NOAs and Zhu algebras

History:

Math –	Zhu '90	PhD
	Frenkel-Zhu '92	Universal aff. Vir ; WZW
	Wang '93	Vir min. mods
	Kac-Wang '93	Super-Zhu
	Li '94	PhD
	Zhu '96	Modular invariance
	Dong-Li-Mason '96	Higher Zhu
Physics –	Gepner-Witten '86	WZW (SV decoupling)
	Feigin-Nakanishi-Ooguri '92	Vir min mods (annihilating ideals)

Literature:

Brungs-Nahm hep-th/9811239 Zhu = deform. of norm. ord.

Nagatomo-Tsuchiya math.QA/0206223 Zhu = zero mode algebra

Kac-Raina-Rozhkovskaya Ch. 18 Zhu prod. = GCR

DR-Wood 1501.07318, 1606.04187 (Appendices)

Let V be an (even) VOA with $V = \bigoplus_{\Delta \in \mathbb{N}} V_\Delta$, where V_Δ is the L_0-eigenspace of eigenvalue Δ.

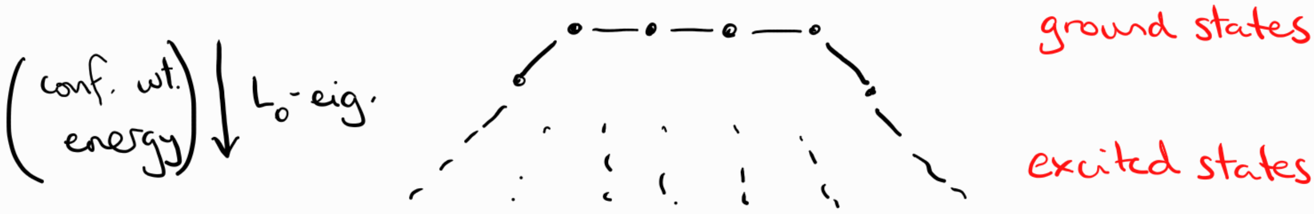
For $A \in V_\Delta$, we write $A(z) \equiv Y(A,z)$ and

$$A(z) = \sum_{n\in\mathbb{Z}}' A_n z^{-n-\Delta}.$$

① The idea

VOA mods can often be identified from their "ground states".

This is true for irreducible modules!



<u>Def</u>: A VOA-mod. M is <u>lower bounded</u> if $\exists\, \Delta \in \mathbb{C}$ s.t.

$$M = \bigoplus_{m \in \mathbb{N}} M_{\Delta + m},$$

where $M_{\Delta+m}$ is the (generalised) L_0-eigenspace of eig. $\Delta+m$

<u>Def</u>: $v \in M$ is a <u>ground state</u> if $A_n v = 0 \quad \forall\, A \in V,\ n > 0$.

Easy facts:

- If M is lower bounded, with lower bound Δ, then every $v \in M_\Delta$ is a ground state.
- The converse is true when M is irreducible.
- Since $[L_0, A_n] = -nA_n$, every zero mode A_0 preserves the subspace of ground states of any VOA mod.

"Def." The <u>Zhu algebra</u> of a VOA V is the associative algebra of its zero modes acting on arbitrary ground states.

<u>Examples</u>:

- Free boson – Heisenberg VOA H. $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$.

State	Field	Zero mode	Acts on ground states as
Ω	$\mathbb{1}(z)$	$\mathbb{1}$	$\mathbb{1}$
$a_{-1}\Omega$	$a(z)$	a_0	a_0
$a_{-2}\Omega$	$\partial a(z)$	$-a_0$	a_0
$a_{-1}^2\Omega$	$:a(z)a(z):$	$\sum_{r\leq -1} a_r a_{-r} + \sum_{r\geq 0} a_{-r} a_r$	a_0^2

$\rightsquigarrow \mathrm{Zhu}[H] \simeq \mathbb{C}[a_0]$.

- Universal affine VOA $V^k(\mathfrak{g})$. The Zhu algebra is

$$\mathrm{Zhu}[V^k(\mathfrak{g})] \simeq \mathcal{U}(\mathfrak{g}).$$

(The conformal vector $L_{-2}\Omega$ maps to the quadratic Casimir.)

- Simple affine VOA $L_1(\mathfrak{sl}_2) = \dfrac{V^1(\mathfrak{sl}_2)}{\langle e_{-1}^2\Omega\rangle}$.

Since the Zhu-image of $e_{-1}^2\Omega$ is e_0^2, we have

$$\mathrm{Zhu}[L_1(\mathfrak{sl}_2)] \simeq \frac{\mathrm{Zhu}[V^1(\mathfrak{sl}_2)]}{\langle e_0^2\rangle} \simeq \frac{\mathcal{U}(\mathfrak{sl}_2)}{\langle e^2\rangle}.$$

② The point

Easy: If M is an <u>irreducible</u> lower-bounded V-mod., then its subspace of ground states is an <u>irreducible</u> $\mathrm{Zhu}[V]$-mod.

<u>Zhu's big theorem</u>: The converse is true!

So we can classify irreducible lower-bounded V-mods. by classifying irreducible $\mathrm{Zhu}[V]$-mods.

<u>Note</u>: If the VOA is C_2-cofinite (lisse), then all mods are lower bounded. (Abe-Buhl-Dong '02)

Zhu's converse is of course <u>hard</u>:

- Zhu-mod $\rightsquigarrow$ zero-mode action.
- set as ground states $\rightsquigarrow$ +ve-mode action
- now "induce" to get -ve-mode action...
  - ↳ requires imposing <u>all</u> VOA relations and checking that the ground states aren't killed.

<u>Example</u>: $\mathrm{Zhu}[L_1(sl_2)] \simeq \dfrac{\mathcal{U}(sl_2)}{\langle e^2 \rangle}$ so its irreps are the irreps of sl_2 on which e^2 acts as 0. There are 2:

trivial fundamental

$\Rightarrow$ there are only 2 lower-bounded irreps of $L_1(sl_2)$.

③ The algebra

We finally stop cheating and explain the product

What is the natural product?

The obvious one is the commutator

$$[A_0, B_0] = \oint_0 \oint_w A(z) B(w) z^{\Delta_A - 1} w^{\Delta_B - 1} \frac{dz}{2\pi i} \frac{dw}{2\pi i}$$

$$= \sum_{j \geq 0} \binom{\Delta_A - 1}{j} \left(A_{-\Delta_A + j + 1} B \right)_0 .$$

This is an identity of zero modes, but it isn't associative (and it doesn't invoke ground states).

Better is the generalised commutation relation

$$\sum_{j \geq 0} \Big(\underbrace{A_{-j} B_j}_{0 \; \forall j > 0} + \underbrace{B_{-j-1} A_{j+1}}_{0 \; \forall j} \Big) = \oint_0 \oint_w A(z) B(w) z^{\Delta_A} w^{\Delta_B - 1} (z-w)^{-1} \frac{dz}{2\pi i} \frac{dw}{2\pi i}$$

$$= \sum_{j \geq 0} \binom{\Delta_A}{j} \left(A_{-\Delta_A + j} B \right)_0 .$$

When acting on ground states, this simplifies to

$$(*) \qquad A_0 B_0 = \sum_{j \geq 0} \binom{\Delta_A}{j} \left(A_{-\Delta_A + j} B \right)_0 .$$

In Zhu'96, the product is introduced as

$$A * B = \oint_0 A(z) B \frac{(1+z)^{\Delta_A}}{z} \frac{dz}{2\pi i},$$

but without this "zero-mode motivation".

these RHSs are the same!

But, Zhu has a problem. Unlike the zero-mode product, his $*$-product doesn't satisfy the basic commutation relations. eg, in $\mathrm{Zhu}[V^k(\mathfrak{sl}_2)]$,

$$h * e - e * h = 4e + 2\partial e \quad \underline{\text{not}} \quad 2e.$$

Unlike the zero-mode product, $*$ is not associative:

$$(h*e)*e - h*(e*e) = 2 :\partial e e: + 2 :ee:,$$

This product fails to account for the kernel of the map from the VOA to its zero modes.

eg. $(\partial e)_0 = -e_0 \implies e + \partial e$ is in the kernel.

Let's help him out by getting the zero-mode product from another generalised commutation relation:

$$\sum_{j\geq 0}(j+1)\left(A_{-j}B_j + B_{-j-2}A_{j+2}\right) = \oint_0\oint_w A(z)B(w)\, z^{\Delta_A+1} w^{\Delta_B-1}(z-w)^{-2}\frac{dz}{2\pi i}\frac{dw}{2\pi i}$$

$0\ \forall j>0$ $\quad$ $0\ \forall j$

$$= \sum_{j\geq 0}\binom{\Delta_A+1}{j}\left(A_{-\Delta_A+j-1}B\right)_0$$

$$(*') \qquad \implies A_0 B_0 = \sum_{j\geq 0}\binom{\Delta_A+1}{j}\left(A_{-\Delta_A+j-1}B\right)_0 .$$

Subtract $(*)$ and $(*')$ and we get

$$(\circ) \qquad \sum_{j\geq 0}\binom{\Delta_A}{j}\left(A_{-\Delta_A+j-1}B\right)_0 = 0 .$$

yes these are

In Zhu '96, a "circle product" is defined by

$$A \circ B = \oint_0 A(z) B \frac{(1+z)^{\Delta_A}}{z^2} \frac{dz}{2\pi i}, \qquad \longleftarrow \text{the same}$$

but without any "zero-mode motivation".

He then defines the Zhu algebra to be

$$\mathrm{Zhu}[V] = \frac{V}{V \circ V}, \qquad \begin{aligned} &\longleftarrow A \mapsto A_0 \\ &\longleftarrow \ker(A \mapsto A_0) \end{aligned}$$

equipped with the $*$-product. He then proves that commutation relations and associativity are satisfied...

(4) Final thoughts

I'm not sure that Zhu proved that $V \circ V$ is equal to the kernel of the zero-mode map. Why couldn't there be more relations (eg. from other GCRs)? Nagatomo–Tsuchiya proved that there aren't.

Either way, Zhu's definition is arguably more elegant, but is significantly more complicated, than the zero-mode approach. This is particularly obvious when it comes to practice: zero-mode computations are orders of magnitude more efficient and so should always be preferred!

$\square$