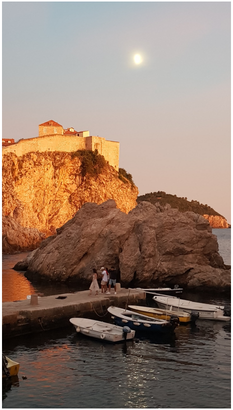
Inverse quantum hamiltonian reduction: a primer

David Ridout
University of Melbourne

June 26, 2023
Representation theory XVIII
Dubrovnik, Croatia

[arXiv:2007.00396, 2303.03713 [math.QA]]



Outline

1. Ancient history
2. Universal inverse QHR
3. Almost irreducibility
4. Simple inverse QHR
5. Classifying irreducibles
6. Outlook

History / Motivation

Once upon a time, I decided to solve the problem [Koh-Sorba'88] of why the Verlinde formula failed for $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$, for fractional k,

$$\textit{ie.,} \quad \mathsf{k} = -2 + \frac{\mathsf{u}}{\mathsf{v}}, \ \mathsf{u},\mathsf{v} \geqslant 2, \ (\mathsf{u},\mathsf{v}) = 1.$$

The cause was simply sloppiness with convergence regions [DR'08].

But how can we fix the Verlinde formula?

History / Motivation

Once upon a time, I decided to solve the problem [Koh-Sorba'88] of why the Verlinde formula failed for $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$, for fractional k,

$$\textit{ie.,} \quad \mathsf{k} = -2 + \frac{\mathsf{u}}{\mathsf{v}}, \ \mathsf{u}, \mathsf{v} \geqslant 2, \ (\mathsf{u}, \mathsf{v}) = 1.$$

The cause was simply sloppiness with convergence regions [DR'08].

But how can we fix the Verlinde formula? Relaxed hw modules!

- These were classified in [Adamović–Milas'95].
- They are necessary ingredients of the $(N=2)_\mathsf{k}$–$\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ Kazama–Suzuki correspondence [Feigin–Semikhatov–Tipunin'97].
- They naturally arise in fusion products of hw modules [Gaberdiel'01].
- They are necessary to get a full coset spectrum, *eg.* [DR'10]

$$\mathsf{L}_{-1/2}(\mathfrak{sl}_2) \xrightarrow{\text{H-coset}} \mathsf{Sing}(2) \xrightarrow{\text{s.c.ext.}} \mathsf{Trip}(2).$$

These modules indeed fix the Verlinde formula [Creutzig–DR'12,'13]. While verifying this, we needed to compute their characters.

The relaxed triangular decomposition of $\widehat{\mathfrak{sl}}_2 = \widehat{\mathfrak{sl}}_2^< \oplus \widehat{\mathfrak{sl}}_2^0 \oplus \widehat{\mathfrak{sl}}_2^>$ has

$$\widehat{\mathfrak{sl}}_2^< = \mathrm{span}\{E_n, F_n, H_n \ : \ n \in \mathbb{Z}_{<0}\}, \quad \widehat{\mathfrak{sl}}_2^0 = \mathrm{span}\{E_0, F_0, H_0, K\}$$
$$\widehat{\mathfrak{sl}}_2^> = \mathrm{span}\{E_n, F_n, H_n \ : \ n \in \mathbb{Z}_{>0}\}, \quad \widehat{\mathfrak{h}} = \mathrm{span}\{H_0, K\}.$$

A relaxed hw vector is an eigenvector of $\widehat{\mathfrak{h}}$ that is annihilated by $\widehat{\mathfrak{sl}}_2^>$.
A relaxed hw module is one that it is generated by a relaxed hw vector.

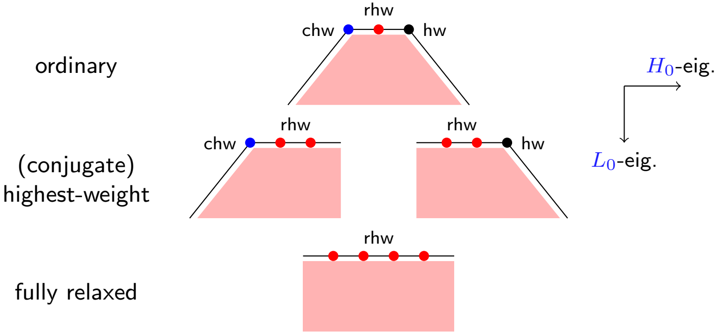
The relaxed triangular decomposition of $\widehat{\mathfrak{sl}}_2 = \widehat{\mathfrak{sl}}_2^< \oplus \widehat{\mathfrak{sl}}_2^0 \oplus \widehat{\mathfrak{sl}}_2^>$ has

$$\widehat{\mathfrak{sl}}_2^< = \operatorname{span}\{E_n, F_n, H_n \ : \ n \in \mathbb{Z}_{<0}\}, \quad \widehat{\mathfrak{sl}}_2^0 = \operatorname{span}\{E_0, F_0, H_0, K\}$$

$$\widehat{\mathfrak{sl}}_2^> = \operatorname{span}\{E_n, F_n, H_n \ : \ n \in \mathbb{Z}_{>0}\}, \quad \widehat{\mathfrak{h}} = \operatorname{span}\{H_0, K\}.$$

A relaxed hw vector is an eigenvector of $\widehat{\mathfrak{h}}$ that is annihilated by $\widehat{\mathfrak{sl}}_2^>$.
A relaxed hw module is one that it is generated by a relaxed hw vector.

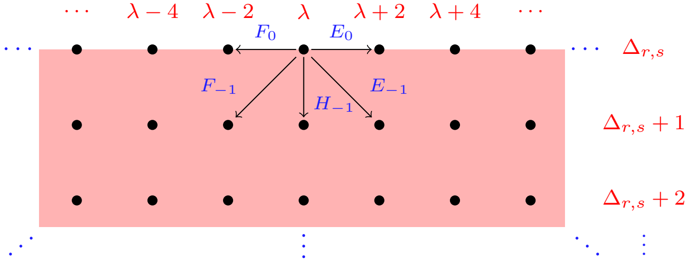
A hw module is thus a relaxed hw module, but there are more...



For fractional k, the fully relaxed $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-irreps are parametrised by the $\mathfrak{sl}_2$-wt (mod $2\mathbb{Z}$) and conformal wt of the top space [Adamović–Milas'95, DR–Wood'15]:

$$\mathcal{R}_{[\lambda];r,s}, \quad [\lambda] \in \mathbb{C}/2\mathbb{Z},\ r = 1, \dots, \mathsf{u}-1,\ s = 1, \dots, \mathsf{v}-1$$

$$\text{and} \quad \Delta_{r,s} = \frac{(\mathsf{v}r - \mathsf{u}s)^2 - \mathsf{v}^2}{4\mathsf{uv}}.$$



The characters of the $\mathcal{R}_{[\lambda];r,s}$ are [Creutzig–DR'13, Kawasetsu–DR'18]

$$\mathrm{ch}\big[\mathcal{R}_{[\lambda];r,s}\big](z;q) = \frac{\chi^{\mathsf{Vir.}}_{r,s}(\mathsf{q})}{\eta(\mathsf{q})^2} \sum_{n\in\mathbb{Z}} \mathsf{z}^{\lambda+2n},$$

where $\chi^{\mathsf{Vir.}}_{r,s}$ is the character of the hw irrep $\mathcal{L}_{r,s}$ of the Virasoro minimal model vertex operator algebra $\mathsf{Vir}_{\mathsf{u},\mathsf{v}}$.

$\mathsf{Vir}_{\mathsf{u},\mathsf{v}}$ is of course the quantum hamiltonian reduction of $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$. And the hw $\mathsf{Vir}_{\mathsf{u},\mathsf{v}}$-irrep is the "−" QHR of the hw $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-irrep of hw

$$\lambda_{r,s} = r - 1 - \tfrac{\mathsf{u}}{\mathsf{v}} s.$$

The characters of the $\mathcal{R}_{[\lambda];r,s}$ are [Creutzig–DR'13, Kawasetsu–DR'18]

$$\mathrm{ch}\left[\mathcal{R}_{[\lambda];r,s}\right](z;q) = \frac{\chi^{\mathsf{Vir.}}_{r,s}(\mathsf{q})}{\eta(\mathsf{q})^2} \sum_{n\in\mathbb{Z}} \mathsf{z}^{\lambda+2n},$$

where $\chi^{\mathsf{Vir.}}_{r,s}$ is the character of the hw irrep $\mathcal{L}_{r,s}$ of the Virasoro minimal model vertex operator algebra $\mathsf{Vir}_{\mathsf{u},\mathsf{v}}$.

$\mathsf{Vir}_{\mathsf{u},\mathsf{v}}$ is of course the quantum hamiltonian reduction of $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$. And the hw $\mathsf{Vir}_{\mathsf{u},\mathsf{v}}$-irrep is the "$-$" QHR of the hw $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-irrep of hw

$$\lambda_{r,s} = r - 1 - \tfrac{\mathsf{u}}{\mathsf{v}} s.$$

A beautiful observation deserves a beautiful explanation and one was subsequently found in [Adamović'17].

We call this explanation inverse quantum hamiltonian reduction.

Inverse quantum hamiltonian reduction

Recall that quantum hamiltonian reduction is a functor mapping $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)$-modules to $\mathsf{Vir}^{\mathsf{k}}$-modules:

$$\mathcal{M} \longmapsto \mathsf{H}^0_{BRST}(\mathcal{M} \otimes \mathsf{bc}, Q).$$

It is not invertible (it has a nonzero kernel).

The idea behind inverse QHR goes back to [Semikhatov'94]. The screening operators of the Wakimoto and Feigin–Fuchs free field realisations

$$\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2) \xhookrightarrow{\text{Wak.}} \beta\gamma \otimes \mathsf{H} \quad \text{and} \quad \mathsf{Vir}^{\mathsf{k}} \xhookrightarrow{\text{FF}} \mathsf{H}$$

are compatible, once the ghosts have been “bosonised” (*cf.* Fehily's talk tomorrow) à la [Friedan–Martinec–Shenker'86]:

$$\beta\gamma \xhookrightarrow{\text{FMS}} \Pi.$$

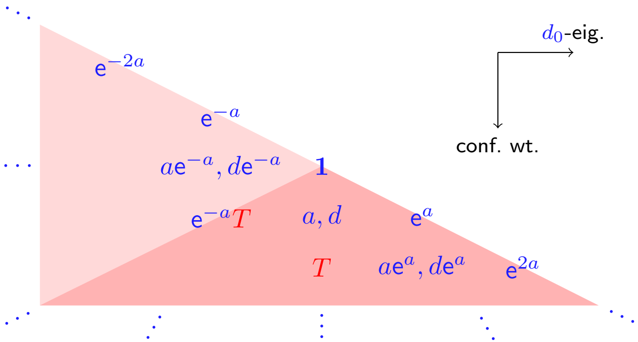
The upshot is an embedding

$$\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2) \longhookrightarrow \Pi \otimes \mathsf{Vir}^{\mathsf{k}}.$$

$\Pi = \langle a, d, \mathsf{e}^{na} \ : \ n \in \mathbb{Z} \rangle$ is a partial compactification of 2 free bosons:

$$a(z)a(w) \sim d(z)d(w) \sim a(z)\mathsf{e}^{na}(w) \sim \mathsf{e}^{ma}(z)\mathsf{e}^{na}(w) \sim 0,$$
$$a(z)d(w) \sim \frac{2}{(z-w)^2}, \quad d(z)\mathsf{e}^{na}(w) \sim \frac{2n\,\mathsf{e}^{na}(w)}{z-w}.$$

The embedding $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2) \longrightarrow \Pi \otimes \mathsf{Vir}^{\mathsf{k}}$ is conformal if the wt of e^{na} is n.



Explicitly, $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2) \hookrightarrow \Pi \otimes \mathsf{Vir}^{\mathsf{k}}$ is given by

$$E \mapsto \mathsf{e}^{a} \otimes \mathbf{1}, \quad H \mapsto (\tfrac{1}{2}\mathsf{k}\, a + d) \otimes \mathbf{1},$$
$$F \mapsto (\mathsf{k}+2)\mathsf{e}^{-a} \otimes T - \tfrac{1}{4} {:}gg\, \mathsf{e}^{-a}{:} \otimes \mathbf{1} + \tfrac{1}{2}(\mathsf{k}+1){:}\partial g\, \mathsf{e}^{-a}{:} \otimes \mathbf{1},$$

where $g = \frac{1}{2}\mathsf{k}\, a - d$.

It follows that any $\mathsf{Vir}^{\mathsf{k}}$-module may be tensored with a Π-module and restricted to get a $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)$-module. This restricted tensoring is our inverse QHR functor (actually a collection of functors) [Adamović'17].

Explicitly, $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2) \longhookrightarrow \Pi \otimes \mathsf{Vir}^{\mathsf{k}}$ is given by

$$E \mapsto \mathsf{e}^{a} \otimes \mathbf{1}, \quad H \mapsto (\tfrac{1}{2}\mathsf{k}\, a + d) \otimes \mathbf{1},$$
$$F \mapsto (\mathsf{k}+2)\mathsf{e}^{-a} \otimes T - \tfrac{1}{4}{:}gg\,\mathsf{e}^{-a}{:} \otimes \mathbf{1} + \tfrac{1}{2}(\mathsf{k}+1){:}\partial g\,\mathsf{e}^{-a}{:} \otimes \mathbf{1},$$

where $g = \frac{1}{2}\mathsf{k}\, a - d$.

It follows that any $\mathsf{Vir}^{\mathsf{k}}$-module may be tensored with a Π-module and restricted to get a $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)$-module. This restricted tensoring is our inverse QHR functor (actually a collection of functors) [Adamović'17].

We propose to call them Adamović functors.

As Π admits no hw modules, but only fully relaxed modules, these functors naturally construct fully relaxed $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)$-modules!

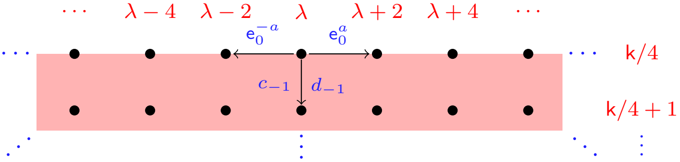
The life of Π

We are interested in the action of the Adamović functors on a $\mathsf{Vir}^{\mathsf{k}}$-irrep. Let $\mathcal{L}_\Delta$ denote the $\mathsf{Vir}^{\mathsf{k}}$-irrep of conformal wt Δ.

The relaxed hw Π-modules all have the form [Berman–Dong–Tan'01]

$$\Pi_{[\lambda]} = \Pi\, \mathsf{e}^{(\lambda/2+\mathsf{k}/4)a - d/2}, \quad [\lambda] \in \mathbb{C}/2\mathbb{Z}.$$

- Every relaxed hw Π-module is fully relaxed and irreducible.
- Every $\lambda \in [\lambda]$ corresponds to a relaxed hw vector in $\Pi_{[\lambda]}$.
- The coefficient of a is chosen so that the eigenvalue of $H_0 = \frac{1}{2}\mathsf{k}c_0 + d_0$ on the generator is λ.
- Other choices for the coefficient of d give spectral flow twists of $\Pi_{[\lambda]}$.



Almost irreducibility

Question: Is $\Pi_{[\lambda]} \otimes \mathcal{L}_\Delta$ irreducible as a $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)$-module?

$E_0 = \mathsf{e}_0^a \otimes \mathbf{1}$ acts injectively, but F_0 acts polynomially on the top space:

$$E_0 F_0 v_\lambda = \Big(\tfrac{1}{4}\big(\mathsf{k}(\mathsf{k}+2) - \lambda(\lambda-2)\big) + (\mathsf{k}+2)\Delta \Big) v_\lambda.$$

Thus, $\Pi_{[\lambda]} \otimes \mathcal{L}_\Delta$ is reducible for at least one $[\lambda] \in \mathbb{C}/2\mathbb{Z}$ because its top space contains a conjugate hw vector.

But, maybe it is irreducible for "almost all" $[\lambda]$...

Almost irreducibility

Question: Is $\Pi_{[\lambda]} \otimes \mathcal{L}_\Delta$ irreducible as a $\mathsf{V}^\mathsf{k}(\mathfrak{sl}_2)$-module?

$E_0 = \mathsf{e}_0^a \otimes \mathbf{1}$ acts injectively, but F_0 acts polynomially on the top space:

$$E_0 F_0 v_\lambda = \Big(\tfrac{1}{4}\big(\mathsf{k}(\mathsf{k}+2) - \lambda(\lambda-2)\big) + (\mathsf{k}+2)\Delta\Big) v_\lambda.$$

Thus, $\Pi_{[\lambda]} \otimes \mathcal{L}_\Delta$ is reducible for at least one $[\lambda] \in \mathbb{C}/2\mathbb{Z}$ because its top space contains a conjugate hw vector.

But, maybe it is irreducible for "almost all" $[\lambda]$...

Definition: An $\mathbb{N}$-graded module is almost irreducible if

- The top space generates the module.
- The top space has nonzero intersection with every nonzero submodule.

Almost-irreducibility is very natural for VOA module categories:

- Zhu's induction functor [Zhu'96] produces almost-irreducible modules.
- In fact, an almost-irreducible VOA module is irreducible iff its top space is irreducible for its Zhu algebra [Adamović–Kawasetsu–DR'20].
- The localisation of an irreducible hw VOA module (wrt an appropriate set of zero modes) produces an almost-irreducible fully relaxed VOA module (or 0) [Kawasetsu–DR'19].

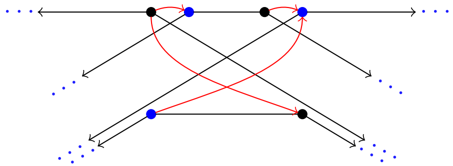
But, it doesn't mean that everything happens in the top space...

Almost-irreducibility is very natural for VOA module categories:

- Zhu's induction functor [Zhu'96] produces almost-irreducible modules.
- In fact, an almost-irreducible VOA module is irreducible iff its top space is irreducible for its Zhu algebra [Adamović–Kawasetsu–DR'20].
- The localisation of an irreducible hw VOA module (wrt an appropriate set of zero modes) produces an almost-irreducible fully relaxed VOA module (or 0) [Kawasetsu–DR'19].

But, it doesn't mean that everything happens in the top space...

eg., $\mathsf{V}^{-1}(\mathfrak{sl}_2)$ has almost-irreducible fully relaxed modules with structure



Theorem: [Adamović'17, Adamović–Kawasetsu–DR'20]

- $\Pi_{[\lambda]} \otimes \mathcal{L}_\Delta$ is almost irreducible as a $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)$-module.
- For fixed $\Delta \in \mathbb{C}$, it is irreducible for all but one or two $[\lambda] \in \mathbb{C}/2\mathbb{Z}$.

Proof:

1. Note that the explicit embedding $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2) \hookrightarrow \Pi \otimes \mathsf{Vir}^{\mathsf{k}}$ is "triangular".
2. Prove that $\Pi_{[\lambda]}$ is almost irreducible as a module over the subalgebra generated by $E \sim \mathsf{e}^a$ and $H \sim d$.
3. Use $F \sim T$ and the irreducibility of $\mathcal{L}_\Delta$ to conclude. ∎

Theorem: [Adamović'17, Adamović–Kawasetsu–DR'20]

- $\Pi_{[\lambda]} \otimes \mathcal{L}_\Delta$ is almost irreducible as a $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)$-module.
- For fixed $\Delta \in \mathbb{C}$, it is irreducible for all but one or two $[\lambda] \in \mathbb{C}/2\mathbb{Z}$.

Proof:

1. Note that the explicit embedding $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2) \hookrightarrow \Pi \otimes \mathsf{Vir}^{\mathsf{k}}$ is "triangular".
2. Prove that $\Pi_{[\lambda]}$ is almost irreducible as a module over the subalgebra generated by $E \sim \mathsf{e}^a$ and $H \sim d$.
3. Use $F \sim T$ and the irreducibility of $\mathcal{L}_\Delta$ to conclude. ∎

Corollary: For all but one or two $[\lambda] \in \mathbb{C}/2\mathbb{Z}$ and all $\Delta \in \mathbb{C}$:

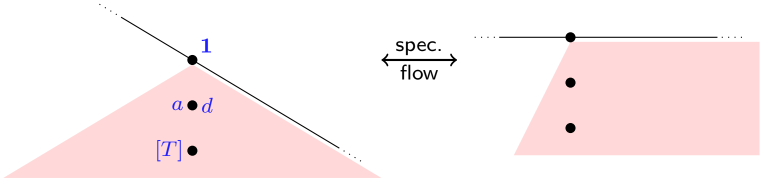
- $\Pi_{[\lambda]} \otimes \mathcal{L}_\Delta \cong \mathcal{R}_{[\lambda];\Delta}$.
- $\mathrm{ch}\big[\mathcal{R}_{[\lambda];\Delta}\big](z;q) = \dfrac{\chi_\Delta^{\mathsf{Vir.}}(\mathsf{q})}{\eta(\mathsf{q})^2} \sum_{n \in \mathbb{Z}} \mathsf{z}^{\lambda+2n}$.

Inverse QHR for $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$

Since $\mathsf{V}^\mathsf{k}(\mathfrak{sl}_2) \hookrightarrow \Pi \otimes \mathsf{Vir}^\mathsf{k} \twoheadrightarrow \Pi \otimes \mathsf{Vir}_\mathsf{k}$ is nonzero, it descends to an embedding of $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ iff the image is irreducible as a $\mathsf{V}^\mathsf{k}(\mathfrak{sl}_2)$-module.

Theorem: [Adamović'17] $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2) \hookrightarrow \Pi \otimes \mathsf{Vir}_\mathsf{k}$ iff $\mathsf{k} \notin \mathbb{N}$.

Proof: $\Pi \otimes \mathsf{Vir}_\mathsf{k}$ is a spectral flow of a fully relaxed $\mathsf{V}^\mathsf{k}(\mathfrak{sl}_2)$-module.

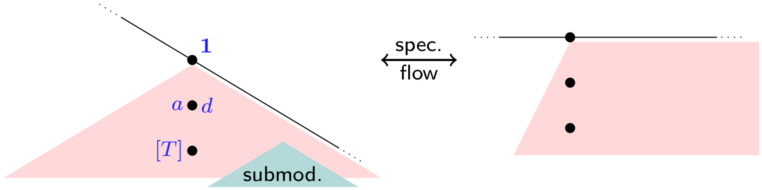


Inverse QHR for $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$

Since $\mathsf{V}^\mathsf{k}(\mathfrak{sl}_2) \hookrightarrow \Pi \otimes \mathsf{Vir}^\mathsf{k} \twoheadrightarrow \Pi \otimes \mathsf{Vir}_\mathsf{k}$ is nonzero, it descends to an embedding of $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ iff the image is irreducible as a $\mathsf{V}^\mathsf{k}(\mathfrak{sl}_2)$-module.

Theorem: [Adamović'17] $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2) \hookrightarrow \Pi \otimes \mathsf{Vir}_\mathsf{k}$ iff $\mathsf{k} \notin \mathbb{N}$.

Proof: $\Pi \otimes \mathsf{Vir}_\mathsf{k}$ is a spectral flow of a fully relaxed $\mathsf{V}^\mathsf{k}(\mathfrak{sl}_2)$-module.

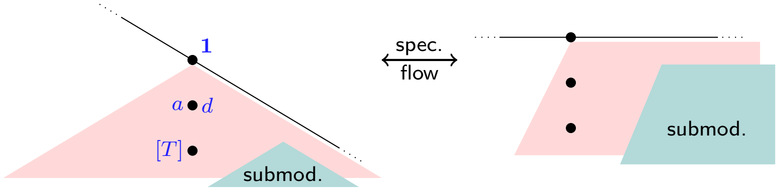


Inverse QHR for $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$

Since $\mathsf{V}^\mathsf{k}(\mathfrak{sl}_2) \hookrightarrow \Pi \otimes \mathsf{Vir}^\mathsf{k} \twoheadrightarrow \Pi \otimes \mathsf{Vir}_\mathsf{k}$ is nonzero, it descends to an embedding of $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ iff the image is irreducible as a $\mathsf{V}^\mathsf{k}(\mathfrak{sl}_2)$-module.

Theorem: [Adamović'17] $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2) \hookrightarrow \Pi \otimes \mathsf{Vir}_\mathsf{k}$ iff $\mathsf{k} \notin \mathbb{N}$.

Proof: $\Pi \otimes \mathsf{Vir}_\mathsf{k}$ is a spectral flow of a fully relaxed $\mathsf{V}^\mathsf{k}(\mathfrak{sl}_2)$-module.

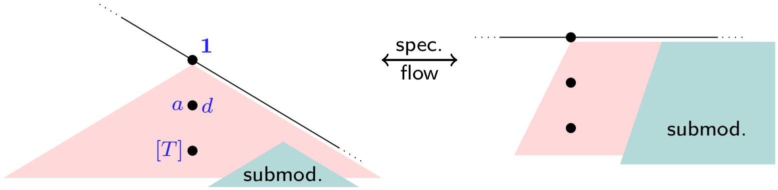


Inverse QHR for $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$

Since $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2) \hookrightarrow \Pi \otimes \mathsf{Vir}^{\mathsf{k}} \twoheadrightarrow \Pi \otimes \mathsf{Vir}_{\mathsf{k}}$ is nonzero, it descends to an embedding of $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ iff the image is irreducible as a $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)$-module.

Theorem: [Adamović'17] $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2) \hookrightarrow \Pi \otimes \mathsf{Vir}_{\mathsf{k}}$ iff $\mathsf{k} \notin \mathbb{N}$.

Proof: $\Pi \otimes \mathsf{Vir}_{\mathsf{k}}$ is a spectral flow of a fully relaxed $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)$-module.

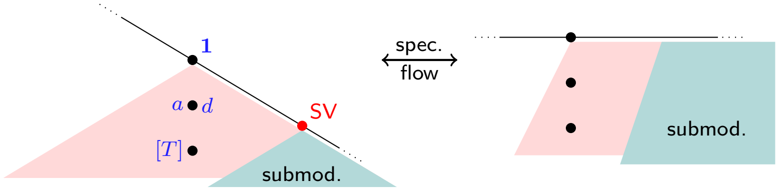


Inverse QHR for $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$

Since $\mathsf{V}^\mathsf{k}(\mathfrak{sl}_2) \hookrightarrow \Pi \otimes \mathsf{Vir}^\mathsf{k} \twoheadrightarrow \Pi \otimes \mathsf{Vir}_\mathsf{k}$ is nonzero, it descends to an embedding of $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ iff the image is irreducible as a $\mathsf{V}^\mathsf{k}(\mathfrak{sl}_2)$-module.

Theorem: [Adamović'17] $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2) \hookrightarrow \Pi \otimes \mathsf{Vir}_\mathsf{k}$ iff $\mathsf{k} \notin \mathbb{N}$.

Proof: $\Pi \otimes \mathsf{Vir}_\mathsf{k}$ is a spectral flow of a fully relaxed $\mathsf{V}^\mathsf{k}(\mathfrak{sl}_2)$-module.

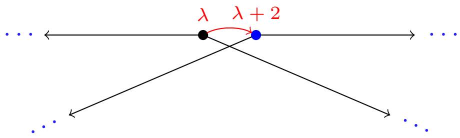


So, the image has a nonzero proper submodule iff $\mathsf{V}^\mathsf{k}(\mathfrak{sl}_2)$ has a singular vector of the form $E^n_{-1}\mathbf{1}$. ∎

Corollary: [Adamović–Kawasetsu-DR'20, Kawasetsu–DR'18]
For k fractional, all but two $[\lambda] \in \mathbb{C}/2\mathbb{Z}$ and all $r = 1, \dots, \mathsf{u}-1$ and $s = 1, \dots, \mathsf{v}-1$:

- $\Pi_{[\lambda]} \otimes \mathcal{L}_{\Delta_{r,s}} \cong \mathcal{R}_{[\lambda];r,s}$.
- $\mathrm{ch}\big[\mathcal{R}_{[\lambda];r,s}\big](z;q) = \dfrac{\chi^{\mathsf{Vir.}}_{r,s}(\mathsf{q})}{\eta(\mathsf{q})^2} \displaystyle\sum_{n\in\mathbb{Z}} \mathsf{z}^{\lambda+2n}$.

Moreover, for fixed r and s, the two reducible $\mathcal{R}_{[\lambda];r,s}$ have the structure



$(\lambda = \lambda_{r,s}, \lambda_{\mathsf{u}-r,\mathsf{v}-s})$.

The big takeaway here is for classifying fully relaxed $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-irreps:

Either we have a very simple singular vector in $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)$, in which case we can use Zhu technology, or we can use inverse QHR!

Classifications?

So for $k \notin \mathbb{N}$, inverse QHR constructs almost-irreducible fully relaxed $L_k(\mathfrak{sl}_2)$-modules from irreducible Vir_k-modules.

Does this construction produce all the fully relaxed irreducibles?

The answer is easily seen to be "yes" if we use the known classification. This classification is easy for generic k but not so easy for fractional k.

Classifications?

So for $\mathsf{k} \notin \mathbb{N}$, inverse QHR constructs almost-irreducible fully relaxed $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-modules from irreducible Vir_k-modules.

Does this construction produce all the fully relaxed irreducibles?

The answer is easily seen to be "yes" if we use the known classification. This classification is easy for generic k but not so easy for fractional k.

We prefer to prove the fractional classification using inverse QHR.

Theorem: [Adamović–Kawasetsu–DR'23]
For fractional k, the Adamović functors construct all irreducible fully relaxed $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-modules (up to isomorphism).

Proof: Show that if $\chi \neq 0$ is in the maximal ideal of Vir^k, then

- $\mathsf{e}^{na} \otimes \chi$ is in the maximal ideal of $\mathsf{V}^\mathsf{k}(\mathfrak{sl}_2)$, for $n \gg 0$.
- χ acts nontrivially on $\mathcal{L}_\Delta \implies \mathsf{e}^{na} \otimes \chi$ acts nontrivially on $\mathcal{R}_{[\lambda];\Delta}$. ■

We can lift this classification of irreducible fully relaxed modules to irreducible relaxed hw modules as follows: [Adamović–Kawasetsu–DR'23]

- Every irreducible hw $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-module, with an infinite-dimensional top space, is the quotient of some reducible $\mathcal{R}_{[\lambda];\Delta}$.
- For $\mathsf{k} \notin \mathbb{N}$, every irreducible hw $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-module is the spectral flow of one with an infinite-dimensional top space.

The hardest part of the proofs was done in [Kawasetsu–DR'19].

We can lift this classification of irreducible fully relaxed modules to irreducible relaxed hw modules as follows: [Adamović–Kawasetsu–DR'23]

- Every irreducible hw $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-module, with an infinite-dimensional top space, is the quotient of some reducible $\mathcal{R}_{[\lambda];\Delta}$.
- For $\mathsf{k} \notin \mathbb{N}$, every irreducible hw $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-module is the spectral flow of one with an infinite-dimensional top space.

The hardest part of the proofs was done in [Kawasetsu–DR'19].

But, this classification is not yet explicit. To make it so:

1. For each Δ, find the $[\lambda] \in \mathbb{C}/2\mathbb{Z}$ for which $\mathcal{R}_{[\lambda];\Delta}$ is irreducible.
2. For the remaining $[\lambda]$, identify the irreducible hw quotient.
3. Apply conjugation to get irreducible conjugate hw modules.
4. Apply spectral flow to identify the remaining irreducible hw modules.

The first step requires a computation that will be nontrivial to generalise to higher ranks, *cf.* [Fehily'22].

The weight category

We thus have the classification of irreducible relaxed hw $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-modules.

However, we know [Creutzig–DR'12,'13] that to get a modular category, we need to include spectral flows (as well as other modules).

Definition: Let $\mathscr{W}_\mathsf{k}$ denote the category of generalised weight $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-modules with finite-dimensional weight spaces.

Proposition: [Adamović–Kawasetsu–DR'23]
For fractional k, every simple object of $\mathscr{W}_\mathsf{k}$ is the spectral flow of an irreducible relaxed hw module.

[Futorny'96] shows that these simple objects have "extremal weights": the conformal wts of each H_0-eigenspace are bounded below. The proof then amounts to getting estimates for these weights from the PBW theorem.

Theorem: [Adamović–Kawasetsu–DR'23]
For fractional k, every simple object of $\mathscr{W}_k$ is a spectral flow of an irreducible quotient of a fully relaxed $L_k(\mathfrak{sl}_2)$-module obtained by applying the Adamović functors to the irreducible Vir_k-modules.

This reproduces (and extends!) the classification results of [Adamović–Milas'95, DR-Wood'15] in a way that generalises to higher ranks.

The generalisation to the simple Bershadsky–Polyakov vertex operator algebras is done and combining with [Fehily'22] should give the classification for all nondegenerate subregular W-algebras of type A.

It is interesting that [Arakawa–Futorny–Ramirez'16] and others have constructed examples of higher-rank irreducible weight VOA-modules with infinite-dimensional weight spaces. If these are required to get a "good category", then our methods will need further development.

Outlook

Inverse quantum hamiltonian reduction is a very powerful tool for analysing the relaxed hw modules of W-algebras.

We've seen it in action in the simplest case: $\mathsf{Vir}_{\mathsf{k}}$ to $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$. But it's also been employed (to various degrees) in other examples, *eg.*

- $(N=1)_{\mathsf{k}}$ to $\mathsf{L}_{\mathsf{k}}(\mathfrak{osp}(1|2))$ [Adamović'17].
- $\mathsf{W}_{3,\mathsf{k}}$ to BP_{k} [Adamović–Kawasetsu–DR'20,'23].
- $\mathsf{W}^{\mathsf{k}}_{\mathsf{reg.}}(\mathfrak{sp}(4))$ to $\mathsf{W}^{\mathsf{k}}_{\mathsf{sub.}}(\mathfrak{sp}(r))$ [Beem–Meneghelli'21].
- BP_{k} to $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_3)$ [Adamović–Creutzig–Genra'21].
- $\mathsf{W}^{\mathsf{reg.}}_{\mathsf{k}}(\mathfrak{sl}(n))$ to $\mathsf{W}^{\mathsf{sub.}}_{\mathsf{k}}(\mathfrak{sl}(n))$ [Fehily'22].
- $\mathsf{W}^{\mathsf{k}}_{\mathsf{hook}}(\mathfrak{sl}(n))$ to $\mathsf{W}^{\mathsf{k}}_{\mathsf{hook}}(\mathfrak{sl}(n))$ [Fehily'23].
- $\mathsf{W}^{\mathsf{k}}_{\mathsf{reg.}}(\mathfrak{so}(2n+1))$ to $\mathsf{W}^{\mathsf{k}}_{\mathsf{sub.}}(\mathfrak{so}(2n+1))$ [Fasquel–Nakatsuka'23].

In general, inverse QHR is expected to "invert" the partial QHR of [Madsen–Ragoucy'95, Morgan'15, Genra–Juillard'22].

Moreover, the fully relaxed modules it constructs are, in the known examples, also the standard modules of the VOA [Creutzig–DR'13, DR-Wood'14].

But, classifying irreducibles is just the beginning!

The plan is to lift data (classification, categorical, analytic, *etc.*) from a well understood W-algebra to a not-well understood one.

For example, principal W-algebras are rational and C_2-cofinite for nondegenerate levels [Arakawa'10,'12]. Inverse QHR can lift this to the subregular W-algebras and beyond, perhaps even to the affine VOA itself.

For degenerate levels, we expect that the role of the principal W-algebra will instead be played by the exceptional ones [Arakawa–van Ekeren'19].

An open question is to develop tools to analyse these W-algebras for nonadmissible levels. Sometimes one can use singular vectors [Adamović–Kontrec'19,'20, Adamović–Perse–Vukorepa'21, . . .], but in the most mysterious cases one cannot. Inverse QHR is nevertheless still available...

"*Only one who attempts the absurd is capable of achieving the impossible.*"

— Miguel de Unamuno