Exploring higher-rank logarithmic vertex operator algebras

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Representation theory, Vertex and Chiral Algebras
IMPA

[Joint works with various subsets of D Adamović, T Creutzig, Z Fehily, K Kawasetsu, M Rupert and S Wood.]

Outline

Motivation

I want to understand the representation theory of vertex operator algebras...

rational	factorisable	non-rational
	[positive lattices] [free fermions] [bc-ghosts] [Virasoro minimal models] [compact WZW models] [a few W-algebras]	[free bosons] [Liouville?] [non-compact WZW?]
discrete		continuous
	[symplectic fermions] [triplet models] [polymers, percolation?] [log minimal models?] [SLE?]	[$\beta\gamma$-ghosts] [supergroup WZW] [Nappi–Witten] [fractional-level WZW] [most W-algebras] [spin chains?]
log-rational	logarithmic	generic

A (strongly) rational VOA has a module category that is

- semisimple: modules are completely reducible,
- finite: there are finitely many irreducibles (up to $\cong$),
- q-finite: modules have q-characters $(\operatorname{tr} \mathsf{q}^{L_0 - \mathsf{c}/24})$.

Generalising to the log-rational setting, we lose semisimplicity but keep both finiteness conditions.

However, there aren't many easily accessible examples beyond symplectic fermions (and friends).

Lie-theoretic VOAs usually have even weight-1 fields. In all but a few cases, these break C_2-cofiniteness (*cf.* the free boson).

One is therefore led to explore accessible examples of VOAs with nonsemisimple and nonfinite module categories.

Today: the admissible-level VOAs associated with $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$...

Ancient history

Our story begins in 1986, with the celebrated coset construction of the unitary Virasoro minimal model VOAs [Goddard–Kent–Olive]:

$$\mathsf{M}(\mathsf{k}+2,\mathsf{k}+3) \cong \mathrm{Com}\big(\mathsf{L}_{\mathsf{k}+1}(\mathfrak{sl}_2) \hookrightarrow \mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2) \otimes \mathsf{L}_1(\mathfrak{sl}_2)\big), \quad \mathsf{k} \in \mathbb{N}.$$

Kent asked: does this extend to the nonunitary minimal models?

For $\mathsf{M}(\mathsf{u},\mathsf{u}+\mathsf{v})$, $\mathsf{v} > 1$, this would require making sense of

$$\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2) \text{ with } \mathsf{k}+2 = \frac{\mathsf{u}}{\mathsf{v}} \quad (\mathsf{u} \geqslant 2,\ \mathsf{v} \geqslant 1).$$

These are the admissible levels of [Kac–Wakimoto '88]. For these levels, category $\mathscr{O}_{\mathsf{k}}$ for $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ is semisimple and finite (but not q-finite).

Moreover, the irreducible characters are vector-valued modular forms, suggesting that these admissible-level models might be rational.

In [Verlinde '88], a formula for the fusion coefficients of a rational VOA in terms of the S-matrix was proposed:

$$\mathcal{N}_{ij}{}^{k} = \sum_{\ell} \frac{S_{i\ell} S_{j\ell} S_{k\ell}^{*}}{S_{0\ell}} \in \mathbb{N}.$$

Subsequently, [Moore–Seiberg '88] showed (modulo strong assumptions) that Verlinde's formula follows from categorical self-consistency.

This was eventually proven for all rational VOAs in [Huang '04].

But, it doesn't hold for the admissible-level $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ theories (with $\mathsf{k} \notin \mathbb{N}$): there are always negative fusion coefficients [Koh–Sorba '88].

Much work ensued [BF '90,MW '90,AY '92,R '93,FM '93,A '95,PRY '96,FGP '96,...], mostly in the physics literature, but with no resolution.

In their textbook, [Di Francesco–Mathieu–Sénéchal '97, §18.6] refer to these "fractional-level WZW models" as having an "intrinsic sickness" that needs curing.

$\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ is not rational

Of course, these admissible-level models are just not rational. At the level of modules, this was already established in [Adamović–Milas '95], where $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$, $\mathsf{k} \notin \mathbb{N}$, was shown to admit infinitely many irreducible modules.

[Feigin–Semikhatov–Tipunin '97] rediscovered this infinitude in relation to the Kazama–Suzuki duality with $N = 2$ minimal models. They dubbed them relaxed highest-weight modules and added their spectral flows to the mix.

[Maldacena–Ooguri '00] made relaxed modules and spectral flows the centrepiece of their proposal for the $\mathsf{SL}(2, \mathbb{R})$ WZW model spectrum.

[Gaberdiel '01] proved that for $\mathsf{k} = -\frac{4}{3}$, the category $\mathscr{O}_\mathsf{k}$ is not closed under fusion. Any tensor category involving the highest-weight modules must include relaxed modules, their spectral flows and logarithmic modules.

[DR '10] extended this to $\mathsf{k} = -\frac{1}{2}$, motivated by links to the $\mathsf{c} = -2$ singlet and triplet models:

$$\mathsf{L}_{-1/2}(\mathfrak{sl}_2) \xrightarrow[\text{coset}]{\text{parafermion}} \mathsf{Sing}(1, 2) \xrightarrow[\text{extension}]{\text{simple current}} \mathsf{Trip}(1, 2).$$

A rank-1 story

Call a VOA logarithmic if it admits logarithmic modules, *ie.* ones on which the Virasoro zero mode L_0 acts nonsemisimply.

The best understood logarithmic VOAs are somehow related to $\mathfrak{sl}_2$ — we call them rank-1 logarithmic VOAs.

Here, we review broad features of their module categories by focusing on the examples $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$, with k admissible (but not integral). This includes:

- Relaxed highest-weight modules;
- Spectral flows;
- Logarithmic (aka staggered) modules;
- Modular transformations;
- (Grothendieck) fusion rules;
- Inverse quantum hamiltonian reduction;
- Logarithmic Kazhdan–Lusztig correspondences.

Relaxed highest-weight modules

Textbook examples of VOA modules tend to be irreducible and highest-weight. More generally, we need relaxed highest-weight modules.

Let:

- $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ in $\mathfrak{sl}_2$.
- $\widehat{\mathfrak{sl}}_2 = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ (the affinisation).
- $j_n = j \otimes t^n$, for all $j \in \mathfrak{sl}_2$.

Then, $\widehat{\mathfrak{sl}}_2$ has a generalised triangular decomposition:

$$\widehat{\mathfrak{sl}}_2 = \underbrace{\widehat{\mathfrak{sl}}_2^{<}}_{=\mathfrak{sl}_2 \otimes t^{-1}\mathbb{C}[t^{-1}]} \oplus \underbrace{\widehat{\mathfrak{sl}}_2^{0}}_{=\mathfrak{sl}_2 \oplus \mathbb{C}K} \oplus \underbrace{\widehat{\mathfrak{sl}}_2^{>}}_{=\mathfrak{sl}_2 \otimes t\mathbb{C}[t]}.$$

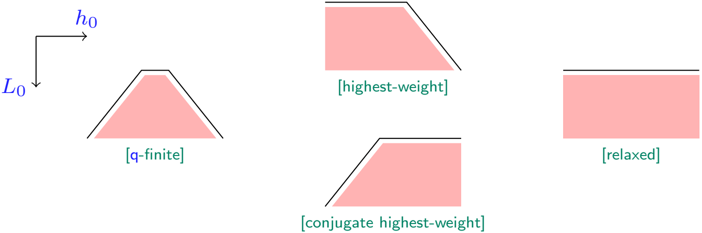
A relaxed highest-weight $\widehat{\mathfrak{sl}}_2$-vector is then a simultaneous eigenvector of h_0 and K that is annihilated by $\widehat{\mathfrak{sl}}_2^{>}$. A relaxed highest-weight $\widehat{\mathfrak{sl}}_2$-module is then a module generated by a single relaxed highest-weight vector.

This restricts to $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-modules ($K$ acts as multiplication by k).

The relaxed $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-modules include highest-weight modules (e_0 annihilates) and their "conjugates" (f_0 annihilates). But, one can have a "top space" without a highest-weight or lowest-weight $\mathfrak{sl}_2$-vector.

The irreducible relaxed highest-weight $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-modules were classified in [Adamović–Milas '95, DR–Wood '15]. With $\mathsf{k}+2=\frac{\mathsf{u}}{\mathsf{v}}$, there are

- $\mathsf{u}-1$ q-finite, irreducible highest-weight modules.
- $(\mathsf{u}-1)(\mathsf{v}-1)$ non-q-finite, irreducible highest-weight modules.
- $(\mathsf{u}-1)(\mathsf{v}-1)$ non-q-finite, irreducible, conjugate highest-weight modules.
- $\frac{1}{2}(\mathsf{u}-1)(\mathsf{v}-1)$ 1-parameter families of other non-q-finite, generically irreducible, relaxed highest-weight modules.



Spectral flow twists

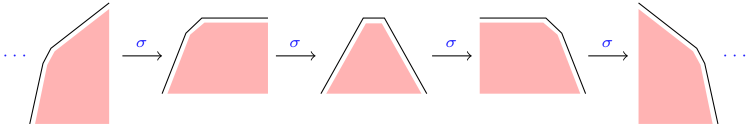
The "spectral flow" automorphisms σ^ℓ, $\ell \in \mathbb{Z}$, of $\widehat{\mathfrak{sl}}_2$ are defined by

$$\sigma^\ell(e_n) = e_{n-\ell}, \quad \sigma^\ell(h_n) = h_n - \delta_{n,0}\ell K, \quad \sigma^\ell(f_n) = f_{n+\ell}, \quad \sigma^\ell(K) = K.$$

We also have $\sigma^\ell(L_0) = L_0 - \frac{1}{2}\ell h_0 + \frac{1}{4}\ell^2 K$.

Since these automorphisms preserve the Cartan subalgebra $\mathbb{C}h_0 \oplus \mathbb{C}K$, twisting by them defines invertible functors on the category of weight $\widehat{\mathfrak{sl}}_2$-modules.

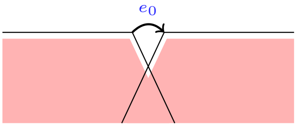
These functors map a $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-module to another: $\mathcal{M} \xrightarrow{\sigma^\ell} \mathcal{M}^\ell \equiv \sigma^\ell(M)$. But, they do not (for $\mathsf{k} \notin \mathbb{N}$) preserve the property of being (relaxed) highest-weight.



It follows from [Futorny–Tsylke '01] that these exhaust the irreducibles in the category $\mathscr{W}_\mathsf{k}$ of weight $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-modules.

Nonsemisimplicity

The category $\mathscr{W}_{\mathsf{k}}$ of weight $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-modules is not semisimple (for $k \notin \mathbb{N}$), *eg.* the 1-parameter relaxed families include nonsplit extensions:



However, $\mathscr{W}_{\mathsf{k}}$ is "almost semisimple": almost all members of these families are irreducible, projective and injective.

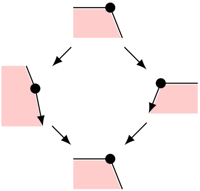
In fact, $\mathscr{W}_{\mathsf{k}}$ decomposes into an uncountably infinite number of semisimple blocks and a finite number of nonsemisimple blocks.

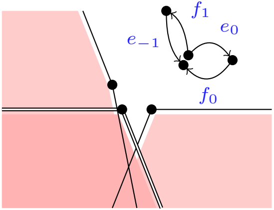
As per [Kac '77], we call these blocks (and their constituent modules) typical (semisimple) and atypical (nonsemisimple). The modules in the 1-parameter families, and their spectral flows, are called the standard modules.

The vacuum module of $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ is atypical.

The atypical blocks of $\mathscr{W}_{\mathsf{k}}$ are naturally harder to understand.

There exist [Gaberdiel '01, Adamović–Milas '09, DR '10, Adamović '17] (indecomposable) logarithmic $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-modules, one for every irreducible atypical module.





L_0 acts with rank-2 Jordan blocks on these modules (but h_0 acts semisimply).

They are the projective covers (and injective hulls) of the atypical irreducibles. [Arakawa–Creutzig–Kawasetsu '??] With respect to the standard $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-modules, they are also tilting modules satisfying BGG reciprocity.

Modularity

For rational VOAs, the irreducible characters (or one-point functions) span a (finite-dimensional) representation of the modular group $\mathsf{SL}(2;\mathbb{Z})$ [Zhu '96].

Despite [Kac–Wakimoto '88], $\mathscr{O}_\mathsf{k}$ is not modular for $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$. The irreducible characters have poles that cause [DR '08] the Verlinde formula to fail.

However, $\mathscr{W}_\mathsf{k}$ is modular! [Creutzig–DR '13]

- The characters of the standard modules are distributions supported at the poles of the Kac–Wakimoto characters.
- They form a (topological) basis for the span of all the characters of $\mathscr{W}_\mathsf{k}$.
- $\mathsf{SL}(2;\mathbb{Z})$ acts on this basis by integral operators (*cf.* Fourier transforms).
- The T-operator is diagonal and unitary. The S-operator is symmetric and unitary; its square is conjugation (à la rational VOAs).
- The obvious generalisation of the Verlinde formula gives nonnegative integer Grothendieck fusion coefficients.
- These fusion coefficients reproduce the Grothendieck fusion rules calculated, for $\mathsf{k} = -\frac{4}{3}$ and $-\frac{1}{2}$, using the NGK algorithm. [Gaberdiel '01, DR '10]

Inverse reduction

An unexpected feature of the modularity of $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ is that the standard characters turn out to be proportional to the irreducible characters of its quantum hamiltonian reduction: a rational Virasoro minimal model.

This means that the kernel of the S-operator of $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ is proportional to the S-matrix of its reduction. Consequently, the fusion coefficients of $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ are expressed in terms of those of its reduction.

This was eventually explained in [Adamović '17], where the standard modules of $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ were constructed as tensor products of irreducible Virasoro modules and modules over the rank-2 indefinite half-lattice VOA of [Berman–Dong–Tan '01].

We call this inverse quantum hamiltonian reduction.

[Adamović '17] also constructed logarithmic $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-modules. However, extracting the full structure of these modules this way remains challenging...

Kazhdan–Lusztig correspondences

Modularity suggests uniform structures for the logarithmic $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-modules.

Explicit constructions for "small" k, *eg.* $\mathsf{k} = -\frac{1}{2}, -\frac{4}{3}$, then yield good conjectures for these structures.

Another source of "small-k" conjectures comes from a logarithmic Kazhdan–Lusztig correspondence with a quantum group at root of unity.

Indeed, the logarithmic structures of $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ and those of its parafermionic coset $\mathrm{Com}\big(\mathsf{H}_1 \hookrightarrow \mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)\big)$ are the same. [Creutzig–Kanade–Linshaw–DR '16]

For $\mathsf{k} = -\frac{1}{2}, -\frac{4}{3}$, the latter is the $\mathfrak{sl}_2$ singlet VOA for $p = 2, 3$. [Adamović '04, DR '10]

Finally, the singlet's weight category is (conjecturally) equivalent to that of the unrolled restricted quantum group of $\mathfrak{sl}_2$ at $q = \mathsf{e}^{\pi \mathsf{i}/p}$.
[DR '13, Costantino–Geer–Patureau-Mirand '14]

The upshot is "easy" quantum group calculations predict the precise structures of the logarithmic $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-modules (for small k). These can be extended to all admissible k using Verlinde computations.

A rank-2 story

A truism of Lie theory is that rank-1 is too easy and it isn't until one masters rank-2 that the generalisation to all cases becomes apparent.

The representation theory of rank-2 logarithmic VOAs is still in its infancy. Nevertheless, there are several tools available to make progress with these cases.

Here, we outline what has been said to date concerning the category $\mathscr{W}_{\mathsf{k}}$ of weight $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_3)$-modules when k is admissible (but not integral):

$$\mathsf{k}+3=\frac{\mathsf{u}}{\mathsf{v}} \quad (\mathsf{u}\geqslant 3,\ \mathsf{v}\geqslant 2).$$

The expectation is that, as with $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$, $\mathscr{W}_{\mathsf{k}}$ is again a (nonsemisimple, nonfinite) modular tensor category satisfying BGG reciprocity.

Relaxed highest-weight modules

The irreducible highest-weight $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_3)$-modules were classified in [Arakawa '12].

The relaxed classification was first achieved using explicit constructions involving Gelfand–Tsetlin combinatorics. [Arakawa–Futorny–Ramirez '16]

It also follows directly from the highest-weight classification using coherent families. [Mathieu '00, Kawasetsu–DR '19]

Either way, there are three classes of irreducibles:

- Doubly atypical — a finite number of highest-weight modules.
- Singly atypical — a finite number of 1-parameter families of "semirelaxed" highest-weight modules (their top spaces are dense in one root direction).
- Typical — a finite number of 2-parameter families of "fully relaxed" highest-weight modules (their top spaces are dense in all root directions).

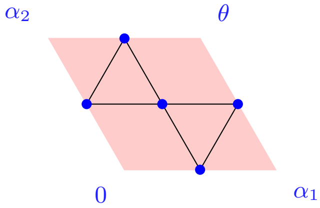
Curiously, there are many families of $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_3)$-modules with infinite-dimensional weight spaces.

Every irreducible in $\mathscr{W}_\mathsf{k}$ is the spectral flow of one of these. [Futorny–Tsylke '01]
Spectral flow automorphisms are indexed by the coweight lattice of $\mathfrak{sl}_3$.

Each fully relaxed family degenerates, for a codimension-1 subset of parameters, into a number of semirelaxed families. These, in turn, degenerate into highest-weight modules on a codimension-2 subset.

A fundamental domain of $\mathfrak{h}^*/\mathsf{Q}$:

- (pink square) = typical
- — = singly atypical,
- • = doubly atypical.



At degenerate parameter values, the relaxed (semirelaxed) modules are reducible but indecomposable. However, they are not logarithmic.

The composition factors of the degenerate reducible modules are known explicitly, as are the spectral flow orbits (these are crucial for modularity). [Kawasetsu–DR–Wood '21]

Going further

The next step would be to understand the logarithmic modules. However, even conjecturing their structures is extremely difficult.

Nevertheless, an explicit construction of some logarithmic modules was recently achieved in [Adamović–Creutzig–Genra '21] using inverse quantum hamiltonian reduction.

Some even have L_0 acting with Jordan blocks of rank 3.

It isn't clear that the structure of the modules can be elucidated from this construction. But, other conjectural methods can be used to make predictions.

In particular, $\mathscr{W}_{\mathsf{k}}$ has been shown to be modular when $\mathsf{k} = -\frac{3}{2}$.

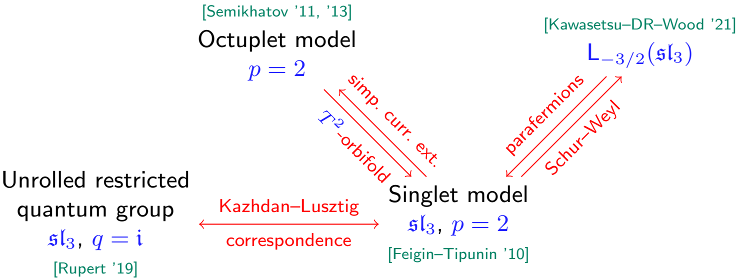
Again, the characters of the standard (fully relaxed) $\mathsf{L}_{-3/2}(\mathfrak{sl}_3)$-modules form a topological basis and the Verlinde formula gives nonnegative fusion coefficients. [Kawasetsu–DR–Wood '21]

The latter predicts the composition factors of fusion products. If $\mathscr{W}_{\mathsf{k}}$ is rigid, then one can identify products that are projective and even logarithmic.

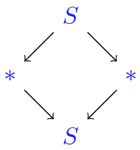
For $\mathsf{k} = -\frac{3}{2}$, we can also appeal to a (conjectural) Kazhdan–Lusztig correspondence because the parafermionic coset $\mathrm{Com}\big(\mathsf{H}_2 \hookrightarrow \mathsf{L}_{-3/2}(\mathfrak{sl}_3)\big)$ is the $\mathfrak{sl}_3$ singlet algebra with $p = 2$ [Adamović–Milas–Wang '20].

The weight module category of the latter is conjecturally equivalent to that of an unrolled restricted quantum group with $q = \mathfrak{i}$. [Creutzig–Rupert '20]

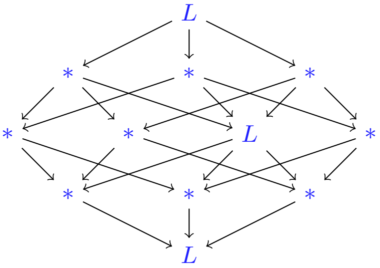
Given the results of [Kawasetsu–DR–Wood '21], this equivalence can be written down explicitly. The structures of the quantum group's projective modules then imply conjectures for those of $\mathscr{W}_{-3/2}$. [Creutzig–DR–Rupert '19]



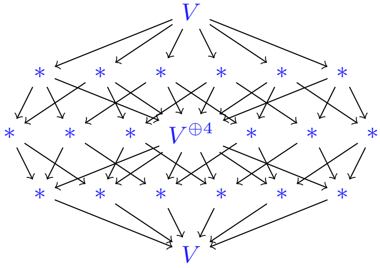
The conjectural Loewy diagrams for the (reducible) projectives of $\mathscr{W}_{-3/2}$ are



projective cover of
semirelaxed irreducible



projective cover of
highest-weight irreducible



projective cover of
vacuum module

The $*$ are all known in terms of spectral flows and D_6-twists of irreducibles. All projectives are self-dual and BGG reciprocity holds.

[Creutzig–DR–Rupert '21]

Note that the Jordan blocks of L_0 have rank at most 3, as per [Adamović–Creutzig–Genra '21].

Outlook

So... where to now?

- Obviously, we need to prove all our conjectures...
- Ideally, we'd like a general theory that proves the Verlinde formula and establishes a theory of (non-C_1-cofinite!) vertex tensor categories for quite general logarithmic VOAs.
- This will surely require many more examples, particularly higher-rank ones, to glean further insights into the general structure.
- One issue with higher-rank examples is the plethora of classes of irreducibles between highest-weight and fully relaxed. Their role in relation to standard modules needs to be pinned down.
- In affine and W-algebraic examples, the structure of the weight category should be controlled by nilpotent orbit classifications.
- Finally, I'd like to know if there are any log-rational examples which can't be realised as simple current extensions of models like those discussed here.

"*Only one who attempts the absurd is capable of achieving the impossible.*"

— Miguel de Unamuno