In which Physics demands new Mathematics

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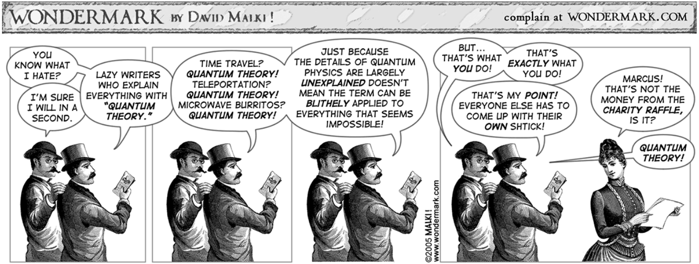
AustMS #67



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Outline

1. What is conformal field theory?
2. Vertex operator algebras and modules
3. Weight modules for $\mathfrak{sl}_2$
4. Weight modules for $\mathfrak{sl}_3$
5. Things to come?

Conformal field theory

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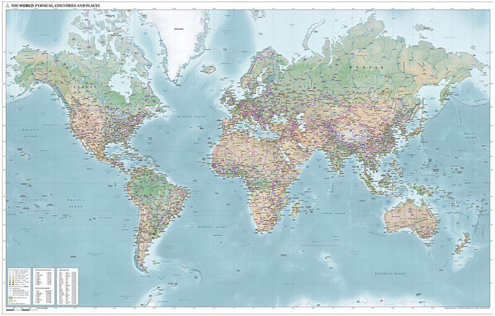

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However, when $d = 2$ ($\mathbb{R}^2 = \mathbb{C}$), the Lie algebra of infinitesimal conformal transformations is infinite-dimensional. It is a direct sum of two copies of the Virasoro algebra $\mathfrak{Vir} = \mathrm{span}\{L_n, C : n \in \mathbb{Z}\}$:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C, \quad [L_m, C] = 0.$$

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The representations are also infinite-dimensional, which makes the mathematics of the quantum state space more interesting (and fun)!

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"*I am an old man and I know that a definition cannot be so complicated.*"
— I. M. Gelfand (after a seminar on vertex algebras)

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[We don't need to know what these adjectives mean except that they are various degrees of "nice".]

Some of the most accessible VOAs are those constructed from a simple Lie algebra. We'll consider $\mathfrak{g} = \mathfrak{sl}_N$, the Lie algebra of traceless $N \times N$ complex matrices, equipped with the matrix commutator

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From this, we construct the (untwisted) affine Kac–Moody algebra

$$\widehat{\mathfrak{sl}}_N = \mathfrak{sl}_N[t, t^{-1}] \oplus \mathbb{C}K,$$
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Since the At^m with $m = 0$ form a copy of $\mathfrak{sl}_N$ in $\widehat{\mathfrak{sl}}_N$, we may induce the trivial $\mathfrak{sl}_N$-module to a "parabolic Verma module" of $\widehat{\mathfrak{sl}}_N$ on which K acts as multiplication by some $k \in \mathbb{C}$.

When $k \neq -n$, this module becomes a VOA: the universal affine VOA of $\mathfrak{sl}_N$ with level k, denoted by $\mathsf{V}^k(\mathfrak{sl}_N)$ [Frenkel–Zhu'92].

Representations

Now that we have a VOA $\mathsf{V}^k(\mathfrak{sl}_N)$, we can ask about its representation theory. This turns out to be complicated because every (smooth) level-k $\widehat{\mathfrak{sl}}_N$-module is a $\mathsf{V}^k(\mathfrak{sl}_N)$-module.

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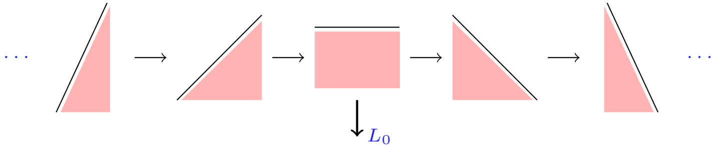
Using [Futorny–Tsylke'01], every irreducible in this category can be twisted by an automorphism to get a lower-bounded one, *ie.* one for which the eigenvalues of the Virasoro zero mode L_0 are bounded below.



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- For $N = 2$, it dates back at least as far as [Gabriel'59].
- For $N = 3$, this is much more recent [Britten–Lemire–Futorny'95].
- For $N > 3$ (and all simple $\mathfrak{g}$), [Fernando'90, Mathieu'00].

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cf., the classification of finite-dimensional modules [Cartan1913].

But physics is more interested in modules for the simple quotient $\mathsf{L}_k(\mathfrak{sl}_N)$.

Theorem [Gorelik–Kac'06]: $\mathsf{V}^k(\mathfrak{sl}_N)$ is a simple VOA unless

$$k + N = \frac{u}{v}, \quad \text{with } u \in \mathbb{Z}_{\geqslant 2} \text{ and } v \in \mathbb{Z}_{\geqslant 1} \text{ coprime.}$$

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This ideal is easy to describe if $v = 1$ and is otherwise difficult.

The VOA $\mathsf{L}_k(\mathfrak{sl}_N)$ with $k \in \mathbb{Z}_{\geqslant 0}$ describes string theory on SU_N [Witten'84].

- The category of weight $\mathsf{L}_k(\mathfrak{sl}_N)$-modules with finite-dimensional weight spaces is finite and semisimple.
- Every irreducible $\mathrm{Zhu}\big[\mathsf{L}_k(\mathfrak{sl}_N)\big]$-module is highest-weight (actually finite-dimensional).

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There are 4D supersymmetric gauge theories whose 2D dual CFTs are described by the $\mathsf{L}_k(\mathfrak{sl}_N)$ with $u = N$ and $v > 1$ [Beem *et al.*'13].

- The category of weight $\mathsf{L}_k(\mathfrak{sl}_N)$-modules with finite-dimensional weight spaces is neither finite nor semisimple.
- There are infinite-dimensional $\mathrm{Zhu}\big[\mathsf{L}_k(\mathfrak{sl}_N)\big]$-modules that are not highest-weight (with respect to any Borel subalgebra).

Weight modules for $\mathfrak{sl}_2$

Physics wants us to understand the representation theory of VOAs like $\mathsf{L}_k(\mathfrak{sl}_N)$. Maths helps us reduce this to understanding weight modules for $\mathfrak{sl}_N$ (with finite-dimensional weight spaces).

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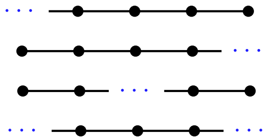

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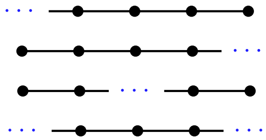

Types 1. and 2. are described by one continuous parameter, type 3. by one discrete parameter and type 4. by two continuous parameters.

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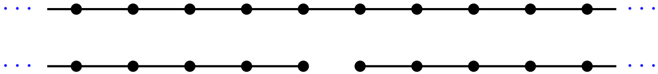

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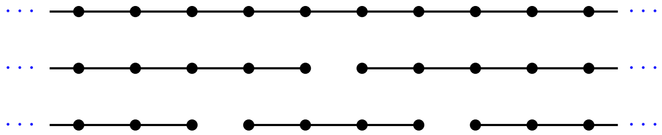

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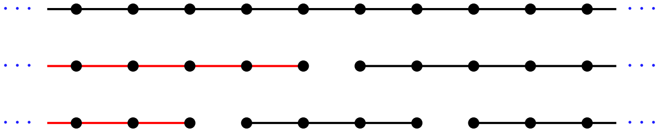

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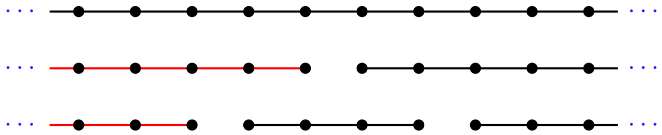

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In fact, these infinite-dimensional highest-weight submodules completely determine the type-4. modules, both reducible and irreducible.

Via degenerations, the infinite-dimensional highest-weight modules thus control the classification of all irreducible weight $\mathfrak{sl}_2$-modules, *cf.* [Duflo'71].

To make this precise, [Mathieu'00] introduced coherent families.

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Theorem [Mathieu'00]: For $\mathfrak{sl}_2$, equivalence classes of infinite-dimensional irreducible highest-weight modules and coherent families are in bijection.

Proof: Construct a coherent family from an infinite-dimensional irreducible highest-weight module using "twisted localisation" functors...

What does this mean for VOAs and physics?

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The classification for $\mathsf{L}_k(\mathfrak{sl}_2)$ is deeper. There are now only finitely many coherent families of relaxed modules and so finitely many highest-weight modules [Adamović–Milas'95, DR–Wood'15, Kawasetsu–DR'19].

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Our understanding of the weight $\mathsf{L}_k(\mathfrak{sl}_2)$-module category is pretty good:

- It is modular, though non-finite and non-semisimple [Creutzig–DR'13].
- It is braided tensor with enough projectives and injectives [Creutzig'23].
- The irreducible fusion rules are known (and nearly proven).
- Rigidity is within our grasp [Orosz Hunziker–Wood'24?].

Weight modules for $\mathfrak{sl}_3$

We can try to understand weight $\mathfrak{sl}_3$-modules by studying the centraliser $\mathrm{U}(\mathfrak{sl}_3)^{\mathfrak{h}}$. However:

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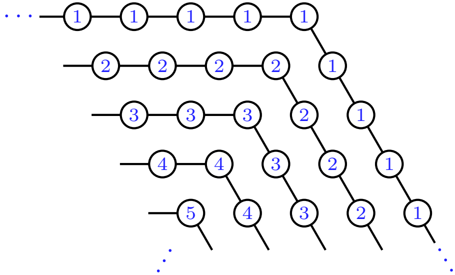

[Futorny'89] wrote down generators and some relations for $\mathrm{U}(\mathfrak{sl}_3)^{\mathfrak{h}}$, which was enough to classify the weight $\mathfrak{sl}_3$-modules with finite-dimensional weight spaces [Britten–Lemire–Futorny'95].

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Theorem [Mathieu'00]: Coherent families are in bijection with equivalence classes of bounded infinite-dimensional highest-weight modules.

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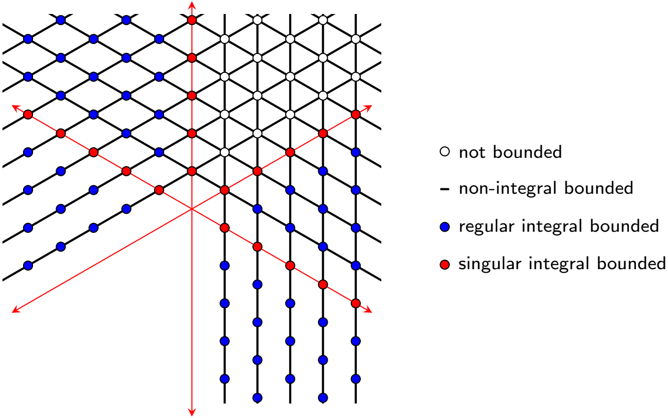

The equivalence classes correspond to orbits under the shifted action of the Weyl group (*ie.* reflections about the red axes).

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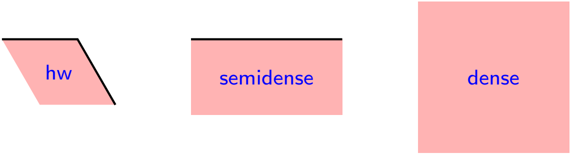

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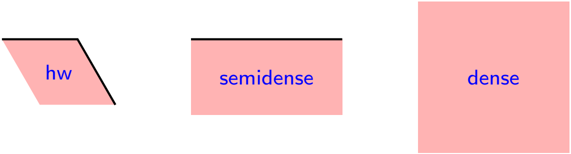

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- A finite number of 2-parameter families of relaxed modules.
- A finite number of 1-parameter families of semirelaxed modules.
- A finite number of highest-weight modules.

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Physics is demanding that we enlarge our module category.

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Up to degenerations and automorphism twists, the irreducible $\mathsf{L}_k(\mathfrak{sl}_3)$-modules with these Zhu images generate a subcategory of weight $\mathsf{L}_k(\mathfrak{sl}_3)$-modules, allowing infinite-dimensional weight spaces when $v \geqslant 3$.

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The Zhu-module subcategory of weight modules should then admit a filtration/poset corresponding to nilpotent orbits:

$$\begin{array}{ccccccc} \text{finite-dim.} & & \text{finite-dim.} & & \text{“slightly infinite-dim.”} & & \\ \text{modules} & \subset & \text{weight spaces} & \subset & \text{weight spaces} & \subset & \cdots \\ \textit{zero} & & \textit{minimal} & & \cdots & & \textit{principal} \end{array}$$

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We hope to connect with this work in the near future.

Either way, there is ample precedent to expect that the mathematical theory that physics is forcing us to develop will be a beautiful one!

"*Only one who attempts the absurd is capable of achieving the impossible.*"

— Miguel de Unamuno