The modular machine

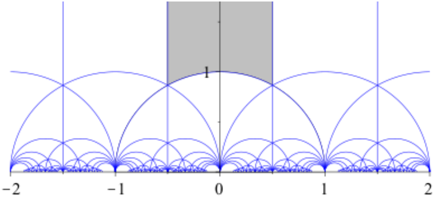
— or —

Even number theory is secretly physics

David Ridout

August 25, 2023

Highlights of Mathematical Physics



Outline

A number theory question...

Question: Is $133\,588$ the sum of 4 squares?

A number theory question...

Question: Is $133\,588$ the sum of 4 squares?
Better Question: Is m the sum of 4 squares?

A number theory question...

Question: Is $133\,588$ the sum of 4 squares?
Better Question: Is m the sum of 4 squares?
Even Better Question: Is m the sum of n squares?

A number theory question...

Question: Is $133\,588$ the sum of 4 squares?
Better Question: Is m the sum of 4 squares?
Even Better Question: Is m the sum of n squares?

Good Question: How many ways can we write m the sum of n squares?
[For our purposes, examples such as $3^2 + 0^2$ and $0^2 + 3^2$ should be counted separately.]

A number theory question...

Question: Is $133\,588$ the sum of 4 squares?
Better Question: Is m the sum of 4 squares?
Even Better Question: Is m the sum of n squares?

Good Question: How many ways can we write m the sum of n squares?
[For our purposes, examples such as 3^2+0^2 and 0^2+3^2 should be counted separately.]

m	0	1	2	3	4	5	6	7	8	9	$\cdots$
$n=0$	1	0	0	0	0	0	0	0	0	0	$\cdots$

A number theory question...

Question: Is $133\,588$ the sum of 4 squares?
Better Question: Is m the sum of 4 squares?
Even Better Question: Is m the sum of n squares?

Good Question: How many ways can we write m the sum of n squares?
[For our purposes, examples such as $3^2 + 0^2$ and $0^2 + 3^2$ should be counted separately.]

m	0	1	2	3	4	5	6	7	8	9	$\cdots$
$n = 0$	1	0	0	0	0	0	0	0	0	0	$\cdots$
$n = 1$	1	2	0	0	2	0	0	0	0	2	$\cdots$

A number theory question...

Question: Is $133\,588$ the sum of 4 squares?
Better Question: Is m the sum of 4 squares?
Even Better Question: Is m the sum of n squares?

Good Question: How many ways can we write m the sum of n squares?
[For our purposes, examples such as $3^2 + 0^2$ and $0^2 + 3^2$ should be counted separately.]

m	0	1	2	3	4	5	6	7	8	9	$\cdots$
$n=0$	1	0	0	0	0	0	0	0	0	0	$\cdots$
$n=1$	1	2	0	0	2	0	0	0	0	2	$\cdots$
$n=2$	1										

A number theory question...

Question: Is $133\,588$ the sum of 4 squares?
Better Question: Is m the sum of 4 squares?
Even Better Question: Is m the sum of n squares?

Good Question: How many ways can we write m the sum of n squares?
[For our purposes, examples such as $3^2 + 0^2$ and $0^2 + 3^2$ should be counted separately.]

m	0	1	2	3	4	5	6	7	8	9	$\cdots$
$n=0$	1	0	0	0	0	0	0	0	0	0	$\cdots$
$n=1$	1	2	0	0	2	0	0	0	0	2	$\cdots$
$n=2$	1	4									

A number theory question...

Question: Is $133\,588$ the sum of 4 squares?
Better Question: Is m the sum of 4 squares?
Even Better Question: Is m the sum of n squares?

Good Question: How many ways can we write m the sum of n squares?
[For our purposes, examples such as $3^2 + 0^2$ and $0^2 + 3^2$ should be counted separately.]

m	0	1	2	3	4	5	6	7	8	9	$\cdots$
$n=0$	1	0	0	0	0	0	0	0	0	0	$\cdots$
$n=1$	1	2	0	0	2	0	0	0	0	2	$\cdots$
$n=2$	1	4	4								

A number theory question...

Question: Is $133\,588$ the sum of 4 squares?
Better Question: Is m the sum of 4 squares?
Even Better Question: Is m the sum of n squares?

Good Question: How many ways can we write m the sum of n squares?
[For our purposes, examples such as $3^2 + 0^2$ and $0^2 + 3^2$ should be counted separately.]

m	0	1	2	3	4	5	6	7	8	9	$\cdots$
$n=0$	1	0	0	0	0	0	0	0	0	0	$\cdots$
$n=1$	1	2	0	0	2	0	0	0	0	2	$\cdots$
$n=2$	1	4	4	0							

A number theory question...

Question: Is $133\,588$ the sum of 4 squares?
Better Question: Is m the sum of 4 squares?
Even Better Question: Is m the sum of n squares?

Good Question: How many ways can we write m the sum of n squares?
[For our purposes, examples such as $3^2 + 0^2$ and $0^2 + 3^2$ should be counted separately.]

m	0	1	2	3	4	5	6	7	8	9	$\cdots$
$n=0$	1	0	0	0	0	0	0	0	0	0	$\cdots$
$n=1$	1	2	0	0	2	0	0	0	0	2	$\cdots$
$n=2$	1	4	4	0	4	8	0	0	4	4	$\cdots$

A number theory question...

Question: Is $133\,588$ the sum of 4 squares?
Better Question: Is m the sum of 4 squares?
Even Better Question: Is m the sum of n squares?

Good Question: How many ways can we write m the sum of n squares?
[For our purposes, examples such as $3^2 + 0^2$ and $0^2 + 3^2$ should be counted separately.]

m	0	1	2	3	4	5	6	7	8	9	$\cdots$
$n=0$	1	0	0	0	0	0	0	0	0	0	$\cdots$
$n=1$	1	2	0	0	2	0	0	0	0	2	$\cdots$
$n=2$	1	4	4	0	4	8	0	0	4	4	$\cdots$
$n=3$	1	6	12	8	6	24	24	0	12	30	$\cdots$
$n=4$	1	8	24	32	24	48	96	64	24	104	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

When counting things, it is wise to consider generating functions:

$$
\begin{array}{r|l}
n=0 & 1+0q+0q^2+0q^3+0q^4+0q^5+0q^6+0q^7+0q^8+0q^9+\cdots \\
n=1 & 1+2q+0q^2+0q^3+2q^4+0q^5+0q^6+0q^7+0q^8+2q^9+\cdots \\
n=2 & 1+4q+4q^2+0q^3+4q^4+8q^5+0q^6+0q^7+4q^8+4q^9+\cdots \\
n=3 & 1+6q+12q^2+8q^3+6q^4+24q^5+24q^6+0q^7+12q^8+30q^9+\cdots \\
n=4 & 1+8q+24q^2+32q^3+24q^4+48q^5+96q^6+64q^7+24q^8+104q^9+\cdots
\end{array}
$$

When counting things, it is wise to consider generating functions:

$$\begin{array}{c|l}
n=0 & 1+0q+0q^2+0q^3+0q^4+0q^5+0q^6+0q^7+0q^8+0q^9+\cdots \\
n=1 & 1+2q+0q^2+0q^3+2q^4+0q^5+0q^6+0q^7+0q^8+2q^9+\cdots \ := \widetilde{\vartheta}_3(q) \\
n=2 & 1+4q+4q^2+0q^3+4q^4+8q^5+0q^6+0q^7+4q^8+4q^9+\cdots \\
n=3 & 1+6q+12q^2+8q^3+6q^4+24q^5+24q^6+0q^7+12q^8+30q^9+\cdots \\
n=4 & 1+8q+24q^2+32q^3+24q^4+48q^5+96q^6+64q^7+24q^8+104q^9+\cdots
\end{array}$$

The $n=1$ generating function is called a theta function.
It converges when $|q|<1$.

When counting things, it is wise to consider generating functions:

$$\begin{array}{c|l} n=0 & 1+0q+0q^2+0q^3+0q^4+0q^5+0q^6+0q^7+0q^8+0q^9+\cdots \\ n=1 & 1+2q+0q^2+0q^3+2q^4+0q^5+0q^6+0q^7+0q^8+2q^9+\cdots := \widetilde{\vartheta}_3(q) \\ n=2 & 1+4q+4q^2+0q^3+4q^4+8q^5+0q^6+0q^7+4q^8+4q^9+\cdots \\ n=3 & 1+6q+12q^2+8q^3+6q^4+24q^5+24q^6+0q^7+12q^8+30q^9+\cdots \\ n=4 & 1+8q+24q^2+32q^3+24q^4+48q^5+96q^6+64q^7+24q^8+104q^9+\cdots \end{array}$$

The $n=1$ generating function is called a theta function.
It converges when $|q|<1$.

This theta function is a helpful gadget because

$$\begin{aligned} \widetilde{\vartheta}_3(q)^2 &= (1+2q+2q^4+2q^9+\cdots)(1+2q+2q^4+2q^9+\cdots) \\ &= 1+ \end{aligned}$$

When counting things, it is wise to consider generating functions:

$$\begin{array}{l|l}
n=0 & 1+0q+0q^2+0q^3+0q^4+0q^5+0q^6+0q^7+0q^8+0q^9+\cdots \\
n=1 & 1+2q+0q^2+0q^3+2q^4+0q^5+0q^6+0q^7+0q^8+2q^9+\cdots := \widetilde{\vartheta}_3(q) \\
n=2 & 1+4q+4q^2+0q^3+4q^4+8q^5+0q^6+0q^7+4q^8+4q^9+\cdots \\
n=3 & 1+6q+12q^2+8q^3+6q^4+24q^5+24q^6+0q^7+12q^8+30q^9+\cdots \\
n=4 & 1+8q+24q^2+32q^3+24q^4+48q^5+96q^6+64q^7+24q^8+104q^9+\cdots
\end{array}$$

The $n=1$ generating function is called a theta function.
It converges when $|q|<1$.

This theta function is a helpful gadget because

$$\begin{aligned}
\widetilde{\vartheta}_3(q)^2 &= (1+2q+2q^4+2q^9+\cdots)(1+2q+2q^4+2q^9+\cdots) \\
&= 1+4q+
\end{aligned}$$

When counting things, it is wise to consider generating functions:

$$
\begin{array}{l|l}
n=0 & 1+0q+0q^2+0q^3+0q^4+0q^5+0q^6+0q^7+0q^8+0q^9+\cdots \\
n=1 & 1+2q+0q^2+0q^3+2q^4+0q^5+0q^6+0q^7+0q^8+2q^9+\cdots := \widetilde{\vartheta}_3(q) \\
n=2 & 1+4q+4q^2+0q^3+4q^4+8q^5+0q^6+0q^7+4q^8+4q^9+\cdots \\
n=3 & 1+6q+12q^2+8q^3+6q^4+24q^5+24q^6+0q^7+12q^8+30q^9+\cdots \\
n=4 & 1+8q+24q^2+32q^3+24q^4+48q^5+96q^6+64q^7+24q^8+104q^9+\cdots
\end{array}
$$

The $n=1$ generating function is called a theta function.
It converges when $|q|<1$.

This theta function is a helpful gadget because

$$
\begin{aligned}
\widetilde{\vartheta}_3(q)^2 &= (1+2q+2q^4+2q^9+\cdots)(1+2q+2q^4+2q^9+\cdots) \\
&= 1+4q+4q^2+
\end{aligned}
$$

When counting things, it is wise to consider generating functions:

$$\begin{array}{c|l}
n=0 & 1+0q+0q^2+0q^3+0q^4+0q^5+0q^6+0q^7+0q^8+0q^9+\cdots \\
n=1 & 1+2q+0q^2+0q^3+2q^4+0q^5+0q^6+0q^7+0q^8+2q^9+\cdots := \widetilde{\vartheta}_3(q) \\
n=2 & 1+4q+4q^2+0q^3+4q^4+8q^5+0q^6+0q^7+4q^8+4q^9+\cdots \\
n=3 & 1+6q+12q^2+8q^3+6q^4+24q^5+24q^6+0q^7+12q^8+30q^9+\cdots \\
n=4 & 1+8q+24q^2+32q^3+24q^4+48q^5+96q^6+64q^7+24q^8+104q^9+\cdots
\end{array}$$

The $n=1$ generating function is called a theta function.
It converges when $|q|<1$.

This theta function is a helpful gadget because

$$\begin{aligned}
\widetilde{\vartheta}_3(q)^2 &= (1+2q+2q^4+2q^9+\cdots)(1+2q+2q^4+2q^9+\cdots) \\
&= 1+4q+4q^2+0q^3+
\end{aligned}$$

When counting things, it is wise to consider generating functions:

$$
\begin{array}{l|l}
n=0 & 1+0q+0q^2+0q^3+0q^4+0q^5+0q^6+0q^7+0q^8+0q^9+\cdots \\
n=1 & 1+2q+0q^2+0q^3+2q^4+0q^5+0q^6+0q^7+0q^8+2q^9+\cdots := \widetilde{\vartheta}_3(q) \\
n=2 & 1+4q+4q^2+0q^3+4q^4+8q^5+0q^6+0q^7+4q^8+4q^9+\cdots \\
n=3 & 1+6q+12q^2+8q^3+6q^4+24q^5+24q^6+0q^7+12q^8+30q^9+\cdots \\
n=4 & 1+8q+24q^2+32q^3+24q^4+48q^5+96q^6+64q^7+24q^8+104q^9+\cdots
\end{array}
$$

The $n=1$ generating function is called a theta function.
It converges when $|q|<1$.

This theta function is a helpful gadget because

$$
\begin{aligned}
\widetilde{\vartheta}_3(q)^2 &= (1+2q+2q^4+2q^9+\cdots)(1+2q+2q^4+2q^9+\cdots) \\
&= 1+4q+4q^2+0q^3+4q^4+8q^5+0q^6+0q^7+4q^8+4q^9+\cdots,
\end{aligned}
$$

When counting things, it is wise to consider generating functions:

$$
\begin{array}{l|l}
n=0 & 1+0q+0q^2+0q^3+0q^4+0q^5+0q^6+0q^7+0q^8+0q^9+\cdots \\
n=1 & 1+2q+0q^2+0q^3+2q^4+0q^5+0q^6+0q^7+0q^8+2q^9+\cdots := \widetilde{\vartheta}_3(q) \\
n=2 & 1+4q+4q^2+0q^3+4q^4+8q^5+0q^6+0q^7+4q^8+4q^9+\cdots \\
n=3 & 1+6q+12q^2+8q^3+6q^4+24q^5+24q^6+0q^7+12q^8+30q^9+\cdots \\
n=4 & 1+8q+24q^2+32q^3+24q^4+48q^5+96q^6+64q^7+24q^8+104q^9+\cdots
\end{array}
$$

The $n=1$ generating function is called a theta function.
It converges when $|q|<1$.

This theta function is a helpful gadget because

$$
\begin{aligned}
\widetilde{\vartheta}_3(q)^2 &= (1+2q+2q^4+2q^9+\cdots)(1+2q+2q^4+2q^9+\cdots) \\
&= 1+4q+4q^2+0q^3+4q^4+8q^5+0q^6+0q^7+4q^8+4q^9+\cdots,
\end{aligned}
$$

which is the $n=2$ generating function.

When counting things, it is wise to consider generating functions:

$$
\begin{array}{c|l}
n=0 & 1+0q+0q^2+0q^3+0q^4+0q^5+0q^6+0q^7+0q^8+0q^9+\cdots \\
n=1 & 1+2q+0q^2+0q^3+2q^4+0q^5+0q^6+0q^7+0q^8+2q^9+\cdots := \widetilde{\vartheta}_3(q) \\
n=2 & 1+4q+4q^2+0q^3+4q^4+8q^5+0q^6+0q^7+4q^8+4q^9+\cdots \\
n=3 & 1+6q+12q^2+8q^3+6q^4+24q^5+24q^6+0q^7+12q^8+30q^9+\cdots \\
n=4 & 1+8q+24q^2+32q^3+24q^4+48q^5+96q^6+64q^7+24q^8+104q^9+\cdots
\end{array}
$$

The $n=1$ generating function is called a theta function.
It converges when $|q|<1$.

This theta function is a helpful gadget because

$$
\begin{aligned}
\widetilde{\vartheta}_3(q)^2 &= (1+2q+2q^4+2q^9+\cdots)(1+2q+2q^4+2q^9+\cdots) \\
&= 1+4q+4q^2+0q^3+4q^4+8q^5+0q^6+0q^7+4q^8+4q^9+\cdots,
\end{aligned}
$$

which is the $n=2$ generating function.

In general, the number of ways to write m as a sum of n squares is the coefficient of q^m in $\widetilde{\vartheta}_3(q)^n$.

Fun with infinite products

When you're handed a polynomial, why not try factorising?
When you're handed a generating function... we can but try...

$\widetilde{\vartheta}_3(q) = 1 + 2q + 2q^4 + 2q^9 + \cdots$

Fun with infinite products

When you're handed a polynomial, why not try factorising?
When you're handed a generating function... we can but try...

$$\begin{aligned}\widetilde{\vartheta}_3(q) &= 1 + 2q + 2q^4 + 2q^9 + \cdots \\ &= (1+q)^2(1 - q^2 + 2q^3 - q^4 + q^6 - 2q^7 + 3q^8 - 2q^9 + \cdots)\end{aligned}$$

Fun with infinite products

When you're handed a polynomial, why not try factorising?
When you're handed a generating function... we can but try...

$$\begin{aligned}\widetilde{\vartheta}_3(q) &= 1 + 2q + 2q^4 + 2q^9 + \cdots \\ &= (1+q)^2(1 - q^2 + 2q^3 - q^4 + q^6 - 2q^7 + 3q^8 - 2q^9 + \cdots) \\ &= (1+q)^2(1-q^2)(1 + 2q^3 - q^4 + 2q^5 + 3q^8 - 2q^9 + \cdots)\end{aligned}$$

Fun with infinite products

When you're handed a polynomial, why not try factorising?
When you're handed a generating function... we can but try...

$$\begin{aligned}\widetilde{\vartheta}_3(q) &= 1 + 2q + 2q^4 + 2q^9 + \cdots \\ &= (1+q)^2(1 - q^2 + 2q^3 - q^4 + q^6 - 2q^7 + 3q^8 - 2q^9 + \cdots) \\ &= (1+q)^2(1-q^2)(1 + 2q^3 - q^4 + 2q^5 + 3q^8 - 2q^9 + \cdots) \\ &= (1+q)^2(1-q^2)(1+q^3)^2(1 - q^4 + 2q^5 - q^6 + 2q^7 - q^8 + \cdots)\end{aligned}$$

Fun with infinite products

When you're handed a polynomial, why not try factorising?
When you're handed a generating function... we can but try...

$$\begin{aligned}\widetilde{\vartheta}_3(q) &= 1 + 2q + 2q^4 + 2q^9 + \cdots \\ &= (1+q)^2(1 - q^2 + 2q^3 - q^4 + q^6 - 2q^7 + 3q^8 - 2q^9 + \cdots) \\ &= (1+q)^2(1-q^2)(1 + 2q^3 - q^4 + 2q^5 + 3q^8 - 2q^9 + \cdots) \\ &= (1+q)^2(1-q^2)(1+q^3)^2(1 - q^4 + 2q^5 - q^6 + 2q^7 - q^8 + \cdots) \\ &= (1+q)^2(1-q^2)(1+q^3)^2(1-q^4)(1 + 2q^5 - q^6 + 2q^7 - q^8 + \cdots)\end{aligned}$$

Fun with infinite products

When you're handed a polynomial, why not try factorising?
When you're handed a generating function... we can but try...

$$\begin{aligned}
\widetilde{\vartheta}_3(q) &= 1 + 2q + 2q^4 + 2q^9 + \cdots \\
&= (1+q)^2(1 - q^2 + 2q^3 - q^4 + q^6 - 2q^7 + 3q^8 - 2q^9 + \cdots) \\
&= (1+q)^2(1-q^2)(1 + 2q^3 - q^4 + 2q^5 + 3q^8 - 2q^9 + \cdots) \\
&= (1+q)^2(1-q^2)(1+q^3)^2(1 - q^4 + 2q^5 - q^6 + 2q^7 - q^8 + \cdots) \\
&= (1+q)^2(1-q^2)(1+q^3)^2(1-q^4)(1 + 2q^5 - q^6 + 2q^7 - q^8 + \cdots) \\
&\overset{??}{=} \prod_{n=1}^{\infty} (1+q^{2n-1})^2(1-q^{2n}).
\end{aligned}$$

Fun with infinite products

When you're handed a polynomial, why not try factorising?
When you're handed a generating function... we can but try...

$$\begin{aligned}
\widetilde{\vartheta}_3(q) &= 1 + 2q + 2q^4 + 2q^9 + \cdots \\
&= (1+q)^2(1 - q^2 + 2q^3 - q^4 + q^6 - 2q^7 + 3q^8 - 2q^9 + \cdots) \\
&= (1+q)^2(1-q^2)(1 + 2q^3 - q^4 + 2q^5 + 3q^8 - 2q^9 + \cdots) \\
&= (1+q)^2(1-q^2)(1+q^3)^2(1 - q^4 + 2q^5 - q^6 + 2q^7 - q^8 + \cdots) \\
&= (1+q)^2(1-q^2)(1+q^3)^2(1-q^4)(1 + 2q^5 - q^6 + 2q^7 - q^8 + \cdots) \\
&\overset{??}{=} \prod_{n=1}^{\infty}(1+q^{2n-1})^2(1-q^{2n}).
\end{aligned}$$

This theta function factorises as an infinite product! It also converges when $|q| < 1$. [How can you tell if an infinite product converges?]

A slight recalibration

For what follows, we'll need to redefine this otherwise extremely beautiful theta function.

A slight recalibration

For what follows, we'll need to redefine this otherwise extremely beautiful theta function. We set

$$\vartheta_3(q) := \widetilde{\vartheta}_3(q^{1/2}) = \sum_{n\in\mathbb{Z}} q^{n^2/2} = \prod_{n=1}^{\infty}(1+q^{n-1/2})^2(1-q^n).$$

A slight recalibration

For what follows, we'll need to redefine this otherwise extremely beautiful theta function. We set

$$\vartheta_3(q) := \widetilde{\vartheta}_3(q^{1/2}) = \sum_{n\in\mathbb{Z}} q^{n^2/2} = \prod_{n=1}^{\infty}(1+q^{n-1/2})^2(1-q^n).$$

This has the small disadvantage of no longer being single-valued:

$$\vartheta_3(\mathrm{e}^{2\pi\mathrm{i}}q) = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2/2}$$

A slight recalibration

For what follows, we'll need to redefine this otherwise extremely beautiful theta function. We set

$$\vartheta_3(q) := \widetilde{\vartheta}_3(q^{1/2}) = \sum_{n\in\mathbb{Z}} q^{n^2/2} = \prod_{n=1}^{\infty}(1+q^{n-1/2})^2(1-q^n).$$

This has the small disadvantage of no longer being single-valued:

$$\vartheta_3(\mathrm{e}^{2\pi\mathrm{i}}q) = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^{n-1/2})^2(1-q^n)$$

A slight recalibration

For what follows, we'll need to redefine this otherwise extremely beautiful theta function. We set

$$\vartheta_3(q) := \widetilde{\vartheta}_3(q^{1/2}) = \sum_{n\in\mathbb{Z}} q^{n^2/2} = \prod_{n=1}^{\infty}(1+q^{n-1/2})^2(1-q^n).$$

This has the small disadvantage of no longer being single-valued:

$$\vartheta_3(\mathrm{e}^{2\pi\mathrm{i}}q) = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^{n-1/2})^2(1-q^n) := \vartheta_4(q).$$

But, instead we get a new theta function!

A slight recalibration

For what follows, we'll need to redefine this otherwise extremely beautiful theta function. We set

$$\vartheta_3(q) := \widetilde{\vartheta}_3(q^{1/2}) = \sum_{n\in\mathbb{Z}} q^{n^2/2} = \prod_{n=1}^{\infty}(1+q^{n-1/2})^2(1-q^n).$$

This has the small disadvantage of no longer being single-valued:

$$\vartheta_3(\mathrm{e}^{2\pi\mathrm{i}}q) = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^{n-1/2})^2(1-q^n) := \vartheta_4(q).$$

But, instead we get a new theta function!

A small variation of this theme even gives us a third theta function:

$$\vartheta_2(q) := \sum_{n\in\mathbb{Z}} q^{(n+1/2)^2/2} = 2q^{1/8}\prod_{n=1}^{\infty}(1+q^n)^2(1-q^n).$$

[Why don't we also consider $\vartheta_1(q) = \sum_{n\in\mathbb{Z}}(-1)^n q^{(n+1/2)^2/2}$?]

Fun with Fourier

Let $f\colon \mathbb{R} \to \mathbb{C}$ be a really nice[1] function. Its Fourier transform is

$$\hat{f}(p) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} p x}\, \mathrm{d}x.$$

[1] [*eg.*, infinitely differentiable and rapidly decaying at $\pm\infty$.]

Fun with Fourier

Let $f\colon \mathbb{R} \to \mathbb{C}$ be a really nice[1] function. Its Fourier transform is

$$\hat{f}(p) = \int_{-\infty}^{\infty} f(x)\, \mathsf{e}^{-2\pi \mathrm{i} p x}\, \mathrm{d}x.$$

Set $F(x) = \sum_{n \in \mathbb{Z}} f(x+n)$ so that F is 1-periodic.

[1] [*eg.*, infinitely differentiable and rapidly decaying at $\pm\infty$.]

Fun with Fourier

Let $f\colon \mathbb{R} \to \mathbb{C}$ be a really nice[1] function. Its Fourier transform is

$$\hat{f}(p) = \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-2\pi \mathrm{i} p x}\,\mathrm{d}x.$$

Set $F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$ so that F is 1-periodic. Its Fourier series is

$$F(x) = \sum_{m\in\mathbb{Z}} c_m \mathrm{e}^{2\pi \mathrm{i} m x}, \quad \text{for some } c_m \in \mathbb{C}.$$

[1] [*eg.*, infinitely differentiable and rapidly decaying at $\pm\infty$.]

Fun with Fourier

Let $f\colon \mathbb{R} \to \mathbb{C}$ be a really nice[1] function. Its Fourier transform is

$$\hat{f}(p) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} p x}\, \mathrm{d}x.$$

Set $F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$ so that F is 1-periodic. Its Fourier series is

$$F(x) = \sum_{m\in\mathbb{Z}} c_m \mathrm{e}^{2\pi \mathrm{i} m x}, \quad \text{for some } c_m \in \mathbb{C}.$$

We compute these constants:

[1] [*eg.*, infinitely differentiable and rapidly decaying at $\pm\infty$.]

Fun with Fourier

Let $f\colon \mathbb{R} \to \mathbb{C}$ be a really nice[1] function. Its Fourier transform is

$$\hat{f}(p) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} p x}\, \mathrm{d}x.$$

Set $F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$ so that F is 1-periodic. Its Fourier series is

$$F(x) = \sum_{m\in\mathbb{Z}} c_m \mathrm{e}^{2\pi \mathrm{i} m x}, \quad \text{for some } c_m \in \mathbb{C}.$$

We compute these constants:

$$c_m = \int_0^1 F(x)\, \mathrm{e}^{-2\pi \mathrm{i} m x}\, \mathrm{d}x$$

[1] [*eg.*, infinitely differentiable and rapidly decaying at $\pm\infty$.]

Fun with Fourier

Let $f\colon \mathbb{R} \to \mathbb{C}$ be a really nice[1] function. Its Fourier transform is

$$\hat{f}(p) = \int_{-\infty}^{\infty} f(x)\, \mathsf{e}^{-2\pi \mathrm{i} p x}\, \mathrm{d}x.$$

Set $F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$ so that F is 1-periodic. Its Fourier series is

$$F(x) = \sum_{m\in\mathbb{Z}} c_m \mathsf{e}^{2\pi \mathrm{i} m x}, \quad \text{for some } c_m \in \mathbb{C}.$$

We compute these constants:

$$c_m = \int_0^1 F(x)\, \mathsf{e}^{-2\pi \mathrm{i} m x}\, \mathrm{d}x = \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n)\, \mathsf{e}^{-2\pi \mathrm{i} m x}\, \mathrm{d}x$$

[1] [*eg.*, infinitely differentiable and rapidly decaying at $\pm\infty$.]

Fun with Fourier

Let $f\colon \mathbb{R} \to \mathbb{C}$ be a really nice[1] function. Its Fourier transform is

$$\hat{f}(p) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} p x}\, \mathrm{d}x.$$

Set $F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$ so that F is 1-periodic. Its Fourier series is

$$F(x) = \sum_{m\in\mathbb{Z}} c_m \mathrm{e}^{2\pi \mathrm{i} m x}, \quad \text{for some } c_m \in \mathbb{C}.$$

We compute these constants:

$$c_m = \int_0^1 F(x)\, \mathrm{e}^{-2\pi \mathrm{i} m x}\, \mathrm{d}x = \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n)\, \mathrm{e}^{-2\pi \mathrm{i} m x}\, \mathrm{d}x$$

$$= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(y)\, \mathrm{e}^{-2\pi \mathrm{i} m y}\, \mathrm{d}y$$

[1] [*eg.*, infinitely differentiable and rapidly decaying at $\pm\infty$.]

Fun with Fourier

Let $f\colon \mathbb{R} \to \mathbb{C}$ be a really nice[1] function. Its Fourier transform is

$$\hat{f}(p) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} p x}\, \mathrm{d}x.$$

Set $F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$ so that F is 1-periodic. Its Fourier series is

$$F(x) = \sum_{m\in\mathbb{Z}} c_m \mathrm{e}^{2\pi \mathrm{i} m x}, \quad \text{for some } c_m \in \mathbb{C}.$$

We compute these constants:

$$\begin{aligned} c_m &= \int_0^1 F(x)\, \mathrm{e}^{-2\pi \mathrm{i} m x}\, \mathrm{d}x = \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n)\, \mathrm{e}^{-2\pi \mathrm{i} m x}\, \mathrm{d}x \\ &= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(y)\, \mathrm{e}^{-2\pi \mathrm{i} m y}\, \mathrm{d}y = \int_{-\infty}^{\infty} f(y)\, \mathrm{e}^{-2\pi \mathrm{i} m y}\, \mathrm{d}y. \end{aligned}$$

[1] [*eg.*, infinitely differentiable and rapidly decaying at $\pm\infty$.]

The c_m are thus equal to $\hat{f}(m)$!

The c_m are thus equal to $\hat{f}(m)$! Substituting back into $F(x)$ now gives

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{m\in\mathbb{Z}} \hat{f}(m)\,\mathrm{e}^{2\pi\mathrm{i}mx}.$$

The c_m are thus equal to $\hat{f}(m)$! Substituting back into $F(x)$ now gives

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{m\in\mathbb{Z}} \hat{f}(m)\,\mathrm{e}^{2\pi \mathrm{i} m x}.$$

The $x = 0$ version is very cool:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m).$$

It's called Poisson resummation.

The c_m are thus equal to $\hat{f}(m)$! Substituting back into $F(x)$ now gives

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{m\in\mathbb{Z}} \hat{f}(m)\, \mathrm{e}^{2\pi \mathrm{i} m x}.$$

The $x = 0$ version is very cool:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m).$$

It's called Poisson resummation.

[You might like to experiment with this to see what it has to do with Dirac delta functions/combs.]

Back to theta functions

One really nice function is $f(x) = \mathrm{e}^{-ax^2}$, $\operatorname{Re} a > 0$.

Back to theta functions

One really nice function is $f(x) = \mathrm{e}^{-ax^2}$, $\operatorname{Re} a > 0$. We have

$$\hat{f}(p) = \sqrt{\frac{\pi}{a}}\,\mathrm{e}^{-\pi^2 p^2/a},$$

so that Poisson resummation becomes

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{-an^2} \overset{\text{PS}}{=} \sqrt{\frac{\pi}{a}} \sum_{m\in\mathbb{Z}} \mathrm{e}^{-\pi^2 m^2/a}, \quad \operatorname{Re} a > 0.$$

Back to theta functions

One really nice function is $f(x) = \mathrm{e}^{-ax^2}$, $\mathrm{Re}\, a > 0$. We have

$$\hat{f}(p) = \sqrt{\frac{\pi}{a}}\, \mathrm{e}^{-\pi^2 p^2/a},$$

so that Poisson resummation becomes

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{-an^2} \overset{\text{PS}}{=} \sqrt{\frac{\pi}{a}} \sum_{m\in\mathbb{Z}} \mathrm{e}^{-\pi^2 m^2/a}, \quad \mathrm{Re}\, a > 0.$$

If we now set $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$ in our favourite theta function, we get

$$\vartheta_3(\tau) := \vartheta_3(\mathrm{e}^{2\pi\mathrm{i}\tau}) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\pi\mathrm{i}\tau n^2}$$

Back to theta functions

One really nice function is $f(x) = \mathrm{e}^{-ax^2}$, $\operatorname{Re} a > 0$. We have

$$\hat{f}(p) = \sqrt{\frac{\pi}{a}}\, \mathrm{e}^{-\pi^2 p^2/a},$$

so that Poisson resummation becomes

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{-an^2} \overset{\text{PS}}{=} \sqrt{\frac{\pi}{a}} \sum_{m\in\mathbb{Z}} \mathrm{e}^{-\pi^2 m^2/a}, \quad \operatorname{Re} a > 0.$$

If we now set $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ in our favourite theta function, we get

$$\vartheta_3(\tau) := \vartheta_3(\mathrm{e}^{2\pi\mathrm{i}\tau}) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\pi\mathrm{i}\tau n^2} \overset{\text{PS}}{=} \frac{1}{\sqrt{-\mathrm{i}\tau}} \sum_{m\in\mathbb{Z}} \mathrm{e}^{-\pi\mathrm{i} m^2/\tau}$$

Back to theta functions

One really nice function is $f(x) = \mathrm{e}^{-ax^2}$, $\mathrm{Re}\, a > 0$. We have

$$\hat{f}(p) = \sqrt{\frac{\pi}{a}}\, \mathrm{e}^{-\pi^2 p^2/a},$$

so that Poisson resummation becomes

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{-an^2} \overset{\text{PS}}{=} \sqrt{\frac{\pi}{a}} \sum_{m\in\mathbb{Z}} \mathrm{e}^{-\pi^2 m^2/a}, \quad \mathrm{Re}\, a > 0.$$

If we now set $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ in our favourite theta function, we get

$$\vartheta_3(\tau) := \vartheta_3(\mathrm{e}^{2\pi\mathrm{i}\tau}) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\pi\mathrm{i}\tau n^2} \overset{\text{PS}}{=} \frac{1}{\sqrt{-\mathrm{i}\tau}} \sum_{m\in\mathbb{Z}} \mathrm{e}^{-\pi\mathrm{i}m^2/\tau} = \frac{1}{\sqrt{-\mathrm{i}\tau}} \vartheta_3(\tfrac{-1}{\tau}).$$

Back to theta functions

One really nice function is $f(x) = \mathrm{e}^{-ax^2}$, $\operatorname{Re} a > 0$. We have

$$\hat{f}(p) = \sqrt{\frac{\pi}{a}}\,\mathrm{e}^{-\pi^2 p^2/a},$$

so that Poisson resummation becomes

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{-an^2} \overset{\text{PS}}{=} \sqrt{\frac{\pi}{a}} \sum_{m\in\mathbb{Z}} \mathrm{e}^{-\pi^2 m^2/a}, \quad \operatorname{Re} a > 0.$$

If we now set $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ in our favourite theta function, we get

$$\vartheta_3(\tau) := \vartheta_3(\mathrm{e}^{2\pi\mathrm{i}\tau}) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\pi\mathrm{i}\tau n^2} \overset{\text{PS}}{=} \frac{1}{\sqrt{-\mathrm{i}\tau}} \sum_{m\in\mathbb{Z}} \mathrm{e}^{-\pi\mathrm{i}m^2/\tau} = \frac{1}{\sqrt{-\mathrm{i}\tau}}\vartheta_3(\tfrac{-1}{\tau}).$$

Note that $|q| < 1 \iff \operatorname{Im}\tau > 0 \iff \operatorname{Re} a = \operatorname{Re}(-\pi\mathrm{i}\tau) > 0$.

Multiplying q by $e^{2\pi i}$ is the same as adding 1 to τ.

Multiplying q by $e^{2\pi i}$ is the same as adding 1 to τ. We therefore have two transformation formulae:

$$\vartheta_3(\tfrac{-1}{\tau}) = \sqrt{-i\tau}\,\vartheta_3(\tau), \qquad \vartheta_3(\tau+1) = \vartheta_4(\tau).$$

Multiplying q by $e^{2\pi i}$ is the same as adding 1 to τ. We therefore have two transformation formulae:

$$\vartheta_3(\tfrac{-1}{\tau}) = \sqrt{-i\tau}\,\vartheta_3(\tau), \qquad \vartheta_3(\tau+1) = \vartheta_4(\tau).$$

Similar shenanigans give transformations for our other theta functions:

$$\vartheta_2(\tfrac{-1}{\tau}) = \sqrt{-i\tau}\,\vartheta_4(\tau), \qquad \vartheta_2(\tau+1) = e^{\pi i/4}\,\vartheta_2(\tau),$$
$$\vartheta_4(\tfrac{-1}{\tau}) = \sqrt{-i\tau}\,\vartheta_2(\tau), \qquad \vartheta_4(\tau+1) = \vartheta_3(\tau).$$

Multiplying q by $\mathrm{e}^{2\pi\mathrm{i}}$ is the same as adding 1 to τ. We therefore have two transformation formulae:

$$\vartheta_3(\tfrac{-1}{\tau}) = \sqrt{-\mathrm{i}\tau}\,\vartheta_3(\tau), \qquad \vartheta_3(\tau+1) = \vartheta_4(\tau).$$

Similar shenanigans give transformations for our other theta functions:

$$\vartheta_2(\tfrac{-1}{\tau}) = \sqrt{-\mathrm{i}\tau}\,\vartheta_4(\tau), \qquad \vartheta_2(\tau+1) = \mathrm{e}^{\pi\mathrm{i}/4}\,\vartheta_2(\tau),$$
$$\vartheta_4(\tfrac{-1}{\tau}) = \sqrt{-\mathrm{i}\tau}\,\vartheta_2(\tau), \qquad \vartheta_4(\tau+1) = \vartheta_3(\tau).$$

These formulae can be summarised by declaring that $\big[\vartheta_2(\tau), \vartheta_3(\tau), \vartheta_4(\tau)\big]$ is a vector-valued modular form of weight $\frac{1}{2}$.

Multiplying q by $\mathrm{e}^{2\pi\mathrm{i}}$ is the same as adding 1 to τ. We therefore have two transformation formulae:

$$\vartheta_3(\tfrac{-1}{\tau}) = \sqrt{-\mathrm{i}\tau}\,\vartheta_3(\tau), \qquad \vartheta_3(\tau+1) = \vartheta_4(\tau).$$

Similar shenanigans give transformations for our other theta functions:

$$\vartheta_2(\tfrac{-1}{\tau}) = \sqrt{-\mathrm{i}\tau}\,\vartheta_4(\tau), \qquad \vartheta_2(\tau+1) = \mathrm{e}^{\pi\mathrm{i}/4}\,\vartheta_2(\tau),$$
$$\vartheta_4(\tfrac{-1}{\tau}) = \sqrt{-\mathrm{i}\tau}\,\vartheta_2(\tau), \qquad \vartheta_4(\tau+1) = \vartheta_3(\tau).$$

These formulae can be summarised by declaring that $\big[\vartheta_2(\tau), \vartheta_3(\tau), \vartheta_4(\tau)\big]$ is a vector-valued modular form of weight $\frac{1}{2}$.

[This is the start of a beautiful story into which we sadly have not the time to delve...]

Multiplying q by $e^{2\pi i}$ is the same as adding 1 to τ. We therefore have two transformation formulae:

$$\vartheta_3(\tfrac{-1}{\tau}) = \sqrt{-i\tau}\,\vartheta_3(\tau), \qquad \vartheta_3(\tau+1) = \vartheta_4(\tau).$$

Similar shenanigans give transformations for our other theta functions:

$$\vartheta_2(\tfrac{-1}{\tau}) = \sqrt{-i\tau}\,\vartheta_4(\tau), \qquad \vartheta_2(\tau+1) = e^{\pi i/4}\,\vartheta_2(\tau),$$
$$\vartheta_4(\tfrac{-1}{\tau}) = \sqrt{-i\tau}\,\vartheta_2(\tau), \qquad \vartheta_4(\tau+1) = \vartheta_3(\tau).$$

These formulae can be summarised by declaring that $\big[\vartheta_2(\tau), \vartheta_3(\tau), \vartheta_4(\tau)\big]$ is a vector-valued modular form of weight $\frac{1}{2}$.

[This is the start of a beautiful story into which we sadly have not the time to delve...]

Exercise: Show that $\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ satisfies

$$2\eta(q)^3 = \vartheta_2(q)\vartheta_3(q)\vartheta_4(q), \quad \eta(\tfrac{-1}{\tau}) = \sqrt{-i\tau}\,\eta(\tau), \quad \eta(\tau+1) = e^{\pi i/12}\,\eta(\tau).$$

At last some physics!

Recall that workhorse of undergraduate physics, the harmonic oscillator:

$$\ddot{x}(t) + \omega^2 x(t) = 0.$$

At last some physics!

Recall that workhorse of undergraduate physics, the harmonic oscillator:

$$\ddot{x}(t) + \omega^2 x(t) = 0.$$

After quantising, the hamiltonian operator takes the form

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2, \qquad [x, p] = \mathrm{i}\hbar.$$

At last some physics!

Recall that workhorse of undergraduate physics, the harmonic oscillator:

$$\ddot{x}(t) + \omega^2 x(t) = 0.$$

After quantising, the hamiltonian operator takes the form

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2, \qquad [x, p] = \mathrm{i}\hbar.$$

The energies may be extracted using ladder operators (*aka*. Lie algebra representation theory) or Hermite functions. Either way, the answer is

$$E_n = \hbar\omega(n + \tfrac{1}{2}), \quad n \in \mathbb{N}.$$

At last some physics!

Recall that workhorse of undergraduate physics, the harmonic oscillator:

$$\ddot{x}(t) + \omega^2 x(t) = 0.$$

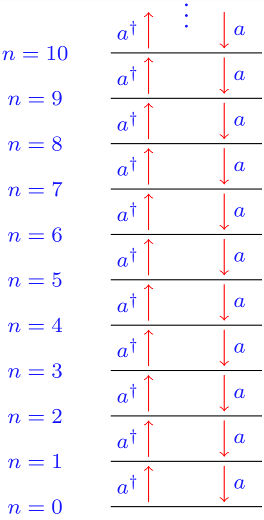
After quantising, the hamiltonian operator takes the form

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2, \qquad [x, p] = \mathrm{i}\hbar.$$

The energies may be extracted using ladder operators (*aka*. Lie algebra representation theory) or Hermite functions. Either way, the answer is

$$E_n = \hbar\omega(n + \tfrac{1}{2}), \quad n \in \mathbb{N}.$$

Even better, the spectrum is nondegenerate, meaning that each energy eigenvalue has a one-dimensional eigenspace.



Annihilation operator: a

Creation operator: $a^\dagger$

$E_n = \hbar\omega(n + \frac{1}{2})$

Let's compute the partition function of the quantum harmonic oscillator:

$$Z(T) = \mathrm{tr}\ \mathsf{e}^{-H/kT}$$

Let's compute the partition function of the quantum harmonic oscillator:

$$Z(T) = \mathrm{tr}\ \mathrm{e}^{-H/kT} = \sum_{n=0}^{\infty} \mathrm{e}^{-E_n/kT}$$

Let's compute the partition function of the quantum harmonic oscillator:

$$Z(T) = \operatorname{tr} \mathrm{e}^{-H/kT} = \sum_{n=0}^{\infty} \mathrm{e}^{-E_n/kT} = \sum_{n=0}^{\infty} \mathrm{e}^{-\hbar\omega(n+1/2)/kT}.$$

Let's compute the partition function of the quantum harmonic oscillator:

$$Z(T) = \mathrm{tr}\ \mathrm{e}^{-H/kT} = \sum_{n=0}^{\infty} \mathrm{e}^{-E_n/kT} = \sum_{n=0}^{\infty} \mathrm{e}^{-\hbar\omega(n+1/2)/kT}.$$

Now put $q = \mathrm{e}^{-\hbar\omega/kT}$ for clarity, noting that $|q| < 1$ (in fact $0 < q < 1$).

Let's compute the partition function of the quantum harmonic oscillator:

$$Z(T) = \mathrm{tr}\ \mathrm{e}^{-H/kT} = \sum_{n=0}^{\infty} \mathrm{e}^{-E_n/kT} = \sum_{n=0}^{\infty} \mathrm{e}^{-\hbar\omega(n+1/2)/kT}.$$

Now put $q = \mathrm{e}^{-\hbar\omega/kT}$ for clarity, noting that $|q| < 1$ (in fact $0 < q < 1$). Then,

$$Z(q) = \sum_{n=0}^{\infty} q^{n+1/2} = \frac{q^{1/2}}{1-q},$$

which is...

Let's compute the partition function of the quantum harmonic oscillator:

$$Z(T) = \mathrm{tr}\ \mathrm{e}^{-H/kT} = \sum_{n=0}^{\infty} \mathrm{e}^{-E_n/kT} = \sum_{n=0}^{\infty} \mathrm{e}^{-\hbar\omega(n+1/2)/kT}.$$

Now put $q = \mathrm{e}^{-\hbar\omega/kT}$ for clarity, noting that $|q| < 1$ (in fact $0 < q < 1$). Then,

$$Z(q) = \sum_{n=0}^{\infty} q^{n+1/2} = \frac{q^{1/2}}{1-q},$$

which is... quite underwhelming.

Let's compute the partition function of the quantum harmonic oscillator:

$$Z(T) = \mathrm{tr}\ \mathrm{e}^{-H/kT} = \sum_{n=0}^{\infty} \mathrm{e}^{-E_n/kT} = \sum_{n=0}^{\infty} \mathrm{e}^{-\hbar\omega(n+1/2)/kT}.$$

Now put $q = \mathrm{e}^{-\hbar\omega/kT}$ for clarity, noting that $|q| < 1$ (in fact $0 < q < 1$). Then,

$$Z(q) = \sum_{n=0}^{\infty} q^{n+1/2} = \frac{q^{1/2}}{1-q},$$

which is... quite underwhelming.

The problem here is that we're doing boring ol' quantum mechanics. To get the good stuff, we need some quantum field theory!

So let's kick it up a notch!

The free bosonic string

We'd like to study the quantum field theory, actually conformal field theory (CFT) underlying the massless spinless noninteracting bosonic string (and on a one-dimensional spacetime no less)!

The free bosonic string

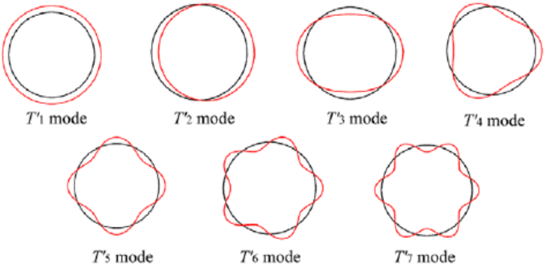
We'd like to study the quantum field theory, actually conformal field theory (CFT) underlying the massless spinless noninteracting bosonic string (and on a one-dimensional spacetime no less)!

But it's getting late and we're all pretty tired on a Friday afternoon, so let me just say that the string behaves like an infinite set of independent harmonic oscillators, one for each vibration mode.

The free bosonic string

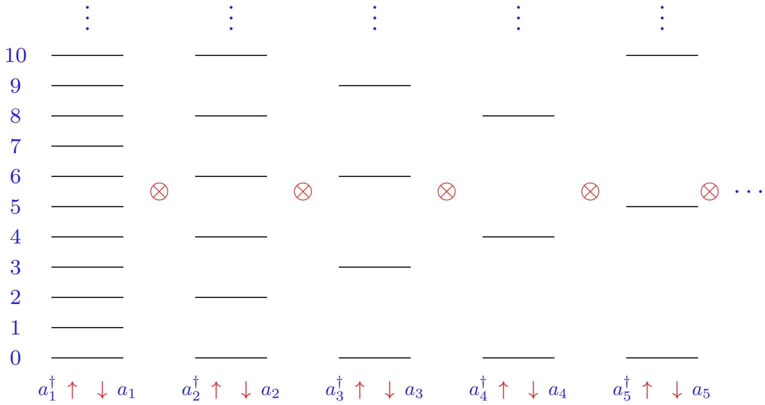
We'd like to study the quantum field theory, actually conformal field theory (CFT) underlying the massless spinless noninteracting bosonic string (and on a one-dimensional spacetime no less)!

But it's getting late and we're all pretty tired on a Friday afternoon, so let me just say that the string behaves like an infinite set of independent harmonic oscillators, one for each vibration mode.

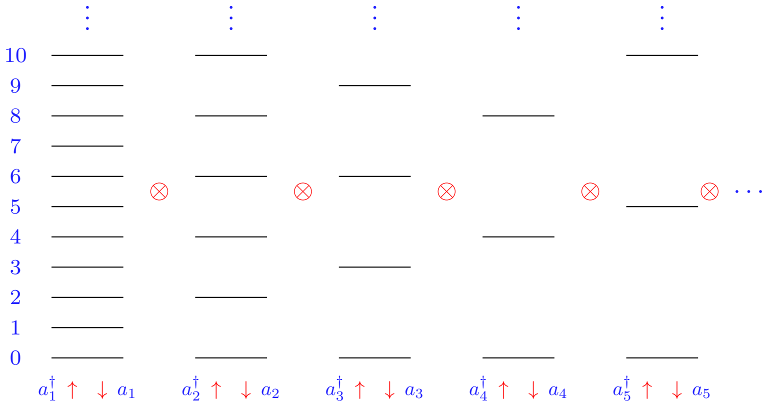


[This is because the equation of motion of the string is the wave equation — see MAST90069.]

Since collections of independent systems are modelled by tensor products, the spectrum of the bosonic string is as follows [ignoring zero-point energies!]:



Since collections of independent systems are modelled by tensor products, the spectrum of the bosonic string is as follows [ignoring zero-point energies!]:



The upshot is that the partition function of the bosonic string is the product of the partition functions of its harmonic oscillator components.

So let's compute the stringy partition function...

So let's compute the stringy partition function...

We have a harmonic oscillator of frequency $n\omega$, for each $n \in \mathbb{Z}_{>0}$, so

$$Z_{BS}(q) = Z_{HO}(q) Z_{HO}(q^2) Z_{HO}(q^3) \cdots \qquad (\text{since } q = \mathrm{e}^{-\hbar\omega/kT})$$

So let's compute the stringy partition function...

We have a harmonic oscillator of frequency $n\omega$, for each $n \in \mathbb{Z}_{>0}$, so

$$Z_{BS}(q) = Z_{HO}(q)Z_{HO}(q^2)Z_{HO}(q^3)\cdots \qquad \text{(since } q = \mathrm{e}^{-\hbar\omega/kT}\text{)}$$

$$\Rightarrow \quad Z_{BS}(q) = \prod_{n=1}^{\infty} Z_{HO}(q^n) = \prod_{n=1}^{\infty} \frac{q^{n/2}}{1-q^n}$$

So let's compute the stringy partition function...

We have a harmonic oscillator of frequency $n\omega$, for each $n \in \mathbb{Z}_{>0}$, so

$$Z_{BS}(q) = Z_{HO}(q)Z_{HO}(q^2)Z_{HO}(q^3)\cdots \qquad \text{(since } q = \mathrm{e}^{-\hbar\omega/kT}\text{)}$$

$$\Rightarrow \quad Z_{BS}(q) = \prod_{n=1}^{\infty} Z_{HO}(q^n) = \prod_{n=1}^{\infty} \frac{q^{n/2}}{1-q^n} = \frac{q^{\frac{1}{2}\sum_{n=1}^{\infty} n}}{\prod_{n=1}^{\infty}(1-q^n)}.$$

So let's compute the stringy partition function...

We have a harmonic oscillator of frequency $n\omega$, for each $n \in \mathbb{Z}_{>0}$, so

$$Z_{BS}(q) = Z_{HO}(q)Z_{HO}(q^2)Z_{HO}(q^3)\cdots \qquad (\text{since } q = \mathrm{e}^{-\hbar\omega/kT})$$

$$\Rightarrow \quad Z_{BS}(q) = \prod_{n=1}^{\infty} Z_{HO}(q^n) = \prod_{n=1}^{\infty} \frac{q^{n/2}}{1-q^n} = \frac{q^{\frac{1}{2}\sum_{n=1}^{\infty} n}}{\prod_{n=1}^{\infty}(1-q^n)}.$$

We seem to have a problem...

So let's compute the stringy partition function...

We have a harmonic oscillator of frequency $n\omega$, for each $n \in \mathbb{Z}_{>0}$, so

$$Z_{BS}(q) = Z_{HO}(q)Z_{HO}(q^2)Z_{HO}(q^3)\cdots \qquad (\text{since } q = \mathrm{e}^{-\hbar\omega/kT})$$

$$\Rightarrow \quad Z_{BS}(q) = \prod_{n=1}^{\infty} Z_{HO}(q^n) = \prod_{n=1}^{\infty} \frac{q^{n/2}}{1-q^n} = \frac{q^{\frac{1}{2}\sum_{n=1}^{\infty} n}}{\prod_{n=1}^{\infty}(1-q^n)}.$$

We seem to have a problem... and the solution is to regularise!

So let's compute the stringy partition function...

We have a harmonic oscillator of frequency $n\omega$, for each $n \in \mathbb{Z}_{>0}$, so

$$Z_{BS}(q) = Z_{HO}(q) Z_{HO}(q^2) Z_{HO}(q^3) \cdots \qquad \text{(since } q = \mathrm{e}^{-\hbar\omega/kT}\text{)}$$

$$\Rightarrow \quad Z_{BS}(q) = \prod_{n=1}^{\infty} Z_{HO}(q^n) = \prod_{n=1}^{\infty} \frac{q^{n/2}}{1-q^n} = \frac{q^{\frac{1}{2}\sum_{n=1}^{\infty} n}}{\prod_{n=1}^{\infty}(1-q^n)}.$$

We seem to have a problem... and the solution is to regularise!

Set $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. This converges for $\mathrm{Re}\, s > 1$, *eg.* $\zeta(2) = \frac{\pi^2}{6}$, but

So let's compute the stringy partition function...

We have a harmonic oscillator of frequency $n\omega$, for each $n \in \mathbb{Z}_{>0}$, so

$$Z_{BS}(q) = Z_{HO}(q) Z_{HO}(q^2) Z_{HO}(q^3) \cdots \qquad (\text{since } q = \mathrm{e}^{-\hbar\omega/kT})$$

$$\Rightarrow \quad Z_{BS}(q) = \prod_{n=1}^{\infty} Z_{HO}(q^n) = \prod_{n=1}^{\infty} \frac{q^{n/2}}{1-q^n} = \frac{q^{\frac{1}{2}\sum_{n=1}^{\infty} n}}{\prod_{n=1}^{\infty}(1-q^n)}.$$

We seem to have a problem... and the solution is to regularise!

Set $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. This converges for $\operatorname{Re} s > 1$, *eg.* $\zeta(2) = \frac{\pi^2}{6}$, but

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2}\,\Gamma(1-s)\zeta(1-s)$$

So let's compute the stringy partition function...

We have a harmonic oscillator of frequency $n\omega$, for each $n \in \mathbb{Z}_{>0}$, so

$$Z_{BS}(q) = Z_{HO}(q) Z_{HO}(q^2) Z_{HO}(q^3) \cdots \qquad \text{(since } q = e^{-\hbar\omega/kT}\text{)}$$

$$\Rightarrow \quad Z_{BS}(q) = \prod_{n=1}^{\infty} Z_{HO}(q^n) = \prod_{n=1}^{\infty} \frac{q^{n/2}}{1-q^n} = \frac{q^{\frac{1}{2}\sum_{n=1}^{\infty} n}}{\prod_{n=1}^{\infty}(1-q^n)}.$$

We seem to have a problem... and the solution is to regularise!

Set $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. This converges for $\operatorname{Re} s > 1$, *eg.* $\zeta(2) = \frac{\pi^2}{6}$, but

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2}\,\Gamma(1-s)\zeta(1-s) \Rightarrow \quad \zeta(-1) = -\frac{1}{12}.$$

So let's compute the stringy partition function...

We have a harmonic oscillator of frequency $n\omega$, for each $n \in \mathbb{Z}_{>0}$, so

$$Z_{BS}(q) = Z_{HO}(q)Z_{HO}(q^2)Z_{HO}(q^3)\cdots \qquad \text{(since } q = \mathrm{e}^{-\hbar\omega/kT}\text{)}$$

$$\Rightarrow \quad Z_{BS}(q) = \prod_{n=1}^{\infty} Z_{HO}(q^n) = \prod_{n=1}^{\infty} \frac{q^{n/2}}{1-q^n} = \frac{q^{\frac{1}{2}\sum_{n=1}^{\infty} n}}{\prod_{n=1}^{\infty}(1-q^n)}.$$

We seem to have a problem... and the solution is to regularise!

Set $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. This converges for $\operatorname{Re} s > 1$, *eg.* $\zeta(2) = \frac{\pi^2}{6}$, but

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2}\,\Gamma(1-s)\zeta(1-s) \Rightarrow \quad \zeta(-1) = -\frac{1}{12}.$$

The (zeta-function-regularised) stringy partition function is thus

$$Z_{BS}(q) = \frac{q^{\frac{1}{2}\zeta(-1)}}{\prod_{n=1}^{\infty}(1-q^n)} = \frac{1}{q^{1/24}\prod_{n=1}^{\infty}(1-q^n)} = \frac{1}{\eta(q)}.$$

The fermionic string

Lest ye think that yon partition function be only coincidentally a modular form, we can check another example: the fermionic string.

The fermionic string

Lest ye think that yon partition function be only coincidentally a modular form, we can check another example: the fermionic string.

Without getting bogged down in details, the essential differences are:

The fermionic string

Lest ye think that yon partition function be only coincidentally a modular form, we can check another example: the fermionic string.

Without getting bogged down in details, the essential differences are:

- The energy spectrum respects the Pauli exclusion principle.

The fermionic string

Lest ye think that yon partition function be only coincidentally a modular form, we can check another example: the fermionic string.

Without getting bogged down in details, the essential differences are:

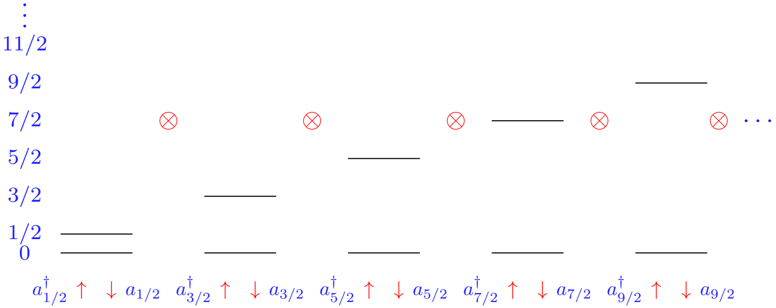
- The energy spectrum respects the Pauli exclusion principle.
- Being spin-$\frac{1}{2}$ modifies the (relative) excited state energies.

The fermionic string

Lest ye think that yon partition function be only coincidentally a modular form, we can check another example: the fermionic string.

Without getting bogged down in details, the essential differences are:

- The energy spectrum respects the Pauli exclusion principle.
- Being spin-$\frac{1}{2}$ modifies the (relative) excited state energies.

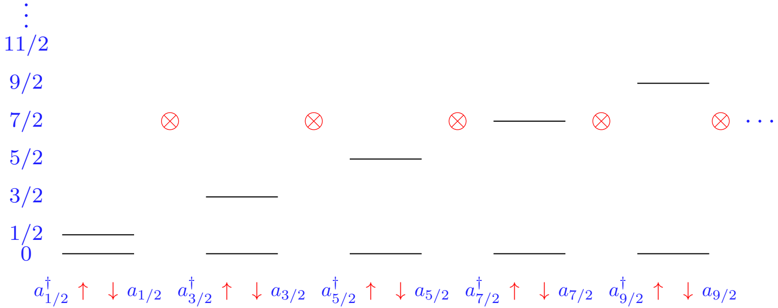


The fermionic string

Lest ye think that yon partition function be only coincidentally a modular form, we can check another example: the fermionic string.

Without getting bogged down in details, the essential differences are:

- The energy spectrum respects the Pauli exclusion principle.
- Being spin-$\frac{1}{2}$ modifies the (relative) excited state energies.



[Again, we're ignoring the zero-point energies because that requires more work!]

The partition function of the n-th fermionic “harmonic oscillator” is thus $1 + q^{n-1/2}$ [neglecting the zero-point energies], so that of the fermionic string is

The partition function of the n-th fermionic "harmonic oscillator" is thus $1 + q^{n-1/2}$ [neglecting the zero-point energies], so that of the fermionic string is

$$Z_{FS}(q) = q^{E_0} \prod_{n=1}^{\infty} (1 + q^{n-1/2})$$

The partition function of the n-th fermionic "harmonic oscillator" is thus $1 + q^{n-1/2}$ [neglecting the zero-point energies], so that of the fermionic string is

$$Z_{FS}(q) = q^{E_0} \prod_{n=1}^{\infty} (1 + q^{n-1/2}) = q^{E_0} \sqrt{\frac{\prod_{n=1}^{\infty} (1 + q^{n-1/2})^2 (1 - q^n)}{\prod_{n=1}^{\infty} (1 - q^n)}}.$$

The partition function of the n-th fermionic "harmonic oscillator" is thus $1 + q^{n-1/2}$ [neglecting the zero-point energies], so that of the fermionic string is

$$Z_{FS}(q) = q^{E_0} \prod_{n=1}^{\infty} (1 + q^{n-1/2}) = q^{E_0} \sqrt{\frac{\prod_{n=1}^{\infty} (1 + q^{n-1/2})^2 (1 - q^n)}{\prod_{n=1}^{\infty} (1 - q^n)}}.$$

The (regularised) zero-point energy turns out to be $E_0 = -\frac{1}{48}$, hence

$$Z_{FS}(q) = \sqrt{\frac{\prod_{n=1}^{\infty} (1 + q^{n-1/2})^2 (1 - q^n)}{q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)}}$$

The partition function of the n-th fermionic "harmonic oscillator" is thus $1 + q^{n-1/2}$ [neglecting the zero-point energies], so that of the fermionic string is

$$Z_{FS}(q) = q^{E_0} \prod_{n=1}^{\infty} (1 + q^{n-1/2}) = q^{E_0} \sqrt{\frac{\prod_{n=1}^{\infty}(1 + q^{n-1/2})^2 (1 - q^n)}{\prod_{n=1}^{\infty}(1 - q^n)}}.$$

The (regularised) zero-point energy turns out to be $E_0 = -\frac{1}{48}$, hence

$$Z_{FS}(q) = \sqrt{\frac{\prod_{n=1}^{\infty}(1 + q^{n-1/2})^2 (1 - q^n)}{q^{1/24} \prod_{n=1}^{\infty}(1 - q^n)}} = \sqrt{\frac{\vartheta_3(q)}{\eta(q)}},$$

which is indeed another modular form (as advertised)!

The partition function of the n-th fermionic "harmonic oscillator" is thus $1 + q^{n-1/2}$ [neglecting the zero-point energies], so that of the fermionic string is

$$Z_{FS}(q) = q^{E_0} \prod_{n=1}^{\infty} (1 + q^{n-1/2}) = q^{E_0} \sqrt{\frac{\prod_{n=1}^{\infty}(1 + q^{n-1/2})^2 (1 - q^n)}{\prod_{n=1}^{\infty}(1 - q^n)}}.$$

The (regularised) zero-point energy turns out to be $E_0 = -\frac{1}{48}$, hence

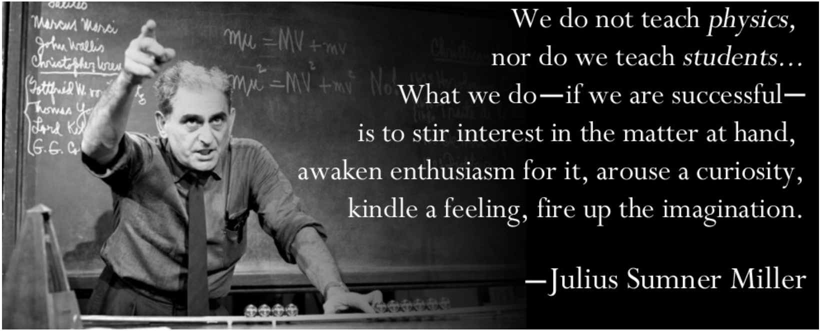
$$Z_{FS}(q) = \sqrt{\frac{\prod_{n=1}^{\infty}(1 + q^{n-1/2})^2 (1 - q^n)}{q^{1/24} \prod_{n=1}^{\infty}(1 - q^n)}} = \sqrt{\frac{\vartheta_3(q)}{\eta(q)}},$$

which is indeed another modular form (as advertised)! Even better:

- Inserting a "fermion number" operator $(-1)^F$ into $Z_{FS}(q)$ results instead in $\sqrt{\vartheta_4(q)/\eta(q)}$ [the superpartition function].
- Exchanging antiperiodic boundary conditions [the Neveu–Schwarz sector] for periodic ones [the Ramond sector] results instead in $\sqrt{\vartheta_2(q)/\eta(q)}$.

Why is it so?

mathematical!
∨



The modular machine

We've seen that stringy quantum field theories, actually CFTs (conformal field theories), have modular forms for partition functions.

The modular machine

We've seen that stringy quantum field theories, actually CFTs (conformal field theories), have modular forms for partition functions.

Is this appearance of number theory in basic physics mere coincidence or is there something deeper going on?

The modular machine

We've seen that stringy quantum field theories, actually CFTs (conformal field theories), have modular forms for partition functions.

Is this appearance of number theory in basic physics mere coincidence or is there something deeper going on?

As always, this ain't no coincidence. It holds true for all CFTs (and is even a theorem for the so-called "strongly rational" ones).

The modular machine

We've seen that stringy quantum field theories, actually CFTs (conformal field theories), have modular forms for partition functions.

Is this appearance of number theory in basic physics mere coincidence or is there something deeper going on?

As always, this ain't no coincidence. It holds true for all CFTs (and is even a theorem for the so-called "strongly rational" ones).

The explanation is a marvellous confluence of conformal physics and the mathematics of complex curves, *aka*. Riemann surfaces.

The modular machine

We've seen that stringy quantum field theories, actually CFTs (conformal field theories), have modular forms for partition functions.

Is this appearance of number theory in basic physics mere coincidence or is there something deeper going on?

As always, this ain't no coincidence. It holds true for all CFTs (and is even a theorem for the so-called "strongly rational" ones).

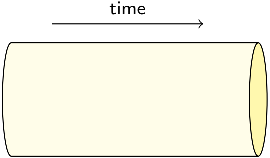
The explanation is a marvellous confluence of conformal physics and the mathematics of complex curves, *aka*. Riemann surfaces.

We won't be able to do justice to this here, but the missing details are (hopefully) covered in MAST90056 and MAST90069.

Our story starts with the following observations:

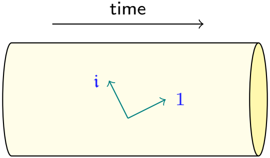
Our story starts with the following observations:

- A string is a circle, so it sweeps out a cylinder as it evolves in time.



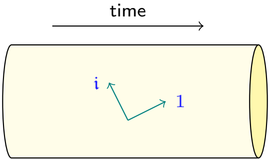
Our story starts with the following observations:

- A string is a circle, so it sweeps out a cylinder as it evolves in time.
- The equations defining the "conformal" nature of the theory reduce to the Cauchy–Riemann equations, so we need to equip the cylinder with a complex structure (essentially, a consistent choice of i).



Our story starts with the following observations:

- A string is a circle, so it sweeps out a cylinder as it evolves in time.
- The equations defining the "conformal" nature of the theory reduce to the Cauchy–Riemann equations, so we need to equip the cylinder with a complex structure (essentially, a consistent choice of i).
- Luckily, there is only one way to do this.

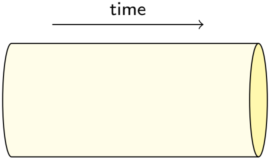


However, partition functions make this game much more complicated:

However, partition functions make this game much more complicated:

- A trace is a sum over a basis of (normalised) eigenvectors $\{v_n\}$:

$$Z(q) = \operatorname{tr} q^H = \sum_n \langle v_n \mid q^H \mid v_n \rangle = \sum_n \langle v_n \mid \mathrm{e}^{-\mathrm{i}Ht/\hbar} \mid v_n \rangle.$$

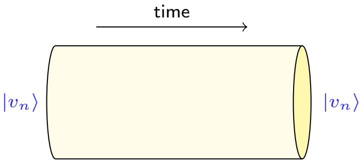


However, partition functions make this game much more complicated:

- A trace is a sum over a basis of (normalised) eigenvectors $\{v_n\}$:

$$Z(q) = \operatorname{tr} q^H = \sum_n \langle v_n \mid q^H \mid v_n \rangle = \sum_n \langle v_n \mid \mathrm{e}^{-\mathrm{i}Ht/\hbar} \mid v_n \rangle.$$

- It therefore sums the evolution from one basis state to itself.

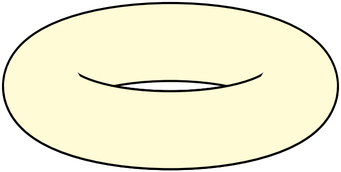


However, partition functions make this game much more complicated:

- A trace is a sum over a basis of (normalised) eigenvectors $\{v_n\}$:

$$Z(q) = \operatorname{tr} q^H = \sum_n \langle v_n \,|\, q^H \,|\, v_n \rangle = \sum_n \langle v_n \,|\, \mathrm{e}^{-\mathrm{i}Ht/\hbar} \,|\, v_n \rangle.$$

- It therefore sums the evolution from one basis state to itself.
- The cylinder is therefore effectively replaced by a torus!

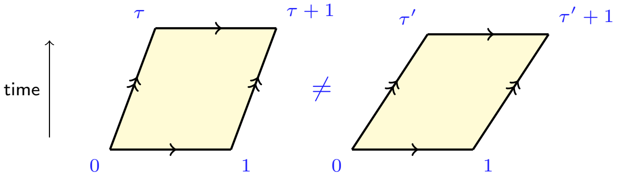


However, partition functions make this game much more complicated:

- A trace is a sum over a basis of (normalised) eigenvectors $\{v_n\}$:

$$Z(q) = \operatorname{tr} q^H = \sum_n \langle v_n \,|\, q^H \,|\, v_n \rangle = \sum_n \langle v_n \,|\, \mathsf{e}^{-\mathrm{i}Ht/\hbar} \,|\, v_n \rangle.$$

- It therefore sums the evolution from one basis state to itself.
- The cylinder is therefore effectively replaced by a torus!
- However, a torus has uncountably many different complex structures, depending on how we glue it together.

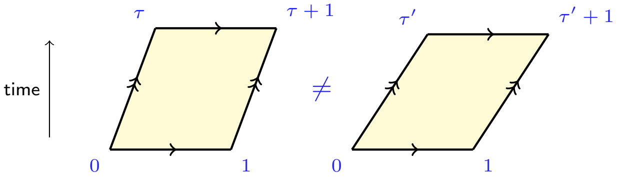


However, partition functions make this game much more complicated:

- A trace is a sum over a basis of (normalised) eigenvectors $\{v_n\}$:

$$Z(q) = \operatorname{tr} q^H = \sum_n \langle v_n \,|\, q^H \,|\, v_n \rangle = \sum_n \langle v_n \,|\, \mathrm{e}^{-\mathrm{i}Ht/\hbar} \,|\, v_n \rangle.$$

- It therefore sums the evolution from one basis state to itself.
- The cylinder is therefore effectively replaced by a torus!
- However, a torus has uncountably many different complex structures, depending on how we glue it together.
- These are classified by $\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$, modulo $\tau \mapsto \frac{-1}{\tau}$, $\tau \mapsto \tau + 1$.



So the partition function of a CFT depends on the choice of τ because q does (hence t or T does), hence on the torus.

So the partition function of a CFT depends on the choice of τ because q does (hence t or T does), hence on the torus.

Being conformal means only the complex structure of the torus matters.

So the partition function of a CFT depends on the choice of τ because q does (hence t or T does), hence on the torus.

Being conformal means only the complex structure of the torus matters.

The partition function must thus be invariant under the modular transformations

$$\tau \mapsto -\frac{1}{\tau} \quad \text{and} \quad \tau \mapsto \tau + 1,$$

because these precisely preserve the complex structure of the torus.

So the partition function of a CFT depends on the choice of τ because q does (hence t or T does), hence on the torus.

Being conformal means only the complex structure of the torus matters.

The partition function must thus be invariant under the modular transformations

$$\tau \mapsto -\frac{1}{\tau} \quad \text{and} \quad \tau \mapsto \tau + 1,$$

because these precisely preserve the complex structure of the torus.

These transformations generate the modular group

$$\mathsf{PSL}(2;\mathbb{Z}) \simeq \left\{ \tau \mapsto \frac{a\tau + b}{c\tau + d} \;:\; a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}.$$

So the partition function of a CFT depends on the choice of τ because q does (hence t or T does), hence on the torus.

Being conformal means only the complex structure of the torus matters.

The partition function must thus be invariant under the modular transformations

$$\tau \mapsto -\frac{1}{\tau} \quad \text{and} \quad \tau \mapsto \tau + 1,$$

because these precisely preserve the complex structure of the torus.

These transformations generate the modular group

$$\mathsf{PSL}(2;\mathbb{Z}) \simeq \left\{ \tau \mapsto \frac{a\tau + b}{c\tau + d} \ : \ a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}.$$

Summary: CFT partition functions are modular forms.

What's next?

- First, I should admit to some flagrant lying...

What's next?

- First, I should admit to some flagrant lying...
- None of the so-called partition functions that we've discussed have actually been modular invariants.

What's next?

- First, I should admit to some flagrant lying...
- None of the so-called partition functions that we've discussed have actually been modular invariants.
- For the bosonic string, I ignored two key points:

  1. Strings not only vibrate, they also move (they have momentum).
  2. Their conformal nature implies a factorisation into independent holomorphic and antiholomorphic sectors.

What's next?

- First, I should admit to some flagrant lying...
- None of the so-called partition functions that we've discussed have actually been modular invariants.
- For the bosonic string, I ignored two key points:
  1. Strings not only vibrate, they also move (they have momentum).
  2. Their conformal nature implies a factorisation into independent holomorphic and antiholomorphic sectors.
- For the fermionic string, there is no momentum (!) but I did neglect the antiholomorphic contributions. However, the antiperiodic boundary conditions mean the complex torus should be replaced by an appropriate "double cover" (which changes the modular group).

What's next?

- First, I should admit to some flagrant lying...
- None of the so-called partition functions that we've discussed have actually been modular invariants.
- For the bosonic string, I ignored two key points:
    1. Strings not only vibrate, they also move (they have momentum).
    2. Their conformal nature implies a factorisation into independent holomorphic and antiholomorphic sectors.
- For the fermionic string, there is no momentum (!) but I did neglect the antiholomorphic contributions. However, the antiperiodic boundary conditions mean the complex torus should be replaced by an appropriate "double cover" (which changes the modular group).
- But in both cases, we can fix it and show that the partition function is indeed modular invariant (for an appropriate definition of modular).

- The modular machine that turns CFTs into modular forms has been good news for number theorists.

- The modular machine that turns CFTs into modular forms has been good news for number theorists.
- This has not only revitalised interest in (vector-valued) modular forms, it also led directly to the concept of a modular tensor category.

- The modular machine that turns CFTs into modular forms has been good news for number theorists.
- This has not only revitalised interest in (vector-valued) modular forms, it also led directly to the concept of a modular tensor category.
- So CFT has therefore also been good news for category theorists.

- The modular machine that turns CFTs into modular forms has been good news for number theorists.
- This has not only revitalised interest in (vector-valued) modular forms, it also led directly to the concept of a modular tensor category.
- So CFT has therefore also been good news for category theorists.
- The recent push to understand more exotic examples CFTs has also revolutionised the study of exotic variants of modular forms, *eg.*
  - Ramanujan's mock modular forms.
  - Partial and false theta functions.
  - Appell-Lerch sums.

- The modular machine that turns CFTs into modular forms has been good news for number theorists.
- This has not only revitalised interest in (vector-valued) modular forms, it also led directly to the concept of a modular tensor category.
- So CFT has therefore also been good news for category theorists.
- The recent push to understand more exotic examples CFTs has also revolutionised the study of exotic variants of modular forms, *eg.*
    - Ramanujan's mock modular forms.
    - Partial and false theta functions.
    - Appell-Lerch sums.
- These examples seem to arise quite naturally in the simplest known examples of the so-called logarithmic CFTs.

- The modular machine that turns CFTs into modular forms has been good news for number theorists.
- This has not only revitalised interest in (vector-valued) modular forms, it also led directly to the concept of a modular tensor category.
- So CFT has therefore also been good news for category theorists.
- The recent push to understand more exotic examples CFTs has also revolutionised the study of exotic variants of modular forms, *eg.*
  - Ramanujan's mock modular forms.
  - Partial and false theta functions.
  - Appell-Lerch sums.
- These examples seem to arise quite naturally in the simplest known examples of the so-called logarithmic CFTs.
- There is currently a focus on understanding these "log-modular" forms and the corresponding log-modular tensor categories... but that's a topic for a completely different talk!

"*Only one who attempts the absurd is capable of achieving the impossible.*"

— Miguel de Unamuno