Inverse quantum hamiltonian reduction

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February 7, 2024
ANZAMP 11, Katoomba

Outline

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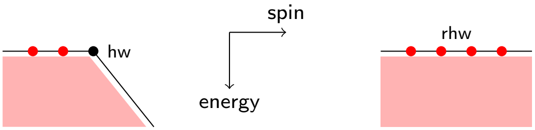
Relaxed modules indeed fix the Verlinde formula [Creutzig–DR'12,'13]. But to verify this, we needed to compute their characters.

A relaxed hw module is just like a hw module, except that the space of ground states need not have a state of maximal charge/spin.



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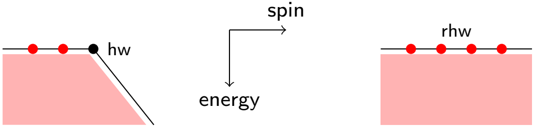

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The $\mathfrak{sl}_2$ WZW model of level $\mathsf{k} = -2 + \frac{\mathsf{u}}{\mathsf{v}}$ has a finite number of hw modules, but a finite number of 1-parameter families of relaxed modules:

- hw: $\mathcal{H}_{r,s}$, $r = 1, \dots, \mathsf{u} - 1$, $s = 0, \dots, \mathsf{v} - 1$.
- relaxed: $\mathcal{R}_{[j];r,s}$, $[j] \in \mathbb{R}/\mathbb{Z}$, $r = 1, \dots, \mathsf{u} - 1$, $s = 1, \dots, \mathsf{v} - 1$.

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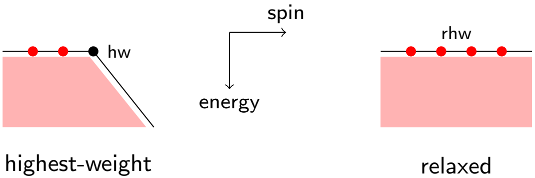

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The relaxed modules have a Kac-table symmetry: $\mathcal{R}_{[j];r,s} = \mathcal{R}_{[j];\mathsf{u}-r,\mathsf{v}-s}$. The hw modules do not.

Curiously, the characters of the $\mathcal{R}_{[j];r,s}$ are [Creutzig–DR'13, Kawasetsu–DR'18]

$$\mathrm{ch}\big[\mathcal{R}_{[j];r,s}\big](z;q) = \mathrm{tr}(\mathsf{z}^{\mathsf{spin}}\mathsf{q}^{\mathsf{energy}}) = \frac{\chi^{\mathsf{Vir.}}_{r,s}(\mathsf{q})}{\eta(\mathsf{q})^2} \sum_{n\in\mathbb{Z}} \mathsf{z}^{j+n},$$

where for $r = 1, \dots, \mathsf{u}-1$ and $s = 1, \dots, \mathsf{v}-1$, $\chi^{\mathsf{Vir.}}_{r,s}(\mathsf{q})$ is the character of the hw irrep $\mathcal{L}_{r,s}$ of the Virasoro minimal model $\mathsf{M}(\mathsf{u},\mathsf{v})$.

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... but first a word from our sponsor: quantum hamiltonian reduction.

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More precisely, let $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)$ ($\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$) be the universal (irreducible) $\mathfrak{sl}_2$ VOA of level $\mathsf{k} = -2 + \frac{\mathsf{u}}{\mathsf{v}}$ and let $\mathsf{Vir}(\mathsf{u}, \mathsf{v})$ ($\mathsf{M}(\mathsf{u}, \mathsf{v})$) be the universal (irreducible) Virasoro VOA of central charge $13 - 6(\frac{\mathsf{u}}{\mathsf{v}} + \frac{\mathsf{v}}{\mathsf{u}})$.

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$$\mathsf{QHR}\big(\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)\big) = \mathsf{Vir}(\mathsf{u},\mathsf{v}), \quad \text{but}$$
$$\mathsf{QHR}\big(\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)\big) = \begin{cases} 0 & \text{if } \mathsf{v} = 1, \\ \mathsf{M}(\mathsf{u},\mathsf{v}) & \text{otherwise.} \end{cases}$$

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These seem to be very hard theoretical questions, but necessary for exploring CFTs with W-algebra symmetries.

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$$\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2) \xhookrightarrow{\text{Wak.}} \beta\gamma \otimes \mathsf{V}(\mathfrak{gl}_1) \quad \text{and} \quad \mathsf{Vir}(\mathsf{u},\mathsf{v}) \xhookrightarrow{\text{FF}} \mathsf{V}(\mathfrak{gl}_1)$$

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The upshot is an embedding

$$\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2) \hookrightarrow \Pi \otimes \mathsf{Vir}(\mathsf{u},\mathsf{v}).$$

If $\mathsf{v} \neq 1$, we also get the irreducible embedding [Adamović'17]:

$$\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2) \hookrightarrow \Pi \otimes \mathsf{M}(\mathsf{u},\mathsf{v}).$$

The point of these embeddings is that any $M(u, v)$-module may be tensored with a Π-module and restricted to get a $L_k(\mathfrak{sl}_2)$-module, $v \neq 1$.

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Better, inverse QHR constructs almost-irreducible relaxed modules, meaning that almost all are irreducible [Adamović–Kawasetsu–DR'20]:

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Even better again, every irreducible relaxed $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-module may be constructed in this way [Adamović–Kawasetsu–DR'23].

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These reducible relaxed $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-modules are indecomposable [*cf.* logCFT] with the following short exact sequences [Kawasetsu–DR'18]:

$$0 \to \mathcal{H}_{r,s} \to \mathcal{R}_{[j(r,s)];r,s} \to \mathrm{c}(\mathcal{H}_{\mathsf{u}-r,\mathsf{v}-s}) \to 0.$$

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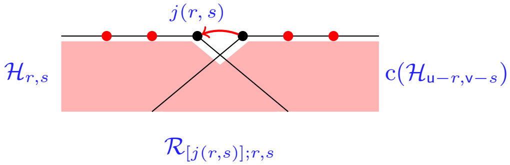

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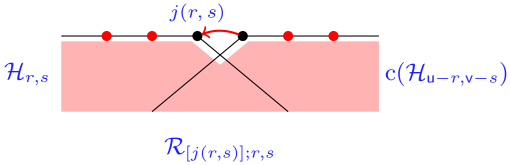

In this way, we also construct all the hw $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-modules!

[Technically, we also need spectral flow here...]

Outlook

Inverse quantum hamiltonian reduction is a very powerful tool for analysing the relaxed modules.

For $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$, $\mathsf{v} \neq 1$, the known Virasoro minimal model spectrum can be used to (re)prove the spectrum and compute all characters.

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The first example to consider is $\mathfrak{sl}_3$ [*cf.* Fasquel's talk for the state of the art]:

$$\mathsf{L}_\mathsf{k}(\mathfrak{sl}_3) \to \mathsf{BP}_\mathsf{k} \to \mathsf{W}_{3,\mathsf{k}}$$

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In general, inverse QHR is expected to "invert" the partial QHR of [Madsen–Ragoucy'95, Morgan'15, Genra–Juillard'22].

But, classifying irreducibles is just the beginning!

The plan is to lift data (classification, categorical, analytic, *etc.*) from a well understood W-algebra to a not-well understood one.

For example, regular W-algebras are rational and C_2-cofinite for nondegenerate levels [Arakawa'10,'12]. Inverse QHR lifts this to understand subregular W-algebras and beyond, perhaps to the affine VOA.

For degenerate levels, we expect that the role of the principal W-algebra will instead be played by the exceptional ones [Arakawa–van Ekeren'19].

An open question is to develop tools to analyse these W-algebras for nonadmissible levels. Sometimes one can use singular vectors [Adamović–Kontrec'19,'20, Adamović–Perše–Vukorepa'21, ...], but in the most mysterious cases one cannot. Inverse QHR is nevertheless still available...

"*Only one who attempts the absurd is capable of achieving the impossible.*"

— Miguel de Unamuno