Representations of affine vertex operator algebras and W-algebras

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[New Directions in Conformal Field Theory]

Outline

Motivation

I want to understand conformal field theory (CFT)...

rational — factorisable — non-rational

[compactified bosons]
[free fermions]
[bc-ghosts]
[minimal models]
[compact WZW models]
[a few W-algebras]

[non-compact bosons]
[Liouville/Toda]
[non-compact WZW?]

discrete - - - continuous

[symplectic fermions]
[triplet models]
[polymers, percolation?]
[log minimal models?]
[SLE/CLE?]

[$\beta\gamma$-ghosts]
[supergroup WZW]
[Nappi–Witten]
[fractional-level WZW]
[most W-algebras]
[spin chains?]

log-rational — logarithmic — generic

CFTs are built from reps of its chiral algebra, *aka*. vertex operator algebra (VOA).

A rational CFT has a VOA module category that is

- semisimple: modules are completely reducible,
- finite: there are finitely many irreducibles (up to $\cong$),
- q-finite: modules have q-characters ($\operatorname{tr} \mathsf{q}^{L_0 - \mathsf{c}/24}$).

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Log-rational means non-semisimple but finite and q-finite.
[But few accessible examples.]

Non-rational means semisimple but not finite (but can be q-finite).
[Usually notoriously difficult.]

Generically, we lose all three conditions. But here we have surprisingly many accessible (and important!) examples... this is log CFT.

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But good news! There's still an awful lot we don't yet understand...

> "... *it is a highly nontrivial problem to construct essentially any example of a vertex operator algebra.*"
>
> "*A significant feature of the theory is that the construction of modules for a vertex operator algebra is more subtle than the construction of the algebra itself.*"

— [Lepowsky–Li'04]

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This goal is still a bit lofty at present! But models with this property may be easier to analyse while exhibiting new features.

These fractional-level models are expected to act as stepping stones to a deeper understanding of physically interesting theories...

Affine VOAs and W-algebras

Input: simple Lie algebra $\mathfrak{g}$, complex number $\mathsf{k} \neq -\mathsf{h}^\vee$.

Construction: induce the trivial $\mathfrak{g}$-module to a level-k $\widehat{\mathfrak{g}}$-module.

Result: the universal affine VOA $\mathsf{V}^\mathsf{k}(\mathfrak{g})$.

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Theorem [Gorelik–Kac'06]: $V^k(\mathfrak{g})$ is not simple iff

$$k + h^\vee = \frac{u}{v}, \quad u \in \mathbb{Z}_{\geqslant 2},\ v \in \mathbb{Z}_{\geqslant 1},\ \gcd\{u, v\} = 1.$$

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Rep theory of $V^k(\mathfrak{g})$ is essentially unconstrained:

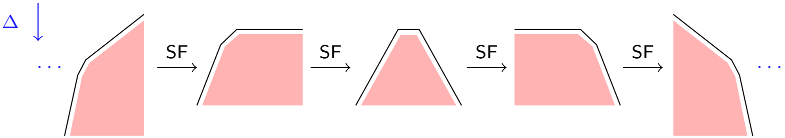
$$V^k(\mathfrak{g})\text{-module} \equiv \text{“smooth” level-}k\ \widehat{\mathfrak{g}}\text{-module}.$$

That of $L_k(\mathfrak{g})$ is much more interesting.

Weight modules

A weight module for $\widehat{\mathfrak{g}}$ is a weight module for $\mathfrak{g}$ on which L_0 acts with finite-rank Jordan blocks.

Every irreducible weight $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$-module is the "spectral flow" of a lower-bounded one. [Futorny–Tsylke'01, Adamović–Kawasetsu–DR'23]



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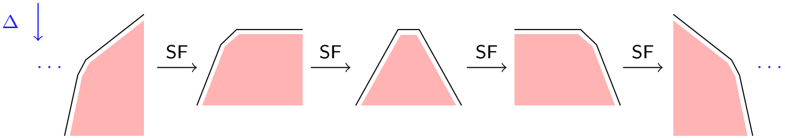
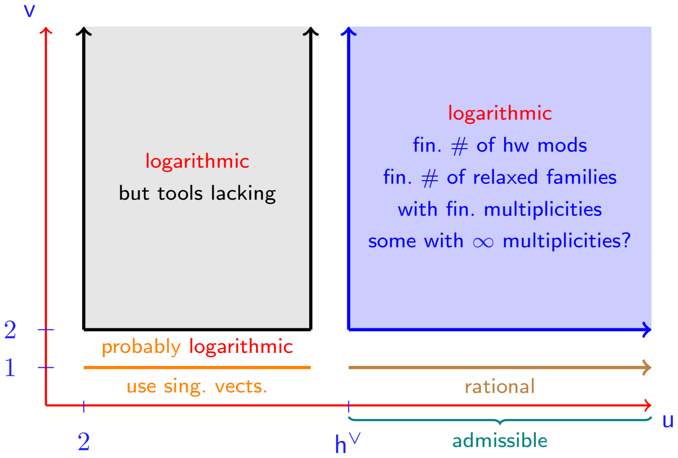


A lower-bounded irreducible is a relaxed highest-weight module [Feigin–Semikhatov–Tipunin'97, DR–Wood'15].
Relaxed means generated by a single weight vector of minimal Δ.

The weight category is modular, *wrt*. generalised characters, and closed under fusion. It's a good candidate for building consistent CFTs.

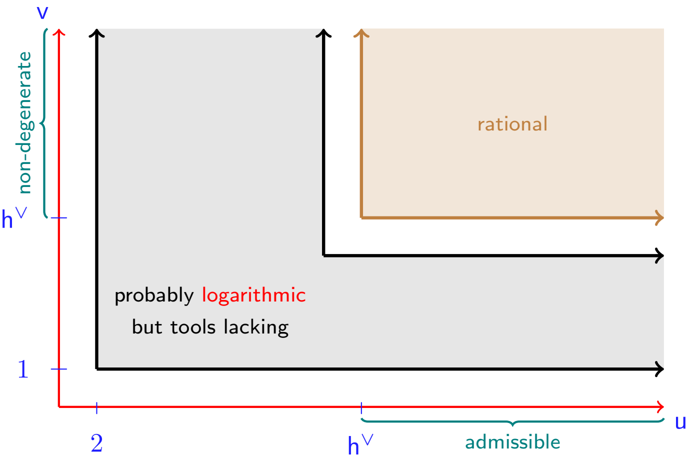
Affine VOA weight category

Given $k + \mathsf{h}^\vee = \frac{\mathsf{u}}{\mathsf{v}}$:

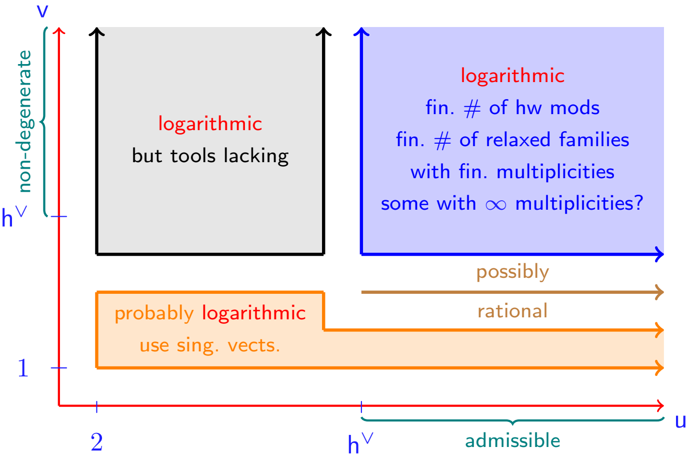


[I'll assume throughout that $\mathfrak{g}$ is simply laced for simplicity.]

Principal W-algebra weight category



Other W-algebra weight category



[This picture is mostly plausible speculation so don't hold me to it...]

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For $L_k(\mathfrak{g})$ with k admissible ($u \geqslant h^\vee$), the irreducible highest-weight modules were classified in [Arakawa'12].

Using coherent families [Mathieu'00], this was lifted to an algorithmic classification of irreducible relaxed highest-weight $L_k(\mathfrak{g})$-modules with finite multiplicities in [Kawasetsu–DR'19].

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In general, there also exist irreducible weight modules with infinite multiplicities, *eg.* when $\mathfrak{g} = \mathfrak{sl}_n$, $n > 1$, and $\mathsf{v} > 2$, some of which admit generalised characters.

The theory of these modules is currently poorly developed...

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But, reducibility occurs at finitely many parameter values in each family. The (finite-multiplicity) weight category is almost semisimple: its blocks are parametrised by a measure space, almost all of which are semisimple.

cf. log-rational VOAs like the triplet algebra, where there are finitely many blocks, more than one of which is non-semisimple [Gaberdiel–Kausch'96].

We call the semisimple (non-semisimple) blocks typical (atypical).

Example: $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$, $\mathsf{u}, \mathsf{v} \geqslant 2$

Let $K_{\mathsf{u},\mathsf{v}} = \{1, \dots, \mathsf{u}-1\} \times \{1, \dots, \mathsf{v}-1\}$ and let $\mathbb{Z}_2$ be generated by $(r,s) \to (\mathsf{u}-r, \mathsf{v}-s)$. Up to spectral flow, there are:
[Adamović–Milas'95, DR–Wood'15, Kawasetsu–DR'19, Adamović–Kawasetsu–DR'23]

- Irreducible highest-weight modules $\mathscr{H}_{r,s}$, for $(r,s) \in K_{\mathsf{u},\mathsf{v}}$;
- Irreducible relaxed highest-weight modules $\mathscr{R}_{[\lambda];r,s}$, for $(r,s) \in K_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_2$ and $[\lambda] \in (\mathbb{C}/2\mathbb{Z}) - \{[\lambda_{r,s}], [\lambda_{\mathsf{u}-r,\mathsf{v}-s}]\}$;
- Reducible relaxed highest-weight modules $\mathscr{R}_{[\lambda_{r,s}];r,s}$ and $\mathscr{R}_{[\lambda_{\mathsf{u}-r,\mathsf{v}-s}];r,s}$, for $(r,s) \in K_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_2$.

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Moreover: [Creutzig–DR'13, Creutzig–Kanade–Liu–DR'18, Arakawa–Creutzig–Kawasetsu'23]

- The irreducible $\mathscr{R}_{[\lambda];r,s}$ are projective and injective (typical);
- The projective covers/injective hulls of the $\mathscr{H}_{r,s}$ are glueings of spectral flows of 2 reducible $\mathscr{R}_{[\lambda];r,s}$ (atypical).

The measure space is (roughly speaking) a countably infinite product of copies of $\mathbb{C}/2\mathbb{Z}$ with the product Haar measure.

Example: $\mathsf{L}_{-3/2}(\mathfrak{sl}_3)$

Up to spectral flow, there are: [Arakawa–Futorny–Ramirez'16, Kawasetsu–DR'19]

- Irreducible highest-weight modules $\mathscr{H}_0$ and $\mathscr{H}_{-\rho/2}$;
- Irreducible "semirelaxed" highest-weight modules $\mathscr{S}_{[\mu]}$, for $[\mu] \in (-\frac{3}{2}\Lambda_1 + \mathbb{C}\alpha_1)/\mathbb{Z}\alpha_1 - \{[-\frac{3}{2}\Lambda_1], [-\frac{1}{2}\rho]\}$;
- Reducible semirelaxed highest-weight modules $\mathscr{S}_{[-3\Lambda_1/2]}$ and $\mathscr{S}_{[-\rho/2]}$;
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Conjecture: [Creutzig–DR–Rupert'21]

- The irreducible $\mathscr{R}_{[\mu]}$ are projective and injective (typical);
- The projective covers/injective hulls of the irreducible $\mathscr{S}_{[\mu]}$, $\mathscr{H}_{-\rho/2}$ and $\mathscr{H}_0$ are explicitly known glueings of 2, 3 and 6 reducible $\mathscr{R}_{[\mu]}$ (atypical of degrees 1, 2 and 2), respectively.

The measure space is a product of countably many copies of $\mathfrak{h}^*/\mathsf{Q}$.

W-algebra weight module categories

A few W-algebras may be constructed from other VOAs, *eg.* affine ones, as cosets (commutants) or ...

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This converts an affine VOA $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$ into a W-algebra $\mathsf{W}^{\mathsf{k}}_f(\mathfrak{g})$, $f \in \mathfrak{g}^{\text{nil.}}$:

- Complete f to an $\mathfrak{sl}_2$-triple $\{f, h, e\}$.
- Tensor $\widehat{\mathfrak{g}}_k$ with pairs of bc-ghosts, one for each positive root, and pairs of $\beta\gamma$-ghosts, one for each root with $\alpha(h) = 1$.
- Construct a fermionic field with conformal weight 1 and (fermionic) ghost number 1:

$$d(z) = \sum_{\alpha>0} \bigl[e^{\alpha}(z) - \langle f | e^{\alpha} \rangle\bigr] c^{\alpha}(z) + [\text{terms in } b^{\alpha}, c^{\alpha}, \beta^{\alpha}, \gamma^{\alpha}].$$

- d_0 is a differential, the fixed-$\#_1$ subspaces of $\mathsf{V}^{\mathsf{k}}(\mathfrak{g}) \otimes (bc)^{\#_1} \otimes (\beta\gamma)^{\#_2}$ define a differential complex, and the non-zero cohomology vanishes?
- The W-algebra $\mathsf{W}^{\mathsf{k}}_f(\mathfrak{g})$ is $H_k^{(0)}$. Its simple quotient is $\mathsf{W}_{\mathsf{k}}^f(\mathfrak{g})$.

Examples

- Taking $f = 0$ results in $\mathsf{W}^{\mathsf{k}}_f(\mathfrak{g}) = \mathsf{V}^{\mathsf{k}}(\mathfrak{g})$ (reduction does nothing).
- Taking $f = \sum_{\alpha \text{ simple}} f^\alpha$ gives the regular W-algebra: $\mathsf{W}^{\mathsf{k}}_{\text{reg.}}(\mathfrak{g})$.
- Taking $f = f^\theta$ gives the minimal W-algebra $\mathsf{W}^{\mathsf{k}}_{\text{min.}}(\mathfrak{g})$.
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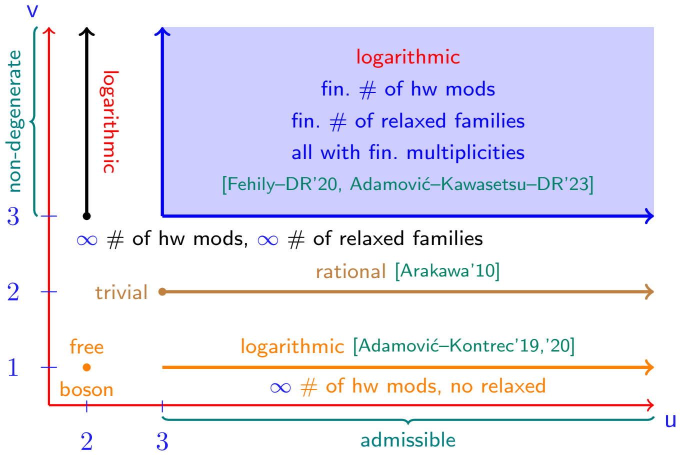
$\mathsf{W}^{\mathsf{k}}_{\text{reg.}}(\mathfrak{sl}_3)$	Zamolodchikov $\mathsf{W}^{\mathsf{k}}_3$
$\mathsf{W}^{\mathsf{k}}_{\text{reg.}}(\mathfrak{sl}_n)$	Casimir of type $(2, 3, 4, \ldots, n)$
$\mathsf{W}^{\mathsf{k}}_{\text{min.}}(\mathfrak{sl}_3) = \mathsf{W}^{\mathsf{k}}_{\text{sub.}}(\mathfrak{sl}_3)$	Bershadsky–Polyakov $\mathsf{W}^{(2),\mathsf{k}}_3$
$\mathsf{W}^{\mathsf{k}}_{\text{reg.}}(\mathfrak{osp}_{1\|2}) = \mathsf{W}^{\mathsf{k}}_{\text{min.}}(\mathfrak{osp}_{1\|2})$	$N = 1$
$\mathsf{W}^{\mathsf{k}}_{\text{reg.}}(\mathfrak{sl}_{2\|1}) = \mathsf{W}^{\mathsf{k}}_{\text{min.}}(\mathfrak{sl}_{2\|1})$	$N = 2$
$\mathsf{W}^{\mathsf{k}}_{\text{min.}}(\mathfrak{osp}_{3\|2}) = \mathsf{W}^{\mathsf{k}}_{\text{sub.}}(\mathfrak{osp}_{3\|2})$	small $N = 3$
$\mathsf{W}^{\mathsf{k}}_{\text{min.}}(\mathfrak{psl}_{2\|2}) = \mathsf{W}^{\mathsf{k}}_{\text{sub.}}(\mathfrak{psl}_{2\|2})$	small $N = 4$
$\mathsf{W}^{\mathsf{k}}_{\text{min.}}(\mathfrak{d}_{2\|1;\alpha})$	big $N = 4$

Example: $W_k^{\text{min.}}(\mathfrak{sl}_3)$ (Bershadsky–Polyakov)

The weight module category ($k + 3 = \frac{u}{v}$) has the following properties.



Inversion by example

Take $L_k(\mathfrak{sl}_2) \overset{\text{QHR}}{\longmapsto} W_k^{\text{reg.}}(\mathfrak{sl}_2) \equiv \text{Vir}_k$ at fractional k:

$$k + 2 = \frac{u}{v}, \quad u, v \in \mathbb{Z}_{\geqslant 2},\ \gcd\{u, v\} = 1.$$

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What can we learn about their representations?

	[ordinary]
	[highest-weight]
	[conjugate highest-weight]
	[relaxed highest-weight]
[ordinary]	[projective/injective]
	[spectral flows]
	[Whittaker]
	[others...]
Vir_k	$\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$

Free-field realisations suggest a path:

- Feigin–Fuchs say $\mathsf{Vir}^{\mathsf{k}} \hookrightarrow \mathsf{H}$.
- Wakimoto says $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2) \hookrightarrow \mathsf{H} \otimes \beta\gamma$.
- Bosonise the ghosts: $\beta\gamma \hookrightarrow \Pi$. [Friedan–Martinec–Shenker'86]
- Trade FF for FMS: $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2) \hookrightarrow \mathsf{Vir}^{\mathsf{k}} \otimes \Pi$.. [Semikhatov'94]
- Prove that $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2) \hookrightarrow \mathsf{Vir}_{\mathsf{k}} \otimes \Pi$ iff $k \notin \mathbb{N}$. [Adamović'17]

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Thus, every $M \in \mathsf{Vir}_{\mathsf{k}}$-mod and $N \in \Pi$-mod yield a representation

$$M \otimes N \in \mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)\text{-mod},$$

by restriction (for $k \notin \mathbb{N}$).

$\mathsf{Vir}_{\mathsf{k}}$ only has ordinary modules and weight Π-modules are spectral flow-relaxed, so we get spectral flows of relaxed $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-modules!

The Adamović functors

$$\mathsf{Vir}_{\mathsf{k}}\text{-mod} \to \mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)\text{-mod}, \qquad \mathscr{H} \mapsto \left(\mathscr{H} \otimes \Pi_\lambda^\ell\right)\downarrow,$$

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Theorem [Adamović–Kawasetsu–DR'20]:

- The image of an Adamović functor is a non-semisimple category.
- If $\mathscr{H}$ is irreducible, then its image under an Adamović functor is almost irreducible. [*cf.*, de Sole–Kac'05]

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Theorem [Adamović–Kawasetsu–DR'23]:

- Every irreducible relaxed $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-module is the image of an irreducible $\mathsf{Vir}_{\mathsf{k}}$-module under an Adamović functor.
- Every irreducible weight $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-module is a quotient of one in the image of an Adamović functor.

Beyond $\mathfrak{sl}_2$

Adamović functors for other simple affine W-(super)algebras are known:

- $(\mathsf{N}=1)_\mathsf{k} \to \mathsf{L}_\mathsf{k}(\mathfrak{osp}_{1|2})$ for k admissible but non-integral.
  [Adamović'17, Kawasetsu–DR'18, Creutzig–Kanade–Liu–DR'19]
- $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2) \to \mathsf{L}_\mathsf{k}(\mathfrak{sl}_{2|1})$ and $(\mathsf{N}=2)_\mathsf{k} \to \mathsf{L}_\mathsf{k}(\mathfrak{sl}_{2|1})$ for $\mathsf{k}+1=\frac{\mathsf{u}}{\mathsf{v}}$ admissible with $\begin{cases} \mathsf{u}\neq 1 \\ \mathsf{v}\neq 1 \end{cases}$.
  [Creutzig–Fasquel–Genra–DR'24]
- $\mathsf{W}^{\text{reg.}}_\mathsf{k}(\mathfrak{sl}_3) \to \mathsf{W}^{\text{min.}}_\mathsf{k}(\mathfrak{sl}_3)$ iff $k+3=\frac{\mathsf{u}}{\mathsf{v}}$ with $\mathsf{v}\geqslant 3$.
  [Adamović–Kawasetsu–DR'20]
- $\mathsf{W}^{\text{min.}}_\mathsf{k}(\mathfrak{sl}_3) \to \mathsf{L}_\mathsf{k}(\mathfrak{sl}_3)$ iff $k+3=\frac{\mathsf{u}}{\mathsf{v}}$ with $\mathsf{v}\geqslant 2$. [Adamović–Creutzig–Genra'21]
- $\mathsf{W}^{\text{reg.}}_\mathsf{k}(\mathfrak{sp}_4) \to \mathsf{W}^{\text{sub.}}_\mathsf{k}(\mathfrak{sp}_4)$ iff $k+3=\frac{\mathsf{u}}{\mathsf{v}}$ with $\mathsf{v}\geqslant 3$. [Fasquel–Fehily–DR'24]
- $\mathsf{W}^{\text{reg.}}_\mathsf{k}(\mathfrak{sl}_n) \to \mathsf{W}^{\text{sub.}}_\mathsf{k}(\mathfrak{sl}_n)$ iff $k+n=\frac{\mathsf{u}}{\mathsf{v}}$ with $\mathsf{v}\geqslant n$. [Fehily'21]

There are also many universal examples being worked out, *eg.*
[Fehily'23, Fasquel–Nakatsuka'23, Creutzig–Fasquel–Linshaw–Nakatsuka'24, Fasquel–Fehily–Nakatsuka'24, ...].

There is clearly a lot still to do...

Conclusions

It seems that the right way to analyse W-algebra CFTs is:

- Start with the regular W-algebra at an admissible but non-degenerate level. These are rational with known representation theories!
- Use inverse reduction to construct the relaxed modules of the subregular W-algebra. Get the other irreducibles as quotients.
- Repeat, working your way up the lattice of nilpotents until the representation theory of the desired W-algebra is known!

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If the level is admissible but degenerate, don't despair: start instead with a rational exceptional W-algebra. [Arakawa–van Ekeren'19, McRae'21]

- When $\mathsf{v} = 1$, $\mathsf{L}_\mathsf{k}(\mathfrak{g})$ is exceptional.
- For $\mathfrak{g} = \mathfrak{sl}_3$, $\mathsf{u} \geqslant 3$ and $\mathsf{v} = 2$, Bershadsky–Polyakov is exceptional.
- For $\mathfrak{g} = \mathfrak{sl}_n$, $\mathsf{u} \geqslant n$ and $\mathsf{v} = n-1$, the subregular is exceptional.

[This needs generalising to the super case...]

Outlook

- Inverse quantum hamiltonian reduction lets us analyse logarithmic CFTs with W-algebra symmetries.
- It allows us to classify irreducible weight modules, compute modular transformations and (Grothendieck) fusion rules.
- These ideas are also relevant to the construction of projective and injective modules, needed for the CFT state space, (genuine) fusion rules, correlation functions and other categorical data.
- It is said that WZW models are the building blocks of rational CFT. If the same is true for admissible-level WZW models and log CFT, then we can expect these methods to generalise widely!
- Either way, the future of these CFTs is looking good...

"*Only those who attempt the absurd will achieve the impossible.*"

— Miguel de Unamuno