like W-algebras), especially when the category is nonsemisimple & nonfinite!

It is however still expected to be tensor, braided, rigid and modular.

But even its abelian structure is hard...